On the convergence of Volterra series of finite dimensional quadratic MIMO systems

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Abstract

In this paper, the Volterra series decomposition of a class of quadratic, time invariant single-input finite dimensional systems is analyzed. The kernels are given by a recursive sequence of linear PDE's in the time domain, and an equivalent algebraic recursion in the Laplace domain. This is used to prove the convergence of the Volterra series to a (possibly weak) trajectory of the system, to provide a practicable value for the radius of convergence of the input in $L^\infty(\mathbb{R}_+)$ and to compute a guaranteed error bound in $L^\infty(\mathbb{R}_+)$ for the truncated series. The result is then extended to MIMO systems. A numerical simulation is performed on an academic SISO example, to illustrate how easily the truncated Volterra series can be implemented.

Index Terms

Volterra series, quadratic systems, convergence, input-output representation.

1- INTRODUCTION

Volterra and Fliess series are functional series expansions of the solution of nonlinear controlled systems. Volterra series were introduced by the Italian mathematician Vito Volterra [Vol59]. They can be viewed as the generalization of the transfer function of a linear system. There exists a vast literature concerning Volterra series. They were extensively studied from the geometrical control point of view for instance in [Bro76], [Gil77]. Fliess series decomposition for nonlinear analytic systems is another class of expansion that has been introduced and developed by Michel Fliess [Fli81]. An extensive review of Fliess and Volterra series can be found in [Isi95], [FLLL83]. Both decomposition are linked, and it is proved in [Fli81] that the Volterra decomposition is a re-ordering of the Fliess series decomposition in the case of analytic systems for which there exist a non empty open ball centered at zero where all the Volterra kernels have a convergent Taylor expansion at zero. Volterra series have also been considered within the framework of the input-output approach. In this context, nonlinear systems systems are expanded into a series of input-output operators (see for example [Boy85], [Rug81], [Sch89]).

These functional series expansions are convenient tools for on-line simulation or system identification [DPO02], but it is often difficult to obtain convergence results and bounds for the series.

This paper is focused on input-output representations into Volterra series. The Volterra series decomposition of a class of quadratic in the state, time invariant, single-input finite dimensional systems is performed and analyzed. The kernels are given by a recursive sequence of linear pde's. This is used to prove the convergence of the Volterra series in $L^\infty(\mathbb{R}_+)$ to a (possibly weak) trajectory of the system. A value for the radius of convergence of the input in $L^\infty(\mathbb{R}_+)$ is computed, and a uniform guaranteed error bound for the truncated series is given. The result is then extended to MIMO systems.

2- VOLTERRA SERIES

2.1- Volterra series of time-variant systems

Following [LL94, p.113], the Volterra series of a time-variant system can be defined as follows.

Definition 1: A causal SISO-system can be described by a Volterra series $\{h_m\}_{m\in\mathbb{N}}$ if there exists functions $h_m : \mathbb{R}_+^{m+1} \to \mathbb{R}$, for $m \in \mathbb{N}$ which are locally bounded, piecewise continuous and such that, for all $T > 0$, there exists $\epsilon(T) > 0$ such that for all piecewise continuous function u satisfying $|u(t)| \leq \epsilon$, $\forall t \in [0,T]$ the series

$$y(t) = h_0(t) + \sum_{m\in\mathbb{N}^*} \int_{[0,t]^m} h_m(t,\tau_{1,m}) \prod_{j=1}^{m} u(\tau_j)\,\mathrm{d}\tau_{1,m} \tag{1}$$

is *normally convergent*, where for $1 \leq p \leq q$,

$$\begin{aligned} \tau_{p,q} &= (\tau_p, \tau_{p+1}, \ldots, \tau_q) \\ \mathrm{d}\tau_{p,q} &= \prod_{j=p}^{q} \mathrm{d}\tau_j \end{aligned} \tag{2}$$

Nevertheless, natural extensions to more general settings can be defined. For example, taking h_m in $L^1_{loc}(\mathbb{R}_+^{m+1})$ or $L^\infty\left(\mathbb{R}_+, L^1_{loc}(\mathbb{R}_+^m)\right)$ still yields well-posed definitions. In this paper, more specific spaces will be introduced in section 2.2.2.

2.2- Volterra series of time-invariant systems and some properties

We refer to [Boy85], [Has99], [Rug81] for developments in this section.

2.2.1- Definition in the time domain and the Laplace domain: For a time-invariant system, the kernels are such that, for $m \in \mathbb{N}^*$,

$$h_m(t,\tau_{1,m}) = \tilde{h}_m(t-\tau_{1,m}), \tag{3}$$

where $t-\tau_{1,m} = (t-\tau_1, \ldots, t-\tau_m)$. Moreover, the *zero-input response* of the system h_0 can be omitted considering the difference output $\tilde{y}(t) = y(t) - h_0(t)$. Then, equation (1) reduces to a sum of standard *multi-convolutions* given by

$$\tilde{y}(t) = \sum_{m\in\mathbb{N}^*} \int_{[0,t]^m} \tilde{h}_m(t-\tau_{1,m}) \prod_{j=1}^{m} u(\tau_j)\,\mathrm{d}\tau_{1,m} = \sum_{m\in\mathbb{N}^*} \int_{[0,t]^m} \tilde{h}_m(t_{1,m}) \prod_{j=1}^{m} u(t-t_j)\,\mathrm{d}t_{1,m} \tag{4}$$

For sake of legibility and conciseness, the tilde of $\tilde{h}_m(t_{1,m})$ will be omitted. This notation is not ambiguous since the number of independent variables in h_m make the time-variant and time-invariant versions distinguishable with $m+1$ and m variables, respectively.

The mono-lateral Laplace transform of the time-invariant kernels is denoted with capital letters and defined by, $\forall m \in \mathbb{N}^*$, $\forall s_{1,m} \in \mathcal{D}_{h_m} \subset \mathbb{C}^m$,

$$H_m(s_{1,m}) = \int_{\mathbb{R}_+^m} h_m(t_{1,m}) \mathrm{e}^{-s_{1,m}\cdot t_{1,m}} \mathrm{d}t_{1,m}, \tag{5}$$

where $\mathcal{D}_{h_m}$ denotes the domain of convergence of the Laplace transform and $s_{1,m}\cdot t_{1,m} = s_1\,t_1 + \cdots + s_m\,t_m$. For stable systems, the kernels H_m are analytic in $\mathcal{D}_{h_m} \subset \left(\mathbb{C}_0^+\right)^m$ where $\mathbb{C}_0^+ = \left\{s \in \mathbb{C} \,\middle|\, \Re\mathrm{e}(s_k) > 0\right\}$.

2.2.2- Functional spaces, characteristic function and a BIBO-convergence theorem:

Definition 2 (Functional spaces): Let $(m,n) \in \mathbb{N}^* \times \mathbb{N}^*$ and $p \in [1,\infty]$. The spaces $\mathcal{V}_p^{m,n}$ and $\mathcal{B}_p^n$ are defined by

$$\mathcal{V}_p^{m,n} = L^1(\mathbb{R}_+^m, \mathbb{R}_p^n) \quad (6)$$

$$\mathcal{B}_p^n = L^\infty(\mathbb{R}_+, \mathbb{R}_p^n) \quad (7)$$

where $\mathbb{R}_p^n$ is the euclidean space of dimension n endowed with the standard p-norm defined by $\|\boldsymbol{x}\|_p = (|x_1|^p + \cdots + |x_N|^p)^{1/p}$ for $p \in [1,\infty[$ and by $\|\boldsymbol{x}\|_\infty = \max(|x_1|,\ldots,|x_N|)$ for $p=\infty$. When $n=1$, all the p-norms are identical so that p is omitted in this case.

Definition 3 (Characteristic function of a time-invariant SISO-system): Let $\{h_m\}_{m\in\mathbb{N}^*}$ be the Volterra series of a time-invariant SISO-system, such that $\forall m \in \mathbb{N}^*, \|h_m\|_{\mathcal{V}^{m,1}} = \int_{\mathbb{R}_+^m} |h_m(t_{1,m})| \mathrm{d}t_{1,m}$ is bounded. The characteristic function φ_h of $\{h_m\}_{m\in\mathbb{N}^*}$ is defined by the power series

$$\varphi_h(z) = \sum_{m\in\mathbb{N}^*} \|h_m\|_{\mathcal{V}^{m,1}} z^m, \quad \forall |z| < \rho, \quad (8)$$

where ρ is the radius of convergence of the power series.

When a system is such that $\rho > 0$, it satisfies the BIBO property given by the following theorem.

Theorem 1: Let $\{h_m\}_{m\in\mathbb{N}^*}$ be the Volterra series of a time-invariant SISO-system such that the characteristic function φ_h has a radius of convergence such that $\rho > 0$. The Volterra series is *convergent* in $\mathcal{B}^1$ for inputs such that $\|u\|_{\mathcal{B}^1} < \rho$. In this case, the output y is bounded and satisfies

$$\|y\|_{\mathcal{B}^1} \leq \varphi_h\big(\|u\|_{\mathcal{B}^1}\big), \quad (9)$$

where it is recalled that $\|u\|_{\mathcal{B}^1} = \sup_{t\in\mathbb{R}_+} |u(t)|$.

This result is quite interesting for system analysis since it is non-local in time. Nevertheless, it requires the determination of the radius of convergence ρ and bounding $\|h_m\|_{\mathcal{V}^{m,1}}$ is not straightforward. This paper copes with this practical problem and establishes practicable BIBO-results.

2.2.3- Interconnection laws: Let $\{f_m\}_{m\in\mathbb{N}^*}$ and $\{g_m\}_{m\in\mathbb{N}^*}$ be the Volterra kernels of two systems, associated to the characteristc functions φ_f and φ_g with radius of convergence ρ_f and ρ_g, respectively.

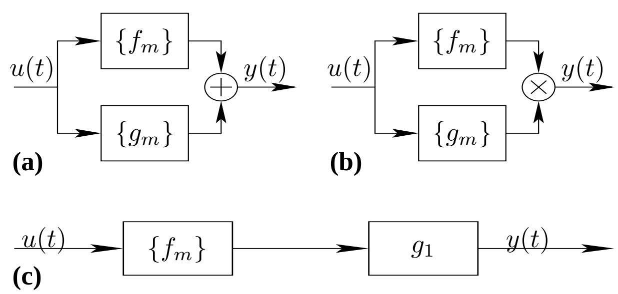

Fig. 1. *Sum (a), product (b), and cascade (c) of two systems*

Connecting these systems through a sum of outputs, a product of outputs or a cascade (Fig. 1a,b,c) still defines a Volterra series [Has99, p. 34,35] with kernels $\{h_m\}_{m\in\mathbb{N}^*}$ such that for $m \in \mathbb{N}^*$,

- Sum (Fig. 1a):

$$h_m(t_{1,m}) = f_m(t_{1,m}) + g_m(t_{1,m}), \qquad \forall t_{1,m} \in \big(\mathbb{R}_+\big)^m \quad (10)$$

$$H_m(s_{1,m}) = F_m(s_{1,m}) + G_m(s_{1,m}), \qquad \forall s_{1,m} \in \mathcal{D}_{f_m} \cap \mathcal{D}_{g_m} \quad (11)$$

$$\varphi_h(z) \leq \varphi_f(z) + \varphi_g(z), \qquad \text{for } |z| \leq \rho_h$$

$$\rho_h \geq \min(\rho_f, \rho_g)$$

- Product (Fig. 1b):

$$h_m(t_{1,m}) = \sum_{k=1}^{m-1} f_k(t_{1,k})\, g_{m-k}(t_{k+1,m}), \qquad \forall t_{1,m} \in \mathbb{R}_+^m \quad (12)$$

$$H_m(s_{1,m}) = \sum_{k=1}^{m-1} F_k(s_{1,k})\, G_{m-k}(s_{k+1,m}), \qquad \forall s_{1,m} \in \bigcap_{1\leq p\leq m-1} \big(\mathcal{D}_{f_k} \times \mathcal{D}_{g_{m-k}}\big) \quad (13)$$

$$\varphi_h(z) \leq \varphi_f(z)\, \varphi_g(z), \qquad \text{for } |z| \leq \rho_h$$

$$\rho_h \geq \min(\rho_f, \rho_g)$$

- Cascade with a linear system (Fig. 1c):

$$h_m(t_{1,m}) = \int_{[0,\min(t_{1,m})]} g_1(\theta_1) f_m(t_{1,m} - \theta_1) \mathrm{d}\theta_1 \tag{14}$$

$$H_m(s_{1,m}) = G_1(\widehat{s_{1,m}}) F_m(s_{1,m}), \qquad \forall s_{1,m} \in \{ s_{1,m} \in \mathcal{D}_{f_m} \,|\, \widehat{s_{1,m}} \in \mathcal{D}_{g_1} \}, \tag{15}$$

$$\varphi_h(z) \leq \|g_1\|_{\mathcal{V}^{1,1}} \varphi_f(z), \qquad \text{for } |z| \leq \rho_h = \rho_f$$

where $\widehat{s_{1,m}}$ denotes the sum of the Laplace variables

$$\widehat{s_{1,m}} = s_1 + \cdots + s_m. \tag{16}$$

This case is such that the output of $\{f_m\}$ feeds a linear system ($g_m = 0, \forall m \geq 2$). The similar (but more complex) relations for the concatenation of two Volterra series are not presented here (see e.g. [Has99]) as they are useless in the following.

3- Quadratic SISO systems

3.1- System under consideration

Let the quadratic ODE system be defined by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \begin{bmatrix} \boldsymbol{x}^T \boldsymbol{E}_1 \boldsymbol{x} \\ \vdots \\ \boldsymbol{x}^T \boldsymbol{E}_N \boldsymbol{x} \end{bmatrix} + \boldsymbol{B}\, u, \tag{17}$$

$$y = \boldsymbol{C}\,\boldsymbol{x}, \tag{18}$$

for $t \in \mathbb{R}^+$ with $\boldsymbol{x}(0) = 0$, where $u(t) \in \mathbb{R}$, $\boldsymbol{x}(t) \in \mathbb{R}^N$ and $\boldsymbol{y}(t) \in \mathbb{R}$ are the *input*, *state* and *output* of the system, respectively. All matrices are real and $\boldsymbol{A}$ is $N \times N$, $\boldsymbol{B}$ is $N \times 1$, $\boldsymbol{C}$ is $1 \times N$, and $\boldsymbol{E}_n$ ($n = 1, \ldots, N$) are $N \times N$.

This type of equations (and also the MIMO case) is for instance frequently encountered in chemical kinetics or biochemical modelling. The well known "law of Mass Action" applied to a set of reaction, each of them being of the form $[A] + [B] \rightleftharpoons [C] + [D]$, leads to dynamical systems with quadratic nonlinearity in the state. This type of nonlinear model is also encountered for instance in the field of epidemiological population dynamics, where the so-called SIR models also show quadratic state nonlinearity. Finally, this system can also be viewed as a 2^{nd} order approximation of a more involved nonlinear system of the form $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{B}\, u$, $y = \boldsymbol{C}\,\boldsymbol{x}$ around the initial state $\boldsymbol{x}(0) = \boldsymbol{0}$.

Definition 4 (Strong and weak solutions): Let $C^1_0(\mathbb{R}_+, \mathbb{R}^N)$ denote the set of all C^1, $\mathbb{R}^N$-valued functions with compact support in $\mathbb{R}_+$.
$(\boldsymbol{x}, y)$ is said to be a *weak solution* of (17-18) in $\mathcal{B}^N_p \times \mathcal{B}^1$ with $p \in [1, \infty]$ iff

$$\forall \boldsymbol{w} \in C^1_0(\mathbb{R}_+, \mathbb{R}^N), \; \int_{\mathbb{R}_+} \dot{\boldsymbol{w}}^T \boldsymbol{x} \mathrm{d}t + \int_{\mathbb{R}_+} \boldsymbol{w}^T \boldsymbol{A}\boldsymbol{x} \mathrm{d}t + \int_{\mathbb{R}_+} \boldsymbol{w}^T \begin{bmatrix} \boldsymbol{x}^T \boldsymbol{E}_1 \boldsymbol{x} \\ \vdots \\ \boldsymbol{x}^T \boldsymbol{E}_N \boldsymbol{x} \end{bmatrix} dt + \int_{\mathbb{R}_+} \boldsymbol{w}^T \boldsymbol{B}\, u \mathrm{d}t = 0, \tag{19}$$

and y satisfies (18). Moreover, $(\boldsymbol{x}, y)$ is said to be a strong solution of (17-18) on $\mathbb{R}_+$ if it is a weak solution and $\boldsymbol{x}$ is $C^1(\mathbb{R}_+, \mathbb{R}^N)$.

3.2- Derivation of the Volterra kernels

3.2.1- Output and state kernels: As the nonlinearity of the system is embedded in the state equation (17), it is quite convenient to consider the Volterra series which maps the input u to each state coordinate x_n. Thus, let $\{g_m\}_{m \in \mathbb{N}^*}$ denote the Volterra series of the SISO-system $\mathcal{S}_{u \mapsto y}$ with input u and output y. Let $\{h^n_m\}_{m \in \mathbb{N}^*}$ for $1 \leq n \leq N$ denote the Volterra series of the SISO-system $\mathcal{S}_{u \mapsto \boldsymbol{x}_n}$ with input u and output x_n. Let $\{\boldsymbol{h}_m\}_{m \in \mathbb{N}^*}$ denote the Volterra series of the SIMO-system $\mathcal{S}_{u \mapsto \boldsymbol{x}}$ with input u and output $\boldsymbol{x}$ so that $\boldsymbol{h}_m = [h^1_m, \ldots, h^N_m]^T$.

From (10-11) and (18), $\{g_m\}_{m \in \mathbb{N}^*}$ are related to $\{\boldsymbol{h}_m\}_{m \in \mathbb{N}^*}$ through the equations, for $m \in \mathbb{N}^*$,

$$g_m(t_{1,m}) = \boldsymbol{C}\, \boldsymbol{h}_m(t_{1,m}) \tag{20}$$

$$G_m(s_{1,m}) = \boldsymbol{C}\, \boldsymbol{H}_m(s_{1,m}) \tag{21}$$

in the time and the Laplace domains, respectively.

3.2.2- Cancelling system: A convenient way to derive a set of equations satisfied by the kernels is to build the system described in Fig. 2. From (17), it is the null-system so that all its Volterra kernels are zero. Writing these zero-kernels from the interconnection laws (10-15) yields the equations satisfied by kernels $\{\boldsymbol{h}_m\}$. In the Laplace domain, these equations are algebraic. In the time domain, they are linear differential.

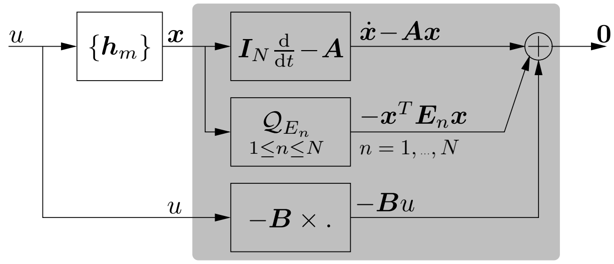

Fig. 2. *Cancelling system for $\mathcal{S}_{u \mapsto \boldsymbol{x}}$*

3.2.3- Kernels in the Laplace domain: In the *upper branch* of the cancelling system (see Fig. 2), the Laplace transform of the linear operator is $s \mapsto \boldsymbol{I}_N s - \boldsymbol{A}$. The kernels of the system $\mathcal{S}_{u \mapsto \dot{\boldsymbol{x}} - \boldsymbol{A}\boldsymbol{x}}$ associated to this upper branch can be derived, for each coordinate $[\dot{\boldsymbol{x}} - \boldsymbol{A}\boldsymbol{x}]_n$ from (11) and (15). They are given by $\left[\widehat{s_{1,m}} \boldsymbol{I_N} - \boldsymbol{A}\right] \boldsymbol{H}_m(s_{1,m})$, for $m \in \mathbb{N}^*$. The *middle branch* corresponds to the quadratic nonlinearities $\mathcal{Q}_{E_n}$. For each sub-system $\mathcal{S}_{u \mapsto -\boldsymbol{x}^T \boldsymbol{E}_n \boldsymbol{x}}$ $(1 \leq n \leq N)$, the kernels are deduced from (13). They are given by $\sum_{k=1}^{m-1} \left(\boldsymbol{H}_k(s_{1,k})\right)^T \boldsymbol{E}_n \boldsymbol{H}_{m-k}(s_{k+1,m})$ for $m \in \mathbb{N}^*$, which is zero for $m = 1$. The *bottom branch* defines the memoryless linear system $\mathcal{S}_{u \mapsto \boldsymbol{B}u}$ associated to the constant kernel $s_1 \mapsto \boldsymbol{B}$ for $m = 1$ and to zero kernels $s_{1,m} \mapsto \mathbf{0}$ for $m \geq 2$. Now, writing from (11) that the sum of these kernels is zero yields the recurrent algebraic equations

$$\boldsymbol{H}_m(s_{1,m}) = \left[\widehat{s_{1,m}} \boldsymbol{I_N} - \boldsymbol{A}\right]^{-1} \boldsymbol{F}_m(s_{1,m}), \qquad \forall m \in \mathbb{N}^* \tag{22}$$

$$\boldsymbol{F}_1(s_1) = \boldsymbol{B} \tag{23}$$

$$\boldsymbol{F}_m(s_{1,m}) = \sum_{k=1}^{m-1} \begin{bmatrix} (\boldsymbol{H}_k(s_{1,k}))^T \boldsymbol{E}_1 \boldsymbol{H}_{m-k}(s_{k+1,m}) \\ \vdots \\ (\boldsymbol{H}_k(s_{1,k}))^T \boldsymbol{E}_N \boldsymbol{H}_{m-k}(s_{k+1,m}) \end{bmatrix}, \qquad \forall m \geq 2 \tag{24}$$

3.2.4- Kernels in the time domain: Using the notation of the time-variant systems $\boldsymbol{h}_m(t, \tau_{1,m})$ rather than the time-invariant version $\boldsymbol{h}_m(t_{1,m})$ with $t_{1,m} = t - \tau_{1,m}$, the time domain versions of (22-24) are

$$\left[\boldsymbol{I_N} \partial_t - \boldsymbol{A}\right] \boldsymbol{h}_m(t, \tau_{1,m}) = \boldsymbol{f}_m(t, \tau_{1,m}), \qquad \forall m \in \mathbb{N}^* \tag{25}$$

$$\boldsymbol{f}_1(t, \tau_1) = \boldsymbol{B} \delta(t - \tau_1) \tag{26}$$

$$\boldsymbol{f}_m(t, \tau_{1,m}) = \sum_{k=1}^{m-1} \begin{bmatrix} (\boldsymbol{h}_k(t, \tau_{1,k}))^T \boldsymbol{E}_1 \boldsymbol{h}_{m-k}(t, \tau_{k+1,m}) \\ \vdots \\ (\boldsymbol{h}_k(t, \tau_{1,k}))^T \boldsymbol{E}_N \boldsymbol{h}_{m-k}(t, \tau_{k+1,m}) \end{bmatrix}, \qquad \forall m \geq 2 \tag{27}$$

The solution is

$$\boldsymbol{h}_1(t, \tau_1) = \mathbf{1}_{\mathbb{R}^+}(t - \tau_1) \, \mathrm{e}^{\boldsymbol{A}(t - \tau_1)} \boldsymbol{B} \tag{28}$$

$$\boldsymbol{h}_m(t, \tau_{1,m}) = \mathbf{1}_{\mathbb{R}^+}\left(t - \max(\tau_{1,m})\right) \int_{\max(\tau_{1,m})}^{t} \mathrm{e}^{\boldsymbol{A}(t - \theta)} \boldsymbol{f}_m(\theta, \tau_{1,m}) \mathrm{d}\theta, \qquad \forall m \geq 2, \tag{29}$$

where $\mathbf{1}_{\mathbb{R}^+}$ denotes the Heaviside function.

3.3- Convergence and guaranteed error bound

In this section, standard p-norms of vectors $\boldsymbol{x}$, matrices $\boldsymbol{M}$ and bilinear forms $\boldsymbol{E}$ are considered for a fixed $p \in [1, \infty]$ and given by,

$$\|\boldsymbol{x}\|_p = \left(|x_1|^p + \cdots + |x_N|^p\right)^{1/p} \qquad \text{if } p \in [1, \infty[, \tag{30}$$

$$\|\boldsymbol{x}\|_p = \max(|x_1|, \ldots, |x_N|) \qquad \text{if } p = \infty, \tag{31}$$

$$\|\boldsymbol{M}\|_p = \sup_{\|\boldsymbol{x}\|_p = 1} \|\boldsymbol{M}\boldsymbol{x}\|_p \tag{32}$$

$$\|\boldsymbol{E}\|_{\mathcal{Q}_p} = \sup_{\|\boldsymbol{x}\|_p = 1, \|\boldsymbol{y}\|_p = 1} |\boldsymbol{y}^T \boldsymbol{E} \boldsymbol{x}|. \tag{33}$$

Theorem 2: Consider system (17) with $\max\Big(\Re\mathrm{e}\big(\mathrm{Spec}\,(\boldsymbol{A})\big)\Big) < 0$. Let $\{\boldsymbol{h}_m\}_{m \in \mathbb{N}^*}$ be the Volterra kernels defined by (28-29). Then, for all $p \in [1, +\infty]$,

$$\|\boldsymbol{h}_m\|_{\mathcal{V}_p^{m,N}} \leq \Phi_m \left(\epsilon_p \, \alpha_p\right)^{m-1} \left(\|\boldsymbol{h}_1\|_{\mathcal{V}_p^{1,N}}\right)^{m-1}, \tag{34}$$

with

$$\begin{aligned}\epsilon_p &= \left\| \left[\|\boldsymbol{E}_1\|_{\mathcal{Q}_p} \ldots \|\boldsymbol{E}_N\|_{\mathcal{Q}_p} \right]^T \right\|_p, & (35)\\ \alpha_p &= \int_{\mathbb{R}_+} \left\| \mathrm{e}^{\boldsymbol{A}\xi} \right\|_p \mathrm{d}\xi < \infty, & (36)\\ \Phi_1 &= 1, & (37)\\ \Phi_m &= \sum_{k=1}^{m-1} \Phi_k \Phi_{m-k} \qquad \forall m \geq 2. & (38)\end{aligned}$$

Note that $\|\boldsymbol{h}_1\|_{\mathcal{V}_p^{1,N}} \leq \alpha_p \|\boldsymbol{B}\|_p$.

Proof:

- **First step:** $\|\boldsymbol{h}_m\|_{\mathcal{V}_p^{m,N}} \leq \epsilon_p \, \alpha_p \sum_{k=1}^{m-1} \|\boldsymbol{h}_k\|_{\mathcal{V}_p^{k,N}} \|\boldsymbol{h}_{m-k}\|_{\mathcal{V}_p^{m-k,N}}$ for $m \geq 2$.

$$\begin{aligned}\|\boldsymbol{h}_m\|_{\mathcal{V}_p^{m,N}} &= \int_{\mathbb{R}_+^m} \left\| \int_{\max(-t_{1,m})}^0 \mathrm{e}^{-\boldsymbol{A}\theta} \boldsymbol{f}_m(\theta, -t_{1,m}) \mathrm{d}\theta \right\|_p \mathrm{d}t_{1,m} \quad \text{from (3) and (29) choosing } t=0\\ &\leq \int_{\mathbb{R}_+^m} \int_0^{\min(t_{1,m})} \left\| \mathrm{e}^{\boldsymbol{A}\xi} \boldsymbol{f}_m(-\xi, -t_{1,m}) \right\|_p \mathrm{d}\xi \mathrm{d}t_{1,m}\\ &\leq \int_{\mathbb{R}_+} \int_{[\xi,+\infty[^m} \left\| \mathrm{e}^{\boldsymbol{A}\xi} \right\|_p \|\boldsymbol{f}_m(-\xi, -t_{1,m})\|_p \mathrm{d}t_{1,m} \mathrm{d}\xi\\ &\leq \int_{\mathbb{R}_+} \left\| \mathrm{e}^{\boldsymbol{A}\xi} \right\|_p \left(\int_{[\xi,+\infty[^m} \|\boldsymbol{f}_m(-\xi, -t_{1,m})\|_p \mathrm{d}t_{1,m} \right) \mathrm{d}\xi,\end{aligned} \tag{39}$$

Now, from (27),

$$\begin{aligned}\|\boldsymbol{f}_m(t, \tau_{1,m})\|_p &= \left\| \sum_{k=1}^{m-1} \begin{bmatrix} (\boldsymbol{h}_k(t,\tau_{1,k}))^T \boldsymbol{E}_1 \boldsymbol{h}_{m-k}(t,\tau_{k+1,m}) \\ \vdots \\ (\boldsymbol{h}_k(t,\tau_{1,k}))^T \boldsymbol{E}_N \boldsymbol{h}_{m-k}(t,\tau_{k+1,m}) \end{bmatrix} \right\|_p\\ &\leq \sum_{k=1}^{m-1} \left\| \begin{bmatrix} (\boldsymbol{h}_k(t,\tau_{1,k}))^T \boldsymbol{E}_1 \boldsymbol{h}_{m-k}(t,\tau_{k+1,m}) \\ \vdots \\ (\boldsymbol{h}_k(t,\tau_{1,k}))^T \boldsymbol{E}_N \boldsymbol{h}_{m-k}(t,\tau_{k+1,m}) \end{bmatrix} \right\|_p\\ &\leq \sum_{k=1}^{m-1} \|\boldsymbol{h}_k(t,\tau_{1,k})\|_p \left\| \left[\|\boldsymbol{E}_1\|_{\mathcal{Q}_p}, \ldots, \|\boldsymbol{E}_N\|_{\mathcal{Q}_p} \right]^T \right\|_p \|\boldsymbol{h}_{m-k}(t,\tau_{k+1,m})\|_p\\ &\leq \left\| \left[\|\boldsymbol{E}_1\|_{\mathcal{Q}_p}, \ldots, \|\boldsymbol{E}_N\|_{\mathcal{Q}_p} \right]^T \right\|_p \left(\sum_{k=1}^{m-1} \|\boldsymbol{h}_k(t,\tau_{1,k})\|_p \|\boldsymbol{h}_{m-k}(t,\tau_{k+1,m})\|_p \right)\end{aligned} \tag{40}$$

Thus, denoting $\epsilon_p = \left\| \left[\|\boldsymbol{E}_1\|_{\mathcal{Q}_p}, \ldots, \|\boldsymbol{E}_N\|_{\mathcal{Q}_p} \right]^T \right\|_p$,

$$\begin{aligned}\int_{[\xi,+\infty[^m} \|\boldsymbol{f}_m(-\xi, -t_{1,m})\|_p \mathrm{d}t_{1,m} &\leq \epsilon_p \int_{[\xi,+\infty[^m} \left(\sum_{k=1}^{m-1} \|\boldsymbol{h}_k(-\xi, -t_{1,k}))\|_p \|\boldsymbol{h}_{m-k}(-\xi, -t_{k+1,m})\|_p \right) \mathrm{d}t_{1,m}\\ &\leq \epsilon_p \sum_{k=1}^{m-1} \left(\int_{[\xi,+\infty[^k} \|\boldsymbol{h}_k(-\xi, -t_{1,k}))\|_p \mathrm{d}t_{1,k} \right) \left(\int_{[\xi,+\infty[^{m-k}} \|\boldsymbol{h}_{m-k}(-\xi, -t_{k+1,m})\|_p \mathrm{d}t_{k+1,m} \right)\\ &\leq \epsilon_p \sum_{k=1}^{m-1} \left(\int_{[\xi,+\infty[^k} \|\boldsymbol{h}_k(t_{1,k} - \xi)\|_p \mathrm{d}t_{1,k} \right) \left(\int_{[\xi,+\infty[^{m-k}} \|\boldsymbol{h}_{m-k}(t_{k+1,m} - \xi)\|_p \mathrm{d}t_{k+1,m} \right)\\ &\leq \epsilon_p \sum_{k=1}^{m-1} \|\boldsymbol{h}_k\|_{\mathcal{V}_p^{k,N}} \|\boldsymbol{h}_{m-k}\|_{\mathcal{V}_p^{m-k,N}}\end{aligned} \tag{41}$$

This completes the proof defining α_p as $\alpha_p = \int_{\mathbb{R}_+} \left\| \mathrm{e}^{\boldsymbol{A}\xi} \right\|_p \mathrm{d}\xi$.

- **Second step:** Recurrence on $\mathcal{H}_m : \|\boldsymbol{h}_m\|_{\mathcal{V}_p^{m,N}} \leq \Phi_m \left(\epsilon_p\, \alpha_p\right)^{m-1} \left(\|\boldsymbol{h}_1\|_{\mathcal{V}_p^{1,N}}\right)^{m-1}$.
  - $\mathcal{H}_1$ is satisfied (with equality) by defining $\Phi_1 = 1$;
  - Now, let $m \geq 2$ and suppose $\mathcal{H}_{m'}$ satisfied for $1 \leq m' \leq m-1$.
    From step 1 and denoting $K_p = \epsilon_p\, \alpha_p$,

$$\begin{aligned}
\|\boldsymbol{h}_m\|_{\mathcal{V}_p^{m,N}} &\leq K_p \sum_{k=1}^{m-1} \|\boldsymbol{h}_k\|_{\mathcal{V}_p^{k,N}} \|\boldsymbol{h}_{m-k}\|_{\mathcal{V}_p^{m-k,N}} \\
&\leq K_p \sum_{k=1}^{m-1} \left(\frac{\Phi_k}{K_p}\left(K_p\, \|\boldsymbol{h}_1\|_{\mathcal{V}_p^{1,N}}\right)^k\right)\left(\frac{\Phi_{m-k}}{K_p}\left(K_p\, \|\boldsymbol{h}_1\|_{\mathcal{V}_p^{1,N}}\right)^{m-k}\right) \\
&\leq \left(K_p\, \|\boldsymbol{h}_1\|_{\mathcal{V}_p^{1,N}}\right)^m \frac{1}{K_p} \sum_{k=1}^{m-1} \Phi_k \Phi_{m-k} \\
&\leq \left(K_p\, \|\boldsymbol{h}_1\|_{\mathcal{V}_p^{1,N}}\right)^m \frac{\Phi_m}{K_p} \qquad (42)
\end{aligned}$$

    where $\Phi_m = \sum_{k=1}^{m-1} \Phi_k \Phi_{m-k}$, which concludes the proof.

■

Definition 5: Consider the system (17) with $\max\Big(\Re\text{e}\big(\text{Spec}\,(\boldsymbol{A})\big)\Big) < 0$ so that $\|\boldsymbol{h}_1\|_{\mathcal{V}_p^{1,N}} < \infty$. Then, the application $Z_p : \mathcal{B}^1 \to \mathbb{R}_+$ is defined by

$$Z_p(u) = \epsilon_p\, \alpha_p\, \|\boldsymbol{h}_1\|_{\mathcal{V}_p^{1,N}}\, \|u\|_{\mathcal{B}^1}. \qquad (43)$$

Corollary 1: Let $p \in [1, \infty]$.

(i) The Volterra series expansion of the state and output of system (17-18) converges in $\mathcal{B}_p^N$ and $\mathcal{B}^1$, respectively, for all input $u \in \mathcal{B}^1$ such that

$$Z_p(u) < 1/4 \qquad (44)$$

where Z_p is given in definition 5.

(ii) For any input $u \in \mathcal{B}^1$ satisfying (44), the sum of the series is a *weak solution* of the system. If u is moreover in $C^0(\mathbb{R}_+, \mathbb{R})$, then the sum of the series is a *strong solution* of system (17-18).

(iii) The output y and the state $\boldsymbol{x}$ are bounded as:

$$\begin{aligned}
\|y\|_{\mathcal{B}^1} &\leq \|\boldsymbol{C}\|_p \|\boldsymbol{x}\|_{\mathcal{B}_p^N}, \qquad (45)\\
\|\boldsymbol{x}\|_{\mathcal{B}_p^N} &\leq \varphi_{\boldsymbol{h},p}\big(\|u\|_{\mathcal{B}^1}\big) \leq \frac{1-\sqrt{1-4Z_p(u)}}{2\epsilon_p \alpha_p}, \qquad (46)
\end{aligned}$$

where ϵ_p and α_p are defined in theorem 2 and $\varphi_{\boldsymbol{h},p}$ denotes the characteristic function

$$\varphi_{\boldsymbol{h},p}(z) = \sum_{m=1}^{\infty} \|\boldsymbol{h}_m\|_{\mathcal{V}_p^{m,N}}\, z^m, \qquad (47)$$

which generalizes definition 3 to SIMO-systems.

Proof:

(i) From (18), $\|y\|_{\mathcal{B}^1} \leq \|\boldsymbol{C}\|_p \|\boldsymbol{x}\|_{\mathcal{B}_p^N}$. Moreover,

$$\begin{aligned}
\|\boldsymbol{x}\|_{\mathcal{B}_p^N} &\leq \sup_{t\in\mathbb{R}_+} \sum_{m=1}^{\infty} \int_{[0,t]^m} \left\|\boldsymbol{h}_m(\tau_{1,m}) \prod_{j=1}^{m} u(t-\tau_j)\right\|_p \\
&\leq \sum_{m=1}^{\infty} \|\boldsymbol{h}_m\|_{\mathcal{V}_p^{m,N}} \Big(\|u\|_{\mathcal{B}^1}\Big)^m \qquad \text{which defines } \varphi_{\boldsymbol{h},p} \\
&\leq \frac{1}{\epsilon_p\, \alpha_p} \sum_{m=1}^{\infty} \Phi_m \Big(\epsilon_p\, \alpha_p\, \|\boldsymbol{h}_1\|_{\mathcal{V}_p^{1,N}}\, \|u\|_{\mathcal{B}^1}\Big)^m \qquad \text{from theorem 2.} \\
&\leq \frac{1}{\epsilon_p\, \alpha_p}\, \Phi\big(Z_p(u)\big) \qquad \text{(see definition 9 in apdx. A)} \qquad (48)
\end{aligned}$$

which is absolutely convergent for $Z_p(u) < 1/4$ from lemma 1**(ii)**.

(ii) Let u be a function in $\mathcal{B}^1$satisfying (44), and $\boldsymbol{x}(t)=\sum_{m=1}^{+\infty}\boldsymbol{w}_m(t)$ with, from (4),

$$\boldsymbol{w}_m(t)=\int_{[0,t]^m}\boldsymbol{h}_m(t-\tau_{1,m})\prod_{j=1}^{m}u(\tau_j)\,\mathrm{d}\tau_{1,m}. \tag{49}$$

From (48), $\boldsymbol{x}$ is in $\mathcal{B}_p^N$, as a normally convergent series in $\mathcal{B}_p^N$. Let $\boldsymbol{w}$ be a test function in $C_0^1(\mathbb{R}_+,\mathbb{R}^N)$. The normal convergence of the series yields

$$\int_{\mathbb{R}_+}\dot{\boldsymbol{w}}^T\boldsymbol{x}dt=\sum_{m=1}^{\infty}\int_{\mathbb{R}_+}\dot{\boldsymbol{w}}^T\boldsymbol{w}_m dt$$

Now, for $m=1$, from standard results for first order linear time invariant systems, $\boldsymbol{w}_1$ is a continuous differentiable function of t with $\boldsymbol{w}_1(0)=\boldsymbol{0}$, and $\dot{\boldsymbol{w}}_1=\boldsymbol{A}\boldsymbol{w}_1+\boldsymbol{B}u$ is in $\mathcal{B}_p^N$. Consequently

$$-\int_{\mathbb{R}_+}\dot{\boldsymbol{w}}^T\boldsymbol{w}_1 dt=\int_{\mathbb{R}_+}\boldsymbol{w}^T v\dot{w}_1 dt=\int_{\mathbb{R}_+}\boldsymbol{w}^T\boldsymbol{A}\boldsymbol{w}_1 dt+\int_{\mathbb{R}_+}\boldsymbol{w}^T\boldsymbol{B}u\,dt.$$

For $m\geq 2$, it comes easily by induction from (29) that $\boldsymbol{h}_m(\tau_{1,m})$ is a continuous function on $\mathbb{R}_+^m$ with zero value on the boundary that can therefore be extended by zero to a continuous function on $\mathbb{R}$. From (25), it comes that $\partial_t\boldsymbol{h}_m(t-\tau_{1,m})$ is also continuous function on $\mathbb{R}_+^m$ with zero value on the boundary. Therefore, $\boldsymbol{w}_m$ is a $C^1(\mathbb{R}^+,\mathbb{R}^N)$ function, and

$$\begin{aligned}
\dot{\boldsymbol{w}}_m(t) &= \int_{[0,t]^m}\partial_t\boldsymbol{h}_m(t-\tau_{1,m})\prod_{j=1}^{m}u(\tau_j)\,\mathrm{d}\tau_{1,m}\\
&= \int_{[0,t]^m}\left(\boldsymbol{A}\boldsymbol{h}_m(t-\tau_{1,m})+\sum_{k=1}^{m-1}\begin{bmatrix}(\boldsymbol{h}_k(t-\tau_{1,k}))^T\boldsymbol{E}_1\boldsymbol{h}_{m-k}(t-\tau_{k+1,m})\\ \vdots\\ (\boldsymbol{h}_k(t-\tau_{1,k}))^T\boldsymbol{E}_N\boldsymbol{h}_{m-k}(t-\tau_{k+1,m})\end{bmatrix}\right)\prod_{j=1}^{m}u(\tau_j)\,\mathrm{d}\tau_{1,m}\\
&= \boldsymbol{A}\boldsymbol{w}_m(t)+\sum_{k=1}^{m-1}\begin{bmatrix}\big(\boldsymbol{w}_k(t)\big)^T\boldsymbol{E}_1\boldsymbol{w}_{m-k}(t)\\ \vdots\\ \big(\boldsymbol{w}_k(t)\big)^T\boldsymbol{E}_N\boldsymbol{w}_{m-k}(t)\end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
-\int_{\mathbb{R}_+}\dot{\boldsymbol{w}}^T\boldsymbol{w}_m dt &= \int_{\mathbb{R}_+}\boldsymbol{w}^T(t)\dot{\boldsymbol{w}}_m\,dt\\
&= \int_{\mathbb{R}_+}\boldsymbol{w}^T(t)\boldsymbol{A}\boldsymbol{w}_m(t)\,dt+\int_{\mathbb{R}_+}\boldsymbol{w}^T(t)\sum_{k=1}^{m-1}\begin{bmatrix}\big(\boldsymbol{w}_k(t)\big)^T\boldsymbol{E}_1\boldsymbol{w}_{m-k}(t)\\ \vdots\\ \big(\boldsymbol{w}_k(t)\big)^T\boldsymbol{E}_N\boldsymbol{w}_{m-k}(t)\end{bmatrix}dt
\end{aligned}$$

Summing the above relations for all $m\geq 1$ and reordering the quadratic terms leads to

$$\int_{\mathbb{R}_+}\dot{\boldsymbol{w}}^T\boldsymbol{x}dt+\int_{\mathbb{R}_+}\boldsymbol{w}^T(t)\boldsymbol{A}\boldsymbol{x}(t)\,dt+\int_{\mathbb{R}_+}\boldsymbol{w}^T\boldsymbol{B}u\,dt+\int_{\mathbb{R}_+}\boldsymbol{w}^T(t)\sum_{k=1}^{m-1}\begin{bmatrix}\big(\boldsymbol{x}(t)\big)^T\boldsymbol{E}_1\boldsymbol{x}(t)\\ \vdots\\ \big(\boldsymbol{x}(t)\big)^T\boldsymbol{E}_N\boldsymbol{x}(t)\end{bmatrix}dt=0,$$

which proves that the series is a weak solution of the system. Moreover if u is a $C^0(\mathbb{R}_+,\mathbb{R}^N)$ function satisfying (44), then from the theory of linear first order systems, $\boldsymbol{w}_1$ is a $C^1(\mathbb{R}_+,\mathbb{R}^N)$ function and from (25), $\sum_{m=1}^{\infty}\dot{\boldsymbol{w}}_m$ is also a normally convergent series of continuous functions in $\mathcal{B}_p^N$. It follows that $(\boldsymbol{x},y)$ is a strong solution of the system.

(iii) Equation (45) is straightforward from (18). Equation (46) is a consequence of (48) and lemma 1**(iii)** (cf. apdx. A).

■

Corollary 2: Let $V_M\boldsymbol{x}$ and $V_M y$ denote the finite M-order approximations

$$V_M\boldsymbol{x}(t) = \sum_{m=1}^{M}\int_{\mathbb{R}^m}\boldsymbol{h}_m(\tau_{1,m})\prod_{j=1}^{m}u(t-\tau_j)\,\mathrm{d}\tau_{1,m}, \tag{50}$$

$$V_M y(t) = \boldsymbol{C}\,V_M\boldsymbol{x} \tag{51}$$

for $M \in \mathbb{N}^*$, defined from the truncated expansion of the Volterra series. Let $\boldsymbol{u} \in \mathcal{B}^1$. If $Z_p(u) < 1/4$ where Z_p is given in definition 5, then

$$\left\|\boldsymbol{x} - V_M \boldsymbol{x}\right\|_{\mathcal{B}_p^N} \leq \frac{R_M \Phi\Big(Z_p(u)\Big)}{\epsilon_p\, \alpha_p} \tag{52}$$

$$\left\|y - V_M y\right\|_{\mathcal{B}^1} \leq \|\boldsymbol{C}\|_p \frac{R_M \Phi\Big(Z_p(u)\Big)}{\epsilon_p\, \alpha_p} \tag{53}$$

where $z \mapsto R_M\Phi(z)$ is defined in lemma 1**(iv)** and is such that, for $z < 1/4$,

$$|R_M\Phi(z)| \leq \Phi_{M+1} \frac{z^{M+1}}{1-4z} \leq \frac{1}{2\sqrt{\pi(M+1)}(2M+1)} \frac{(4z)^{M+1}}{1-4z} \tag{54}$$

Proof: Let $R_M\boldsymbol{x}$ denote the remainder $R_M\boldsymbol{x} = \boldsymbol{x} - V_M\boldsymbol{x}$. Then, from theorem 2,

$$\left\|R_M\boldsymbol{x}\right\|_{\mathcal{B}_p^N} \leq \sum_{m=M+1}^{\infty} \|\boldsymbol{h}_m\|_{\mathcal{V}_p^{m,N}} \Big(\|u\|_{\mathcal{B}^1}\Big)^m \leq \frac{1}{\epsilon_p\, \alpha_p} \sum_{m=M+1}^{\infty} \Phi_m \Big(\epsilon_p\, \alpha_p\, \|\boldsymbol{h}_1\|_{\mathcal{V}_p^{1,N}}\, \|u\|_{\mathcal{B}^1}\Big)^m = \frac{1}{\epsilon_p\, \alpha_p} \sum_{m=M+1}^{\infty} \Phi_m \big(Z_p(u)\big)^m$$

which converges for $Z_p(u) < 1/4$ from lemma 1**(ii)**. Defining $z \mapsto R_M\Phi(z) = \sum_{m=M+1}^{\infty} \Phi_m z^m$ for $|z| < 1/4$ and using lemma 1**(iv)** leads to equation (52). Equation (53) is straightforward. ■

The results in theorem 2, corollary 1 and corollary 2 deserve some comments. For instance, computing the bound of the convergence radius for the single state SISO system **(S1):** $\dot{y} = -ay + ey^2 + bu$ and for **(S2):** $\dot{y} = -ay - ey^2 + bu$ yield the same results (a, b and e are positive constants). For **(S1)**, it is easy to check that the bound for u is the best possible in the sense that both Volterra series expansion of the state and the state of the full nonlinear system blow up for a constant input u greater than the bound given in corollary 2. For **(S2)**, the Volterra series expansion is the same as for the first system but with alternate signs, so that the radius of convergence is the same. However, this nonlinear system has a solution for any constant positive input u.

In the multi-dimensional state case, the bound derived for the input is a "worst case" bound, owing to the use of matrix norms. As a consequence, this bound will be conservative, unless particular features of the system under consideration can be exploited.

4- Quadratic MIMO systems

4.1- System under consideration

Let the quadratic ODE system be defined by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \begin{bmatrix} \boldsymbol{x}^T \boldsymbol{E}_1\, \boldsymbol{x} \\ \vdots \\ \boldsymbol{x}^T \boldsymbol{E}_N\, \boldsymbol{x} \end{bmatrix} + \boldsymbol{B}\,\boldsymbol{u}, \tag{55}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}, \tag{56}$$

for $t \in \mathbb{R}^+$ with $\boldsymbol{x}(0) = \boldsymbol{0}$, where $\boldsymbol{u} \in \mathbb{R}^{N_u}$, $\boldsymbol{x} \in \mathbb{R}^{N_x}$ and $\boldsymbol{y} \in \mathbb{R}^{N_y}$ are the *input*, *state* and *output* vectors of the system, respectively. All matrices are real and $\boldsymbol{A}$ is $N_x \times N_x$, $\boldsymbol{B}$ is $N_x \times N_u$, $\boldsymbol{C}$ is $N_y \times N_x$ and $\boldsymbol{E}_n$ with $n = 1, \ldots, N_x$ are $N_x \times N_x$.

4.2- Volterra series for MIMO systems and notations

Definition 6 (Multi-index, notations and sets): Let $N \in \mathbb{N}^*$ and k be such that $1 \leq k \leq N$. The set $\mathbb{M}_N$ of multi-indexes is defined by

$$\mathbb{M}_N = \mathbb{N}^N \setminus \{(0, \ldots, 0)\}. \tag{57}$$

A multi-index is denoted with an underline; the version without underline denotes the sum of its components, namely

$$\underline{m} = (m_1, \ldots, m_N) \in \mathbb{M}_N, \tag{58}$$

$$m = m_1 + \cdots + m_N. \tag{59}$$

Particular multi-indexes are

$$\underline{1} = (1, 1, \ldots, 1) \in \mathbb{M}_N, \tag{60}$$

$$\underline{1}_k = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{M}_N \tag{61}$$

where only the kth component is equal to 1. Consider two multi-indexes $\underline{p}$ and $\underline{q}$ in $\mathbb{M}_N$. Then, $\tau_{\underline{p},\underline{q}}$ and $\mathrm{d}\tau_{\underline{p},\underline{q}}$ denote

$$\tau_{\underline{p},\underline{q}} = \big(\underbrace{\tau_{(1,p_1)},\tau_{(1,p_1+1)},\cdots,\tau_{(1,q_1)}}_{\text{empty if } q_1<p_1},\ldots,\tau_{(N,p_N)},\tau_{(N,p_N+1)},\cdots,\tau_{(N,q_N)}\big), \tag{62}$$

$$\mathrm{d}\tau_{\underline{p},\underline{q}} = \prod_{i=1}^{N}\prod_{j=1}^{m_i}\mathrm{d}\tau_{i,j} \tag{63}$$

with the convention $\prod_{j=p_1}^{q_1} u_i(t-\tau_{(i,j)}) = 1$ if $q_1 < p_1$. Finally, for $\underline{m} \in \mathbb{M}_N$, the set $\mathbb{M}_{\underline{m}}$ denotes

$$\mathbb{M}_{\underline{m}} = \big\{(\underline{p},\underline{q}) \in \big(\mathbb{M}_N\big)^2 \,\big|\, \underline{p}+\underline{q}=\underline{m}\big\}, \tag{64}$$

where $\underline{p}+\underline{q} = (p_1+q_1,\ldots,p_N+q_N)$.

Definition 7 (Volterra series of a time-invariant MIMO-system): A time-invariant MIMO-system $\mathcal{S}_{\boldsymbol{u}\mapsto\boldsymbol{x}}$ can be represented by a Volterra series if

$$\boldsymbol{x}(t) = \sum_{\underline{m}\in\mathbb{M}_{N_u}} \int_{\mathbb{R}^m} \boldsymbol{h}_{\underline{m}}(\tau_{\underline{1},\underline{m}}) \Big(\prod_{i=1}^{N_u}\prod_{j=1}^{m_i} u_i(t-\tau_{(i,j)})\Big)\,\mathrm{d}\tau_{\underline{1},\underline{m}}. \tag{65}$$

For a given $\underline{m}$, m_i corresponds to the order of the nonlinearity associated to input u_i.

4.3- *Derivation of the Volterra kernels*

4.3.1- Volterra kernels and cancelling system: Let $\{\boldsymbol{g}_{\underline{m}}\}_{\underline{m}\in\mathbb{M}_{N_u}}$ and$\{\boldsymbol{h}_{\underline{m}}\}_{m\in\mathbb{M}_{N_u}}$ denote the Volterra kernels of the MIMO-system $\mathcal{S}_{\boldsymbol{u}\mapsto\boldsymbol{y}}$ and $\mathcal{S}_{\boldsymbol{u}\mapsto\boldsymbol{x}}$ respectively. For these MIMO-systems, (20-21) turn into $\boldsymbol{g}_{\underline{m}}(t_{1,\underline{m}}) = \boldsymbol{C}\,\boldsymbol{h}_{\underline{m}}(t_{1,\underline{m}})$ and $\boldsymbol{G}_{\underline{m}}(s_{1,\underline{m}}) = \boldsymbol{C}\,\boldsymbol{H}_{\underline{m}}(s_{1,\underline{m}})$, for $\underline{m} \in \mathbb{M}_{N_u}$. Moreover, the cancelling system is still described by the block-diagram in Fig. 2 with $\boldsymbol{u}$, $\underline{m}$ instead of u, m.

4.3.2- Kernels in the Laplace and the time domains: Using the natural extension of interconnection laws to the case of multiple input kernels still leads to recurrent algebraic equations in the Laplace domain, given by

$$\boldsymbol{H}_{\underline{m}}(s_{\underline{1},\underline{m}}) = \big[\widehat{s_{\underline{1},\underline{m}}}\boldsymbol{I}_{\boldsymbol{N_x}} - \boldsymbol{A}\big]^{-1}\boldsymbol{F}_{\underline{m}}(s_{\underline{1},\underline{m}}), \qquad \forall \underline{m}\in\mathbb{M}_{N_u} \tag{66}$$

$$\boldsymbol{F}_{\underline{1}_k}(s_{(1,k)}) = \boldsymbol{B}_k, \qquad \text{for } 1\le k\le N_u, \tag{67}$$

$$\boldsymbol{F}_{\underline{m}}(s_{\underline{1},\underline{m}}) = \sum_{(\underline{p},\underline{q})\in\mathbb{M}_{\underline{m}}} \begin{bmatrix} (\boldsymbol{H}_{\underline{p}}(s_{\underline{1},\underline{p}}))^T\boldsymbol{E}_1\boldsymbol{H}_{\underline{q}}(s_{\underline{p}+\underline{1},\underline{m}}) \\ \vdots \\ (\boldsymbol{H}_{\underline{p}}(s_{\underline{1},\underline{p}}))^T\boldsymbol{E}_{N_x}\boldsymbol{H}_{\underline{q}}(s_{\underline{p}+\underline{1},\underline{m}}) \end{bmatrix}, \qquad \forall \underline{m}\in\mathbb{M}_{N_u} \text{ s.t. } m\ge 2 \tag{68}$$

where $\boldsymbol{B}_k$ denotes the kth column of $\boldsymbol{B}$ and $\widehat{s_{\underline{p},\underline{q}}}$ still denotes the sum of all the Laplace variables, namely, $\widehat{s_{\underline{p},\underline{q}}} = \sum_{i=1}^{N_u}\sum_{j=p_i}^{q_i} s_{(i,j)}$.

The time domain versions are given, for $\underline{m} \in \mathbb{M}_{N_u}$ and using the notation of the time-variant systems $\boldsymbol{h}_{\underline{m}}(t,\tau_{\underline{1},\underline{m}})$ rather than the time-invariant version $\boldsymbol{h}_{\underline{m}}(t_{\underline{1},\underline{m}})$, by

$$\big[\boldsymbol{I}_{\boldsymbol{N_x}}\partial_t - \boldsymbol{A}\big]\boldsymbol{h}_{\underline{m}}(t,\tau_{\underline{1},\underline{m}}) = \boldsymbol{f}_m(t,\tau_{\underline{1},\underline{m}}), \tag{69}$$

$$\boldsymbol{f}_{\underline{1}_k}(t,\tau_{(1,k)}) = \boldsymbol{B}_k\,\delta(t-\tau_{(1,k)}), \qquad \text{for } 1\le k\le N_u, \tag{70}$$

$$\boldsymbol{f}_{\underline{m}}(t,\tau_{\underline{1},\underline{m}}) = \sum_{(\underline{p},\underline{q})\in\mathbb{M}_{\underline{m}}} \begin{bmatrix} (\boldsymbol{h}_{\underline{p}}(t,\tau_{\underline{p}}))^T\boldsymbol{E}_1\boldsymbol{H}_{\underline{q}}(t,\tau_{\underline{p}+\underline{1},\underline{m}}) \\ \vdots \\ (\boldsymbol{h}_{\underline{p}}(t,\tau_{\underline{1},\underline{p}}))^T\boldsymbol{E}_N\boldsymbol{H}_{\underline{1},\underline{q}}(t,\tau_{\underline{p}+\underline{1},\underline{m}}) \end{bmatrix}, \qquad \text{if } m\ge 2. \tag{71}$$

The solution is

$$\boldsymbol{h}_{\underline{1}_k}(t,\tau_{(1,k)}) = \mathbf{1}_{\mathbb{R}^+}(t-\tau_{(1,k)})\,\mathrm{e}^{\boldsymbol{A}(t-\tau_{(1,k)})}\boldsymbol{B}_k, \qquad \text{for } 1\le k\le N_u, \tag{72}$$

$$\boldsymbol{h}_{\underline{m}}(t,\tau_{\underline{1},\underline{m}}) = \mathbf{1}_{\mathbb{R}^+}\big(t-\max(\tau_{\underline{1},\underline{m}})\big)\int_{\max(\tau_{\underline{1},\underline{m}})}^{t} \mathrm{e}^{\boldsymbol{A}(t-\theta)}\boldsymbol{f}_{\underline{m}}(\theta,\tau_{\underline{1},\underline{m}})\mathrm{d}\theta, \text{ if } m\ge 2. \tag{73}$$

4.4- *Convergence and guaranteed error bound*

Theorem 3: Consider system (55) with $\max\Big(\Re\mathrm{e}\big(\mathrm{Spec}\,(\boldsymbol{A})\big)\Big) < 0$. Let $\{\boldsymbol{h}_{\underline{m}}\}_{\underline{m}\in\mathbb{M}_M}$ be the Volterra kernels defined by (72-73). Then, for all $p\in[1,\infty]$ and $\underline{m}\in\mathbb{M}_{N_u}$,

$$\|\boldsymbol{h}_{\underline{m}}\|_{\mathcal{V}_p^{m,N_x}} \le \Psi_{\underline{m}}(\epsilon_p\alpha_p)^{m-1}\prod_{k=1}^{N_u}\Big(\|\boldsymbol{h}_{\underline{1}_k}\|_{\mathcal{V}_p^{1,N_x}}\Big)^{m_k}, \tag{74}$$

where

$$\epsilon_p = \left\| \left[\|\boldsymbol{E}_1\|_{\mathcal{Q}_p} \dots \|\boldsymbol{E}_N\|_{\mathcal{Q}_p} \right]^T \right\|_p \quad (75)$$

$$\alpha_p = \int_{\mathbb{R}_+} \left\| \mathrm{e}^{\boldsymbol{A}\xi} \right\|_p \mathrm{d}\xi < \infty, \quad (76)$$

$$\Psi_{\underline{1}_k} = 1, \qquad \text{for } 1 \le k \le N_u, \quad (77)$$

$$\Psi_{\underline{m}} = \sum_{(\underline{p},\underline{q}) \in \mathbb{M}_{\underline{m}}} \Psi_{\underline{p}} \Psi_{\underline{q}} \qquad \text{if } m \ge 2, \quad (78)$$

Proof: The proof is similar to that of theorem 2, starting from (73) rather than (29) and using def. 6. ■

Definition 8 (Extension of definition 5): Consider the system (55) with $\max\Big(\Re\mathrm{e}\big(\mathrm{Spec}\,(\boldsymbol{A})\big)\Big) < 0$ so that $\|\boldsymbol{h}_{\underline{1}_k}\|_{\mathcal{V}_p^{1,N_x}} < \infty$ for $1 \le k \le N_u$. Then, the application $Z_p : \mathcal{B}^{N_u} \to \mathbb{R}_+$ is defined by

$$Z_p(u) = \epsilon_p\, \alpha_p \sum_{k=1}^{N_u} \|\boldsymbol{h}_{\underline{1}_k}\|_{\mathcal{V}_p^{1,N_x}}\, \|u_k\|_{\mathcal{B}^1}. \quad (79)$$

Note that this is an extension of definition 5 since (79) coincides with (43) for the SISO-case $N_u = 1$.

Corollary 3: Let $p \in [1, \infty]$.

(i) The Volterra series expansion of the state and output of system (55-56) converges in $\mathcal{B}_p^{N_x}$ and $\mathcal{B}_p^{N_y}$, respectively, for all input $u \in \mathcal{B}_p^{N_u}$ such that

$$Z_p(u) < 1/4 \quad (80)$$

where Z_p is given in definition 8.

(ii) For any input $u \in \mathcal{B}_p^{N_u}$ satisfying (80), the sum of the series is a *weak solution* of the system. If u is moreover in $C^0(\mathbb{R}_+, \mathbb{R}^{N_u})$, then the sum of the series is a *strong solution* of system (55-56).

(iii) The output $\boldsymbol{y}$ and the state $\boldsymbol{x}$ are bounded as:

$$\|\boldsymbol{y}\|_{\mathcal{B}_p^{N_y}} \le \|\boldsymbol{C}\|_p \|\boldsymbol{x}\|_{\mathcal{B}_p^{N}}, \quad (81)$$

$$\|\boldsymbol{x}\|_{\mathcal{B}_p^{N}} \le \varphi_{\boldsymbol{h},p}\big(\|u_1\|_{\mathcal{B}^1}, \dots, \|u_{N_u}\|_{\mathcal{B}^1}\big) \le \frac{1 - \sqrt{1 - 4Z_p(u)}}{2\epsilon_p\, \alpha_p}, \quad (82)$$

where ϵ_p and α_p are defined in theorem 3 and $\varphi_{\boldsymbol{h},p}$ is the extension of the characteristic function (47), given by

$$\varphi_{\boldsymbol{h},p}(z_{1,N_u}) = \sum_{\underline{m} \in \mathbb{M}_{N_u}} \|\boldsymbol{h}_{\underline{m}}\|_{\mathcal{V}_p^{m,N_x}} \prod_{k=1}^{M} (z_k)^{m_k}. \quad (83)$$

Proof: The proof is similar to that of corollary 1, using lemma 2 instead of lemma 1. ■

Corollary 4: Let $V_M\boldsymbol{x}$ and $V_M\boldsymbol{y}$ denote the finite M-order MIMO approximations

$$V_M\boldsymbol{x}(t) = \sum_{\left\{\substack{\underline{m} \in \mathbb{M}_{N_u} \\ m \ge M+1}\right.} \int_{\mathbb{R}^m} \boldsymbol{h}_{\underline{m}}(\tau_{\underline{1},\underline{m}}) \Big(\prod_{i=1}^{N_u} \prod_{j=1}^{m_i} u_i(t - \tau_{(i,j)}) \Big)\, \mathrm{d}\tau_{\underline{1},\underline{m}}. \quad (84)$$

$$V_M y(t) = \boldsymbol{C}\; V_M\boldsymbol{x} \quad (85)$$

for $M \in \mathbb{N}^*$, defined from the truncated expansion of the MIMO Volterra series (65). Let $\boldsymbol{u} \in \mathcal{B}_p^{N_u}$. If $Z_p(\boldsymbol{u}) < 1/4$ where Z_p is given in definition 8, then

$$\left\| \boldsymbol{x} - V_M\boldsymbol{x} \right\|_{\mathcal{B}_p^{N_x}} \le \frac{R_M \Phi\Big(Z_p(u)\Big)}{\epsilon_p\, \alpha_p} \quad (86)$$

$$\left\| \boldsymbol{y} - V_M\boldsymbol{y} \right\|_{\mathcal{B}^{N_y}} \le \|\boldsymbol{C}\|_p \frac{R_M \Phi\Big(Z_p(u)\Big)}{\epsilon_p\, \alpha_p} \quad (87)$$

where $z \mapsto R_M\Phi(z)$ is defined in lemma 1**(iv)** and is such that, for $z < 1/4$,

$$|R_M\Phi(z)| \le \Phi_{M+1} \frac{z^{M+1}}{1 - 4z} \le \frac{1}{2\sqrt{\pi(M+1)}(2M+1)} \frac{(4z)^{M+1}}{1 - 4z} \quad (88)$$

Proof: The proof is similar to that of corollary 2, using lemma 2 instead of lemma 1. ■

5- SIMULATION

Simulations have been performed on the following two dimensional SISO academic example:

$$\dot{x}_1 = -x_1 + 0.1({x_1}^2 + {x_2}^2) + 0.1x_1x_2 + 3u$$
$$\dot{x}_2 = -2x_2 + 0.2{x_1}^2 + 0.1{x_2}^2 + 0.1x_1x_2 + u$$
$$y = x_2$$

Hence $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $E_1 = \begin{pmatrix} 0.1 & 0.05 \\ 0.05 & 0.1 \end{pmatrix}$ and $E_2 = \begin{pmatrix} 0.2 & 0.05 \\ 0.05 & 0.1 \end{pmatrix}$. The initial state of the system is zero.

Notice that for this system, we can compute $Z_p(u)$ in usual situations where $p = 1$, $p = \infty$ and $p = 2$. The computed values are

$$\alpha_1 = \int_{\mathbb{R}_+} \|\mathrm{e}^{\boldsymbol{A}\xi}\|_1 \,\mathrm{d}\xi = \int_{\mathbb{R}_+} \|\mathrm{e}^{\boldsymbol{A}\xi}\|_\infty \,\mathrm{d}\xi = \int_{\mathbb{R}_+} \|\mathrm{e}^{\boldsymbol{A}\xi}\|_2 \,\mathrm{d}\xi = 1,$$
$$\|\boldsymbol{E}_1\|_{\mathcal{Q}_1} = 0.1, \|\boldsymbol{E}_2\|_{\mathcal{Q}_1} = 0.2, \|\boldsymbol{E}_1\|_{\mathcal{Q}_\infty} = 0.3, \|\boldsymbol{E}_2\|_{\mathcal{Q}_\infty} = 0.4, \|\boldsymbol{E}_1\|_{\mathcal{Q}_2} = 0.15, \|\boldsymbol{E}_2\|_{\mathcal{Q}_2} = 0.3$$
$$K_1 = \epsilon_1\alpha_1 = 0.3,\ K_\infty = \epsilon_\infty\alpha_\infty = 0.4,\ K_2 = \epsilon_2\alpha_2 \simeq 0.335,\ \|\boldsymbol{h}_1\|_{\mathcal{V}_1^{1,2}} = 3.5, \|\boldsymbol{h}_1\|_{\mathcal{V}_\infty^{1,2}} = 3, \|\boldsymbol{h}_1\|_{\mathcal{V}_2^{1,2}} \simeq 3.05$$

The input signal u has been chosen to be a step of magnitude 0.24 at time zero, so that from (43)

$$Z_1(u) \simeq 0.252,\ Z_\infty(u) \simeq 0.288,\ Z_2(u) \simeq 0.245\ .$$

The convergence criterion (44) for the Volterra decomposition of the state and output of the system is met for $p = 2$, and, as all p-norms are equivalent in finite dimensional spaces, the series is convergent for any p-norm although the criterion is not necessarily met. In the same way, higher values for the input step were tested for which the complete nonlinear system was still stable, and the truncated Volterra series was still a good approximation, indicating that our criterion is conservative. However, convergence becomes slower as the value of u increases.

Real time implementation of Volterra decomposition is quite easy for such an LTI system. Let U and X_i denote the (multivariate) Laplace transform of the input and contribution of order i to the state vector of the system. The first order contribution $\boldsymbol{W}_1$ is given by the linear part of the system : $\boldsymbol{W}_1(s_1) = (s_1 I - A)^{-1}BU(s_1) = \boldsymbol{H}_1(s_1)U(s_1)$.

Given the Laplace transforms of all contributions with order $p < m$, denoted as $\boldsymbol{W}_p(s_{1,p}) = \boldsymbol{H}_p(s_{1,p})U(s_1)\ldots U(s_p)$, the order m contribution can be written as

$$\begin{aligned}
\boldsymbol{W}_m(s_{1,m}) &= \left[\widehat{s_{1,m}}\boldsymbol{I_N} - \boldsymbol{A}\right]^{-1}\left[\sum_{k=1}^{m-1}\begin{bmatrix} (\boldsymbol{H}_k(s_{1,k}))^T E_1 \boldsymbol{H}_{m-k}(s_{k+1,m}) \\ \vdots \\ (\boldsymbol{H}_k(s_{1,k}))^T E_N \boldsymbol{H}_{m-k}(s_{k+1,m}) \end{bmatrix}\right] U(s_1)\ldots U(s_m) \\
&= \left[\widehat{s_{1,m}}\boldsymbol{I_N} - \boldsymbol{A}\right]^{-1}\left[\sum_{k=1}^{m-1}\begin{bmatrix} (\boldsymbol{W}_k(s_{1,k}))^T E_1 \boldsymbol{W}_{m-k}(s_{k+1,m}) \\ \vdots \\ (\boldsymbol{W}_k(s_{1,k}))^T E_N \boldsymbol{W}_{m-k}(s_{k+1,m}) \end{bmatrix}\right] \\
&= \sum_{k=1}^{m-1}\left[\left[\widehat{s_{1,m}}\boldsymbol{I_N} - \boldsymbol{A}\right]^{-1}\begin{bmatrix} (\boldsymbol{W}_k(s_{1,k}))^T E_1 \boldsymbol{W}_{m-k}(s_{k+1,m}) \\ \vdots \\ (\boldsymbol{W}_k(s_{1,k}))^T E_N \boldsymbol{W}_{m-k}(s_{k+1,m}) \end{bmatrix}\right] = \sum_{k=1}^{m-1}\boldsymbol{W}_{mk}(s_{1,m}))
\end{aligned}$$

A very simple computation shows that in the time domain, a realization for each $\boldsymbol{w}_{mk}$ is obtained by computing at each time t the N dot products $(\boldsymbol{w}_k(t))^T E_i \boldsymbol{w}_{m-k}(t)$, for $1 \le i \le N$, and filtering the resulting N dimensional vector with the filter with N inputs, N outputs and whose impulse response is $e^{At}\mathbf{1}_{\mathbb{R}+}(t)$, as shown on figure 3. This realization shall be denoted as T.

Figure 5 shows the realization of the order 3 truncated Volterra series for the system. On figure 5, truncated Volterra series of order 1, 2 and 3 are compared with the complete solution. It can be seen in this case that in spite of the slow convergence of the series (three terms needed), the approximation at order 3 is quite satisfactory.

6- CONCLUSION

The convergence of Volterra's series decomposition for a stable system with quadratic state nonlinearity in $L^\infty(\mathbb{R}_+, \mathbb{R}^N)$ has been demonstrated. An algorithm to build the kernels as well as a bound on the input and on the truncation error are given. The resulting truncated system is easy to implement and simulate.

Further work will now firstly consist in extending the result to unstable system using appropriate functional norms (such as norms with exponential weight). Another task will then consist in establishing conditions under which the extension of

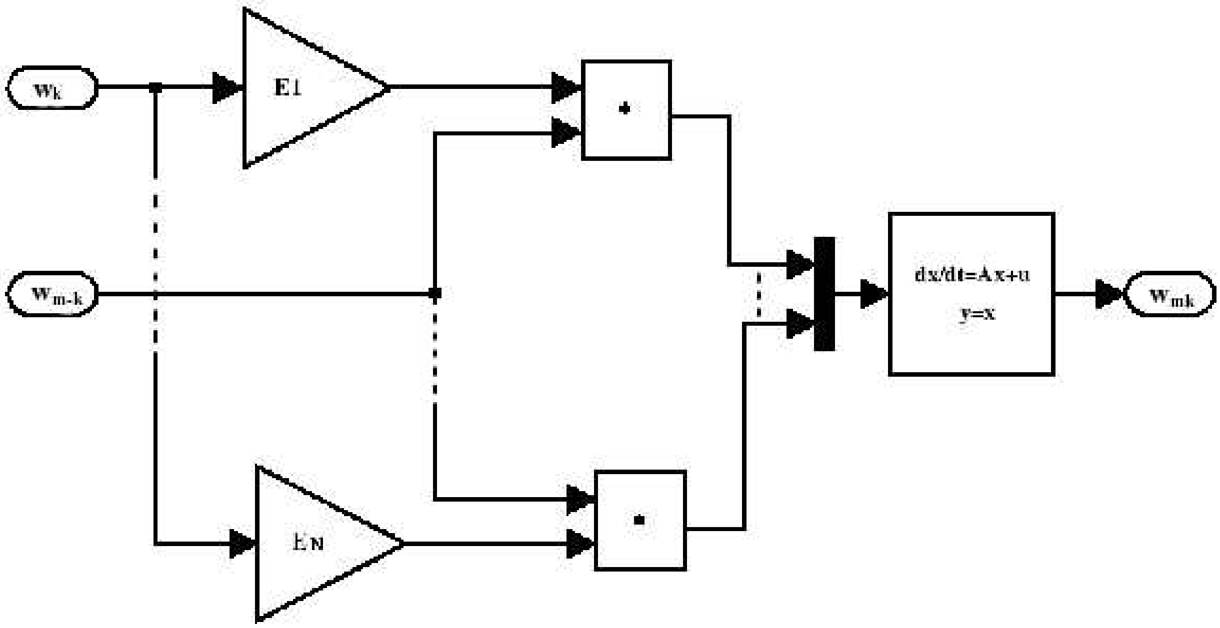

Fig. 3. Realization of $\boldsymbol{w}_{mk}$: the N dot products $(\boldsymbol{w}_k(t))^T E_i \boldsymbol{w}_{m-k}(t)$, for $1 \leq i \leq N$, are computed and the resulting N dimensional vector is filtered with the filter with N inputs, N outputs and whose impulse response is $e^{At}\mathbf{1}_{\mathbb{R}+}(t)$

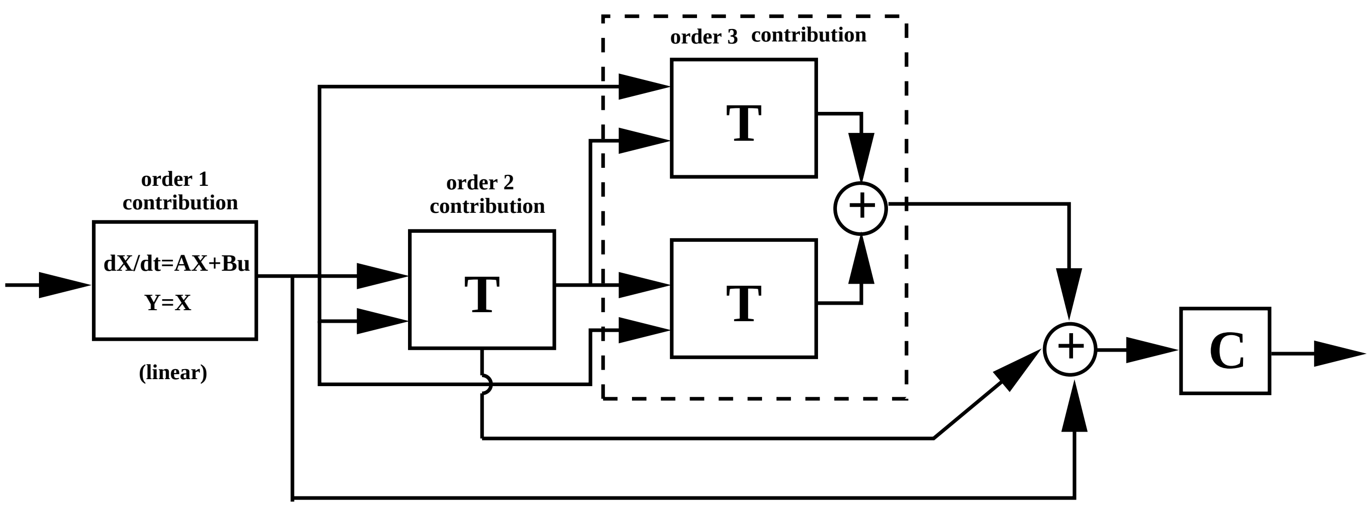

Fig. 4. Order 3 realization of the Volterra series: each term is built using the lower level terms combined through T-boxes

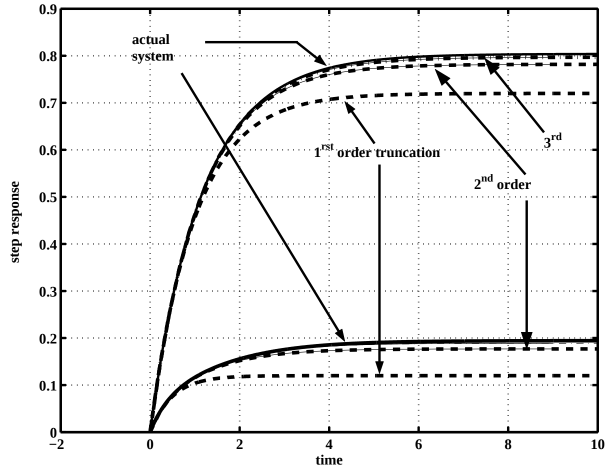

Fig. 5. Comparison between the actual system and the realization of the Volterra series up to order 3 for the two states $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$

these results to a n^{th} order polynomial state nonlinearity is possible. This analysis can also be extended, with greater technical difficulties to be overcome, to some families of infinite dimensional systems such as nonlinear propagation (see *e.g.* [HH04], [Hél06]) or diffusion equations with polynomial in the state diffusion coefficients.

APPENDIX

A- Definition of Φ_m and lemma

Definition 9: Let $N \in \mathbb{N}^*$. The sequence $(\Phi_m)_{m\in\mathbb{N}^*}$ is defined by

$$\Phi_1 = 1, \tag{89}$$

$$\Phi_m = \sum_{k=1}^{m-1} \Phi_k \Phi_{m-k} \qquad \forall m \geq 2 \tag{90}$$

and the formal power series $\Phi(z)$ by

$$\Phi(z) = \sum_{m=1}^{\infty} \Phi_m z^m \tag{91}$$

Lemma 1: The sequence $(\Phi_m)_{m\in\mathbb{N}^*}$ is such that

(i) $\Phi_m = \frac{(4)^{m-1}}{m} \frac{\prod_{k=2}^{m}(k-\frac{3}{2})}{(m-1)!}$,

and the power series $\Phi(z)$ is such that, for $|z| < 1/4$,

(ii) $\Phi(z)$ is absolutely convergent.

(iii) $\Phi(z) = \frac{1}{2}\left(1-\sqrt{1-4z}\right)$

(iv) the remainder $R_M\Phi(z) = \sum_{m=M+1}^{\infty} \Phi_m$ is bounded by

$$|R_M\Phi(z)| \leq \Phi_{M+1}\frac{z^{M+1}}{1-4z} \leq \frac{1}{2\sqrt{\pi(M+1)}(2M+1)} \frac{(4z)^{M+1}}{1-4z} \tag{92}$$

Proof:

(i)-(ii)-(iii) From the definition of Φ_m and $\Phi(z)$,

$$\Phi(z)^2 = \left(\sum_{m=1}^{\infty} \Phi_m z^m\right)^2 = \sum_{m=2}^{\infty}\left(\sum_{k=1}^{m-1}\Phi_k\Phi_{m-k}\right) z^m = \Phi(z) - z. \tag{93}$$

Hence, $\Phi(z)$ is the root of the second order polynomial equation $\Phi(z)^2 - \Phi(z) + z = 0$ such that $\Phi(0) = 0$. The solution is

$$\Phi(z) = \frac{1}{2}\left(1-\sqrt{1-4z}\right). \tag{94}$$

Now, the series expansion of (94) converges absolutely for $|z| < 1/4$ and yields

$$\begin{aligned}
\Phi(z) &= \frac{1}{2}\left(1 - \sum_{m=0}^{\infty} \frac{(-4)^m}{m!} \prod_{k=1}^{m} (\frac{3}{2} - k) z^m\right) \text{ with the convention } \prod_{k=1}^{0} (\frac{3}{2} - k) = 1 \\
&= \frac{1}{2} \sum_{m=1}^{\infty} \frac{4^m}{2(m!)} \prod_{k=2}^{m} (k - \frac{3}{2}) z^m \\
&= \sum_{m=1}^{\infty} \frac{4^{m-1}}{m!} \prod_{k=2}^{n} (k - \frac{3}{2}) z^m \qquad (95)
\end{aligned}$$

so that by identification,

$$\Phi_m = \frac{4^{m-1}}{m} \frac{\prod_{k=2}^{m} (k - \frac{3}{2})}{(m-1)!} \qquad (96)$$

(iv) For $m \in \mathbb{N}^*$, $\frac{4^{-m}\Phi_{m+1}}{4^{1-m}\Phi_m} = \frac{(m-1/2)}{m+1} < 1$ so that $\Phi_{m'} \leq \Phi_m 4^{m'-m}, \forall m' \geq m$. Hence, for $z < 1/4$, the remainder $R_m\Phi(z)$ is absolutely convergent and bounded by

$$|R_m\Phi(z)| \leq \Phi_{m+1} \sum_{m'=m+1}^{\infty} 4^{m'-m-1} z^{m'} = \Phi_{m+1} \frac{z^{m+1}}{1-4z}, \qquad (97)$$

which yields the first inequality in (92).
Now, from the Gamma function identity $\Gamma(m+1/2) = \sqrt{\pi}\big(1.3. \ldots .(2m-1)\big)/2^m$ [AS70, (6.1.12)], Φ_m can be rewritten as

$$\Phi_m = \frac{4^{m-1}}{m} \left(\frac{\Gamma(m+1/2)}{\Gamma(m+1)} \frac{2m}{2m-1}\right). \qquad (98)$$

Using Wallis'formula [AS70, (6.1.49)]

$$\frac{1}{\sqrt{m}}\left(1 - \frac{1}{8m}\right) < \frac{\Gamma(m+1/2)}{\Gamma(m+1)} < \frac{1}{\sqrt{m}} \qquad (99)$$

yields

$$\left(1 - \frac{1}{8m}\right)\alpha_m < \Phi_m < \alpha_m \qquad \text{with } \alpha_m = \frac{4^{m-1}}{\sqrt{\pi}\sqrt{m}\,(m - \frac{1}{2})}. \qquad (100)$$

Finally, using the superior bound α_m of Φ_m in (97) yields the second inequality in (92).

■

B- Definition of $\Psi_{\underline{m}}$ and lemma

Definition 10: The sequence $\left(\Psi_{\underline{m}}\right)_{\underline{m}\in\mathbb{M}_N}$ is defined for $\underline{m} \in \mathbb{M}_N$, by

$$\Psi_{\underline{m}} = 1, \quad \text{if } m = 1, \qquad (101)$$

$$\Psi_{\underline{m}} = \sum_{(\underline{p},\underline{q})\in\mathbb{M}_{\underline{m}}} \Psi_{\underline{p}}\Psi_{\underline{q}} \text{ if } m \geq 2 \qquad (102)$$

and the formal multivariate power series $\Psi(z_{1,N})$ by

$$\Psi(z_{1,N}) = \sum_{\underline{m}\in\mathbb{M}_N} \Psi_{\underline{m}} \prod_{k=1}^{N} (z_k)^{m_k}, \qquad (103)$$

where it is recalled that, for $\underline{m} = (m_1, \ldots, m_M) \in \mathbb{M}_N$, m denotes $m = m_1 + \cdots + m_N$, $z_{1,N}$ and $\widehat{z_{1,N}}$ denote $z_{1,N} = (z_1, \ldots, z_N)$ and $\widehat{z_{1,N}} = z_1 + \cdots + z_N$, respectively. Moreover, $\widetilde{z_{1,N}}$ denotes

$$\widetilde{z_{1,N}} = |z_1| + \cdots + |z_N|. \qquad (104)$$

Lemma 2: The sequence $\left(\Psi_{\underline{m}}\right)_{\underline{m}\in\mathbb{M}_N}$ is such that

(i) $\Psi_{\underline{m}} = \frac{m!}{\underline{m}!}\Phi_m$ where $\frac{m!}{\underline{m}!}$ defines the multinomial coefficient [AS70, 24.1.2] with $\underline{m}! = \prod_{k=1}^{M} m_k!$,

and the power series $\Psi(z_{1,N})$ is such that, for $\widetilde{z_{1,N}} < 1/4$,

(ii) $\Psi(z_{1,N})$ is absolutely convergent,

(iii) $\Psi(z_{1,N}) = \Phi(\widehat{z_{1,N}}) = \frac{1}{2}\left(1 - \sqrt{1 - 4\widehat{z_{1,N}}}\right)$,

(iv) the remainder $R_M\Psi(z_{1,N}) = \displaystyle\sum_{\left\{\substack{\underline{m'}\in\mathbb{M}_N\\ m'\geq M+1}\right.} \Psi_{\underline{m'}} \prod_{k=1}^{N}(z_k)^{m_k}$ is bounded by

$$|R_M\Psi(z_{1,N})| \leq R_M\Phi(\widetilde{z_{1,N}}) \leq \Phi_{M+1}\frac{(\widetilde{z_{1,N}})^{M+1}}{1-4\widetilde{z_{1,N}}} \leq \frac{1}{2\sqrt{\pi(M+1)}(2M+1)}\,\frac{(4\widetilde{z_{1,N}})^{M+1}}{1-4\widetilde{z_{1,N}}}. \tag{105}$$

Proof:

(i)-(ii)-(iii) From the definition of $\Psi_{\underline{m}}$ and $\Psi(z_{1,N})$,

$$\big(\Psi(z_{1,N})\big)^2 = \left(\sum_{\underline{m}\in\mathbb{M}_N} \Psi_{\underline{m}} \prod_{k=1}^{N}(z_k)^{m_k}\right)^2 = \sum_{\left\{\substack{\underline{m}\in\mathbb{M}_N\\ |\underline{m}|\geq 2}\right.} \left(\sum_{(\underline{p},\underline{q})\in\mathbb{M}_{\underline{m}}} \Psi_{\underline{p}}\Psi_{\underline{q}}\right) \prod_{k=1}^{N}(z_k)^{m_k} = \Psi(z_{1,N}) - \widehat{z_{1,N}}. \tag{106}$$

Hence, $\Psi(z_{1,N})$ is the root of the second order polynomial equation $\Psi(z_{1,N})^2 - \Psi(z_{1,N}) + \widehat{z_{1,N}} = 0$ such that $\Psi(z_{1,N}) = 0$ if $\widehat{z_{1,N}} = 0$. The solution is $\Psi(z_{1,N}) = \Phi(\widehat{z_{1,N}})$. Now, $(\widehat{z_{1,N}})^k = (z_1+\cdots+z_M)^k = \displaystyle\sum_{\left\{\substack{\underline{m}\in\mathbb{M}_N\\ m=k}\right.} \frac{m!}{\underline{m}!}\prod_{i=1}^{N}(z_i)^{m_i}$ where $\dfrac{m!}{\underline{m}!}$ defines the multinomial coefficient (see [AS70, 24.1.2]). Hence, the series

$$\Psi(z_{1,N}) = \sum_{m=1}^{+\infty} \Phi_m(\widehat{z_{1,N}})^m = \sum_{\underline{m}\in\mathbb{M}_N} \Phi_m \frac{m!}{\underline{m}!}\prod_{k=1}^{N}(z_k)^{m_k} \tag{107}$$

is absolutely convergent with respect to the mono-index m for $|\widehat{z_{1,N}}| < 1/4$ and is absolutely convergent with respect to the multi-index $\underline{m}$ for $\widetilde{z_{1,N}} < 1/4$. Coefficients $\Psi_{\underline{m}}$ are straightforwardly identified in (107) that completes the proof.

(iv) This point is straightforward.

■

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