Puzzles in pipes with negative curvature: from the Webster PDE to stable numerical simulation in real time^{*}

R. Mignot, T. Hélie[†]

IRCAM & CNRS, UMR 9912: Analysis-Synthesis team. D. Matignon[‡] Université de Toulouse; ISAE, Applied Mathematics training unit.

Abstract

Minimal realizations of a class of delay-differential systems are derived for the digital simulation of waveguides, modelled by the Webster horn equation. Studying their stability is an interesting issue, since *negative* curvatures could lead to unstable systems. Spectral properties of Toeplitz matrix play a key role.

Keywords

Webster equation, delay-differential systems, BIBO-stability, simulation.

1 Introduction

The wave equation with space-varying coefficients that models propagation in horns is the Webster PDE, which is known to be *conservative*, whatever the shape of the horn; hence, stable numerical schemes can be derived for it, e.g. forward Euler on the vector (pressure, flow) with a CFL condition. But, in the case of *negative curvature*, the waveguide decompositions usually used for real-time simulations introduces unstable subsystems. A similar paradox in the case of a convergent cone has been recently fixed in [1] using minimal realization of a delay system. In the present work, the same methodology is applied to a pipe with negative (constant) curvature, the discretization of which now gives rise to a delay-differential system of retarded type.

2 1D propagation in a convex acoustic pipe

Consider a pipe with radius $r(\ell) = \cos(\ell)$ for $\ell \in [-\frac{L}{2}, \frac{L}{2}]$, $L < \pi$ (see Fig. 1a). Inside the pipe, the acoustic pressure p is governed by the (adimensionnal) Webster PDE

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[†]1, pl. Igor Stravinsky, 75004 Paris, France. e-mail: {mignot,helie}@ircam.fr

[‡]10, av. E. Belin, F-31055 Toulouse Cedex 4, France. e-mail: denis.matignon@isae.fr

 $\{\partial_{\ell}^2 + 2\frac{r'(\ell)}{r(\ell)}\partial_{\ell} - \partial_t^2\}p(\ell, t) = 0$, the velocity v by the (adimensionnal) Euler PDE $\partial_t v(\ell, t) = -\partial_\ell p(\ell, t)$, and travelling waves $\phi^{\pm} := r(p \pm v)$ by the coupled equations

$$\partial_{\ell}\phi^{\pm}(\ell,t) \pm \partial_{t}\phi^{\pm}(\ell,t) = \left(r'(\ell) / r(\ell)\right)\phi^{\mp}(\ell,t). \tag{1}$$

The scattering matrix M such that Y(s) = M(s)U(s) with $U(s) = [\hat{\phi}^+(\frac{-L}{2},s), \hat{\phi}^-(\frac{L}{2},s)]^T$ and $Y(s) = [\hat{\phi}^-(\frac{-L}{2},s), \hat{\phi}^+(\frac{L}{2},s)]^T$ in the Laplace domain is meromorphic in $\Gamma(s)^2 = s^2 - 1$, and though defining a BIBO stable system, as expected. But, the standard decomposition of M used for waveguide implementations involves functions of Γ with branching points 1 and -1. Hence, contrarily to flared pipes for which $\Gamma(s)^2 = s^2 + 1$, the analyticity is lost over \mathbb{C}_0^+ : such a decomposition involves unstable sub-systems.

3 Approximation with pieces of conical pipes

Consider the piecewise affine approximations \tilde{r}_N of r with discretization step $\epsilon_N = \frac{L}{N}$. Waves in conical pieces of pipes are assumed to be ideally spherical and ϕ^{\pm} to be



Figure 1: (a) Approximated shape of pipe, (b) Stable realization for simulation.

continuous at junctions. It leads to the network in Fig. 1b where reflection functions are $R_{\alpha}(s) = \frac{\alpha}{s-\alpha}$ with $\alpha_N^i = \frac{\pi}{2\epsilon_N} (1 - \cos \epsilon_N) > 0$ for junctions inside the pipe and with $\alpha_N^e = \frac{\pi}{4\epsilon_N} (1 - \frac{\cos(L/2 - \epsilon_N)}{\cos(L/2)}) < 0$ at both ends. This system corresponds to a delay differential system of retarded type with the following state-space representation

$$s X_N(s) = A_N(e^{-\tau_N s}) X_N(s) + B_N(e^{-\tau_N s}) U(s),$$
 (2)

$$Y(s) = C_N(e^{-\tau_N s})X_N(s) + D_N(e^{-\tau_N s})U(s),$$
(3)

where X_N is defined from $\hat{\phi} = \hat{\phi}^+ + \hat{\phi}^-$ by $X_N(s) = \frac{\Delta_N}{s} [\hat{\phi}(\ell_0, s), \hat{\phi}(\ell_1, s), \dots, \hat{\phi}(\ell_N, s)]^T$, and where $A_N(w) = \Delta_N W_N(w)$, $W_N(w)$ is the $(N+1) \times (N+1)$ -symmetrical Toeplitz matrix such that $[W_N(w)]_{ij} = w^{|i-j|}$, and $\Delta_N = \text{diag}([\alpha_N^e, \alpha_N^i, \dots, \alpha_N^i, \alpha_N^e])$.

For fixed N, the set \mathcal{P}_N of roots of det $(sI_{(N+1)} - A_N(e^{-\tau_N s}))$ is countable, belongs to \mathbb{C}_0^- and the system in Fig. 1b proves to be stable. The theoretical question of the behaviour of the set of poles \mathcal{P}_N as N goes towards infinity is left open, so far. Simulation results will be presented to illustrate and help understand their behaviour.

References

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