Fundamentals of discrete Fourier analysis

summer 2006 lecture on analysis,
modeling and transformation of audio signals

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Contents

1 Discrete time signals

In the following seminar we are going to investigate a number of mathematical tools that may be used for the analysis, modeling or modification of audio signals.

For further information on the theory of discrete time signal porcessing consult [OS75]. Concerning the discrete Fourier transform a comprehensive source of information is [Smi03].

- The signals treated in the seminar will be discrete time signals, which means they have been obtained as the output of a analog to digital converter (see KT1/KT2).

- The sampling process of analog audio signals can be represented mathematically as follows:

Notation	
T	sample interval
$x_a(t)$	continuous time analog signal
n	discrete time
$x(n)$	is the discrete time signal.

$$x(n) = x_a(nT) \quad (1)$$

- In real world systems the time sampling process will always be accompanied by a quantization process.
- We assume the quantization noise is small and will neglect it in the further discussion.

2 Discrete time Fourier analysis

- Analysis of audio signals is most informative if it tells us something about the signal that is close to intuition.
- out intuition works in terms of **audio perception** or in terms of **physical sound sources**.
- one of the main properties of the ear audio perception is that it divides the audio signal into spectral bands and measures the energy in the different bands.
- individual sound sources are distinguished according to the placement of their resonators along the frequency axis.
- both concepts demand for an analysis of **spectral distribution of energy**, however, with differing frequency resolution.
  - taking the perceptual orientation it would be preferable if the frequency analysis resolution would follow the bandwidth of the critical bands in the human auditory system
    $\longrightarrow$ **wavelets** and **auditory filter banks**.
  - because most of the sound sources are either noisy or harmonic the physical point

of view calls for constant frequency resolution (harmonic sources).

- algorithmic design is much simpler with constant frequency resolution, such that this option is most often used in audio processing algorithms.
- we will adopt this view point for the following investigation.

2.1 Fourier analysis

The main tool for spectral analysis is the representation as a Fourier spectrum.

- The Fourier spectrum $X(\omega)$ of a finite signal $x(n)$ can be obtained by means of

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{2}$$

- if the sum in eq. (2) converges then X is the complex amplitude spectrum at frequency ω of the signal $x(n)$.
- eq. (2) is called the **Fourier transform** of the time discrete signal $x(n)$.
- a sufficient condition for convergence of eq. (2) is that $x(n)$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty \tag{3}$$

- we show in section **5.1** that the **inverse Fourier transform** yields

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega. \tag{4}$$

- Replacing the integral by means of the limit of a sequence of finite sums we find the representation

$$x(n) = \lim_{\Delta\omega \to 0} \frac{\Delta_\omega}{2\pi} \sum_{k=\frac{-\pi}{\Delta\omega}}^{\frac{\pi}{\Delta\omega}} X(k\Delta_\omega) e^{j\omega n}. \tag{5}$$

Interpretation:

- $X(\omega)$ indicates the amplitude and phase of a complex sinusoid of frequency ω.
- summing all of these sinusoids and scaling by $\frac{\Delta_\omega}{2\pi}$ recreates the original signal

2.2 Properties of the Fourier transform

- periodic signals are not absolutely summable such that the Fourier transform of a periodic signal is difficult to handle.
- because $e^{j\omega}$ is 2π periodic in ω the Fourier spectrum will also be 2π-periodic.
- for real signals we easily see that

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega} = \overline{\sum_{n=-\infty}^{\infty} x(n)e^{j\omega}} = \overline{X(-\omega)} \tag{6}$$

  the real (imaginary) part of the spectrum is symmetric (antisymmetric) with respect to $\omega = 0$

As a consequence the complete spectral information is contained already in the frequency range between 0 and half the samplerate

2.3 The minimum error property of the Fourier transform

We consider a signal $x(n)$ and search for the amplitude A and phase ϕ of an individual complex exponential of frequency ω such that the squared residual energy

$$E = \sum_{n=0}^{N-1} |(x(n) - Ae^{j(\omega n+\phi)})|^2 \tag{7}$$

$$= \sum_{n=0}^{N-1} (x(n) - Ae^{j(\omega n+\phi)})\overline{(x(n) - Ae^{j(\omega n+\phi)})} \tag{8}$$

becomes minimal (the line over an expression denotes complex conjugation).

Set partial derivatives to zero:
phase :

$$0 = \frac{\partial}{\partial \phi} E \tag{9}$$

$$= \sum_{n=0}^{N-1} -jAe^{-j(\omega n+\phi)}(x(n) - Ae^{j(\omega n+\phi)}) + jAe^{j(\omega n+\phi)}\overline{(x(n) - Ae^{j(\omega n+\phi)})} \quad (10)$$

$$= \sum_{n=0}^{N-1} -e^{-j(\omega n+\phi)}(x(n) - Ae^{j(\omega n+\phi)}) + e^{j(\omega n+\phi)}\overline{(x(n) - Ae^{j(\omega n+\phi)})} \quad (11)$$

$$= \sum_{n=0}^{N-1}(x(n)e^{-j(\omega n+\phi)} - A) - (x(n)e^{j(\omega n+\phi)} - A) \quad (12)$$

$$= \sum_{n=0}^{N-1} x(n)e^{-j(\omega n+\phi)} - x(n)e^{j(\omega n+\phi)} \quad (13)$$

amplitude

$$0 = \frac{\partial}{\partial A}E \quad (14)$$

$$= \sum_{n=0}^{N-1} -e^{-j(\omega n+\phi)}(x(n) - Ae^{j(\omega n+\phi)}) - e^{j(\omega n+\phi)}\overline{(x(n) - Ae^{j(\omega n+\phi)})} \quad (15)$$

$$= \sum_{n=0}^{N-1} +e^{-j(\omega n+\phi)}(x(n) - Ae^{j(\omega n+\phi)}) + e^{j(\omega n+\phi)}\overline{(x(n) - Ae^{j(\omega n+\phi)})} \quad (16)$$

$$= \sum_{n=0}^{N-1} (x(n)e^{-j(\omega n+\phi)} - A) + (x(n)e^{j(\omega n+\phi)} - A) \quad (17)$$

$$2AN = \sum_{n=0}^{N-1} x(n)e^{-j(\omega n+\phi)} + x(n)e^{j(\omega n+\phi)} \quad (18)$$

Adding eq. (13) to eq. (18) yields

$$2AN = \sum_{n=0}^{N-1} 2x(n)e^{-j(\omega n+\phi)} \quad (19)$$

$$Ae^{j\phi} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\omega n} \tag{20}$$

which is directly the FT analysis formula.

As a result we conclude that the FT coefficients minimize the residuum in eq. (7).

3 Discrete Fourier Transform

- The Discrete Fourier Transform deals with a special case of not absolutely summable signals: **the periodic signals**.
- For periodic signals with period N we have $x(n) = x(n + rN)$ for all integer values of r.
- the Fourier representation of periodic functions can be obtained using only basis functions $e^{j\omega n}$ that have the same periodicity as the signal. The basis functions that fulfill this property have frequency

$$\omega = k\Omega_N = k\frac{2\pi}{N} \tag{21}$$

- As shown in section **5.2** for the case of periodic signals we can use the set of basis N-periodic functions $e^{-j\Omega_N kn}$ to represent any periodic signal $x(n)$ according to

$$X(k\Omega_N) = \sum_{n=0}^{N-1} x(n)e^{-j\Omega_N kn} \tag{22}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k\Omega_N)e^{j\Omega_N kn} \tag{23}$$

$$\tag{24}$$

- eq. (22) is called the discrete Fourier transform and eq. (22) the inverse discrete Fourier transform.
- to simplify notation we will use $X(k)$ to denote the amplitude spectrum of the k-th basis function of the DFT.
- a bounded N-periodic signal can always represented using only N basis functions.
- if applied to finite time signals it is convention that the DFT is understood to represent a single period of the periodically extended function.

3.1 DFT for finite signals

- discrete signal processing algorithms on computers can not deal with the continuous FT spectrum.
- the DFT represents signals with a discrete number of basis functions.
- all signal processing algorithms use the DFT to represent the signal.
- For $N = 2^k$ and $k \in \{1, 2, ...\}$ there exist very fast implementations of the DFT which are called **FFT**.
- for finite time signals the N-point DFT calculates the Fourier spectrum of a periodic continuation of the selected part of the signal

Relation between DFT and FT of a finite time signal

- Suppose we have a signal that is time limited with duration K.
- We calculate the DFT and the Fourier transform of this signal and compare the results.

Fourier transform	: $X(\omega)$	$= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$
Discrete FT	: $X(k\Omega_N)$	$= \sum_{n=0}^{N-1} x(n)e^{-j(\Omega_N k)n}$

Table 1: relation between FT and DFT

- comparison shows that the the DFT **samples the FT with a regular grid** at positions $\omega = k\Omega_N$.
- increasing the size N of the DFT will increase the density of the sampling $\longrightarrow$ zero padding!!

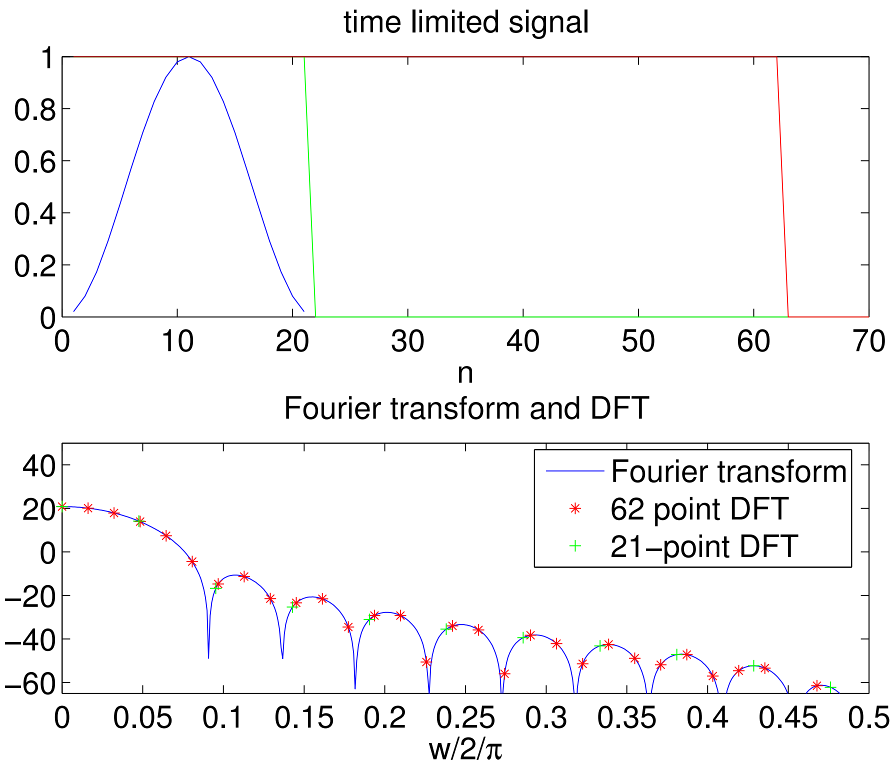

Figure 1: Fourier transform and DFT of a time limited signal

3.2 DFT for infinte signals

- the Fourier transform represents the signal by means of superposition of stationary sinusoids
- For time varying signals the representation with stationary sinusoids does not correlate with perception.
- FT calculates average energy of complete signal.
- perceptual perception *calculates* short time average of energy.
- we can increase the intuitive meaning of the spectral analysis by limiting the analyzed time segment to segments that are perceived as stationary.
- technically this corresponds with the application of a relatively short analysis window to cut the signal into segments that may be considered stationary.

4 Analysis windows

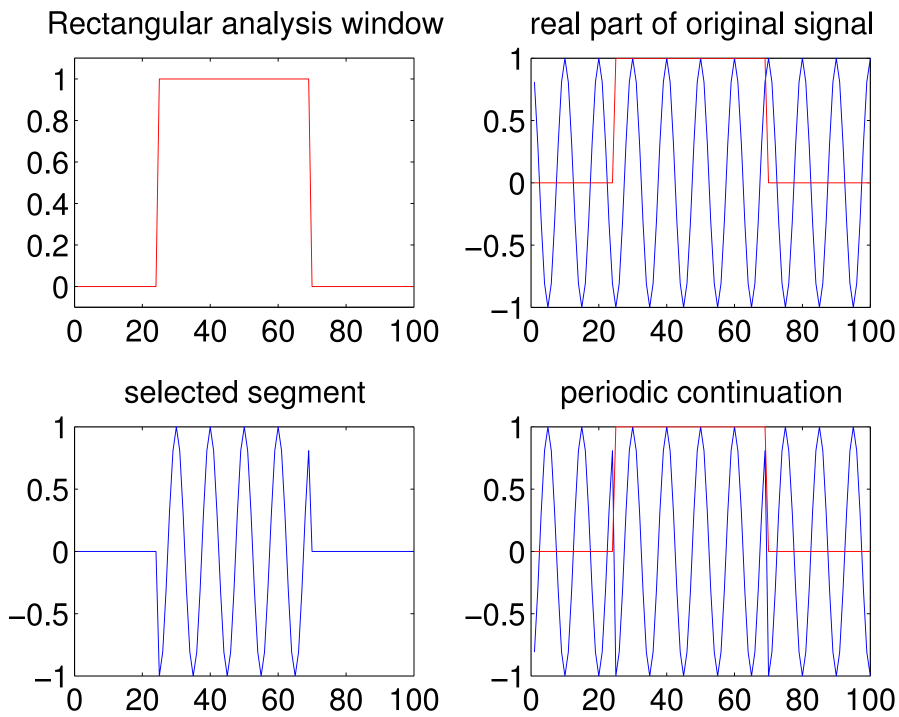

Figure 2: cutting part of an infinite signal and calulation of the DFT yields the spectrum of a periodic continuation of the selected part

- The selection of a part of the signal is called windowing.
- windowing can be understood as a multiplication of the signal with an analysis window.
- The analysis window used in fig. 2 is rectangular and defined as

$$w(n) = \begin{cases} 1 & \text{for} \quad n \geq 0 \quad \& \quad n < 100 \\ 0 & \text{else.} \end{cases} \tag{25}$$

4.1 Analysis of complex exponential

If we are interested to obtain an explicite formular for the DFT analysis of a complex sinusoids with arbitrary analysis window, we have to proceed in three steps.

1. we calculate the Fourier transform of the segment using a rectangular window
2. by means of the convolution theorem we apply he analysis window to the segment
3. we sample the obtained Fourier transform of the windowed signal to obtain the DFT.

Step 1:

we calculate the result for the rectangular analysis window of length M with arbitrary position (note: in all practical implementations Fourier transform basis functions have origin at start of window)

$$x(n) = e^{j(\Omega n+\phi)} \quad (26)$$

$$r(n) = \begin{cases} 1 & \text{for} \quad 0 \leq n < M \\ 0 & \text{else} \end{cases} \quad (27)$$

$$X(\omega) = \sum_{n'=-\infty}^{\infty} r(n'-m)e^{j(\Omega n'+\phi)}e^{-j\omega(n'-m)} \quad (28)$$

$$= \sum_{n=-\infty}^{\infty} e^{j(\Omega(n+m)+\phi)}e^{-j\omega n} = e^{j(\Omega m+\phi)} \sum_{n=-\infty}^{\infty} e^{jn(\Omega-\omega)} \quad (29)$$

$$= e^{j(\Omega m+\phi)} \frac{1-e^{jM(\Omega-\omega)}}{1-e^{j(\Omega-\omega)}} \quad (30)$$

$$= \quad e^{j(\Omega m+\phi)} \frac{e^{j\frac{M}{2}(\Omega-\omega)}}{e^{j\frac{1}{2}(\Omega-\omega)}} \frac{e^{-j\frac{M}{2}(\Omega-\omega)} - e^{j\frac{M}{2}(\Omega-\omega)}}{e^{-j\frac{1}{2}(\Omega-\omega)} - e^{j\frac{1}{2}(\Omega-\omega)}} \tag{31}$$

$$= \quad e^{j(\Omega m+\phi)} e^{j\frac{M-1}{2}(\Omega-\omega)} \frac{2j(e^{-j\frac{M}{2}(\Omega-\omega)} - e^{j\frac{M}{2}(\Omega-\omega)})}{2j(e^{-j\frac{1}{2}(\Omega-\omega)} - e^{j\frac{1}{2}(\Omega-\omega)})} \tag{32}$$

$$= \quad (e^{j((m+\frac{M-1}{2})\Omega+\phi)}) \cdot (e^{-j\frac{M-1}{2}\omega}) \cdot \frac{\sin((\Omega-\omega)\frac{M}{2})}{\sin((\Omega-\omega)\frac{1}{2})} \tag{33}$$

- The first two complex exponential functions have magnitude 1 and affect phase only.
- The last term, the periodic sinc function affects amplitude and phase, phase only by means of sign changes.
- we investigate first the amplitude term and then the phase terms.

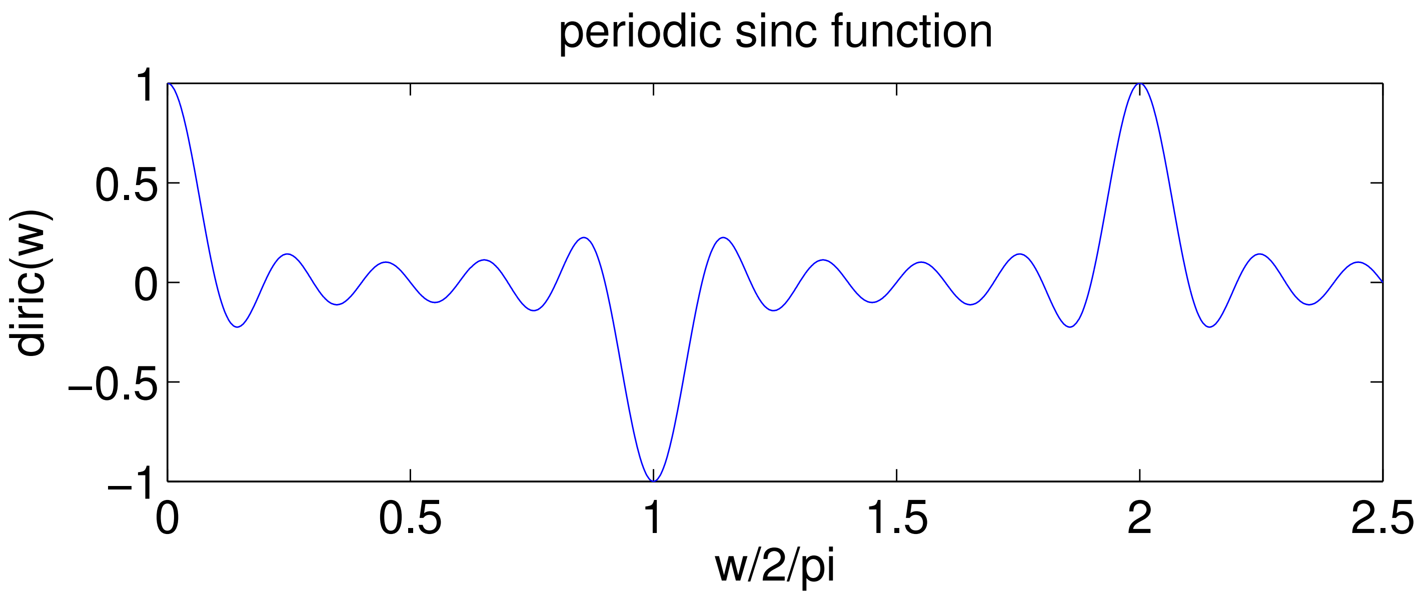

Figure 3: Periodic sinc function $\frac{\sin((\omega)\frac{M}{2})}{M\,\sin((\omega)\frac{1}{2})}$

The result contains this parts which represent different contributions to amplitude and phase spectrum of the sinusoid.

The amplitude spectrum :

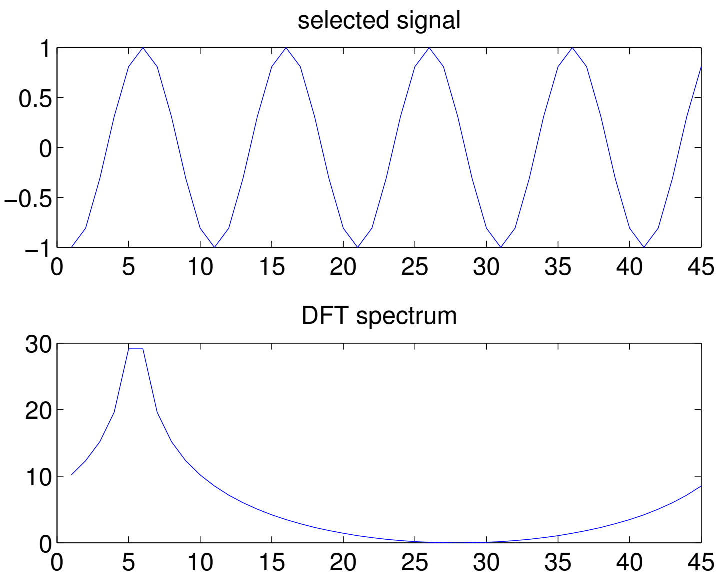

Figure 4: DFT analysis of a sinusoid with rectangular analysis window

is entirely defined by the periodic sinc-function

- $\frac{\sin((\Omega-\omega)\frac{M}{2})}{\sin((\Omega-\omega)\frac{1}{2})}$
- it is independent of the window position!
- is equal to the DFT of the rectangular window itself moved in frequency such that its c enter is exactly at the frequency Ω of the sinusoid.

The phase spectrum has 3 contributions:

1. $\pm\pi$
   according to the sign of the periodic sinc function.
2. $e^{j((m+\frac{M-1}{2})\Omega+\phi)}$
   which is the phase of the original sinusoid at the center position of the window $(m + \frac{M-1}{2})$. This part moves together with the window in sync with frequency of the sinusoid.
3. $e^{-j\frac{M-1}{2}\omega}$
   a linear phase trend that stems from the fact that the origin of the DFT basis functions is not located at the window center but at the start of the center.
   This factor needs to be compensated, because it does not tell us anything related to the signal, but, is entirely due to the mathematical setup.
   If we subtract this phase trend from the phase of the spectrum we find that, besides the sign changes imposed by the periodic sinc, the phase of the whole spectrum is constant and equal to the phase of the sinusoid at the center of the window.

2. Step

For the general case with a analysis window $w(n)$ we make use of the convolution theorem of the Fourier transform

$$x(n) \leftrightarrow X(\omega) \tag{34}$$

$$y(n) \leftrightarrow Y(\omega) \tag{35}$$

$$x(n)y(n) \leftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)Y(\omega-\Omega)\Omega \tag{36}$$

- the DFT of the product of two sequences $x(n)$ and $y(n)$ can be obtained by means of **circular convolution** of the FTs $X(\omega)$ and $Y(\omega)$.
- the multiplication of an arbitrary analysis window $w_a(n)$ with the rectangular window $w_r(n)$ does not change the original window if the length of the $w_r(n)$ is at least the same as the length of $w_a(n)$:

$$w_a(n)w_r(n) = w_a(n) \quad (37)$$

$$\int_{\omega=-\pi}^{\pi} W_a(\Omega)W_r(\omega-\Omega)d\Omega = W_a(\omega) \quad (38)$$

$$(39)$$

- therefore, the convolution with the rectangular window does not change the window spectrum
- convolution with a shifted version of the rectangular window spectrum yields a shifted version of the analysis window spectrum.
- the FT of a complex exponential using an arbitrary analysis window yields a FT that is the FT of the analysis window shifted in frequency to the frequency of the sinusoid.

$$X(\omega) = (e^{j((m+\frac{M-1}{2})\Omega+\phi)}) \cdot (e^{-j\frac{M-1}{2}\omega}) \cdot |W(\omega-\Omega)| \quad (40)$$

- compare: modulation

3. Step

As shown in table (1) the DFT simply samples the Fourier transform

we can conclude, that a sinusoid windowed by an arbitrary analysis window creates a spectrum that is the spectrum of the analysis window itself shifted in frequency to the sinusoids frequency ω. The resulting spectrum is sampled according to the size N of the DF Transform.

4.2 Common analysis windows

A lot of different analysis windows have been proposed. Most of them are combinations of low frequency cosines such that the window spectrum always has lowpass characteristic

- rectangular window

$$w(n) = \begin{cases} 1 & \text{for} \quad n \geq 0 \quad \& \quad n < N \\ 0 & \text{else.} \end{cases} \tag{41}$$

- Hanning window

$$w(n) = \begin{cases} 0.5 - 0.5\cos(2\pi\frac{n}{N-1}) & \text{for} \quad n \geq 0 \quad \& \quad n < N \\ 0 & \text{else.} \end{cases} \tag{42}$$

- Hamming window

$$w(n) = \begin{cases} 0.54 - 0.46\cos(2\pi\frac{n}{N-1}) & \text{for} \quad n \geq 0 \quad \& \quad n < N \\ 0 & \text{else.} \end{cases} \tag{43}$$

- Blackman window

$$w(n) = \begin{cases} 0.42 - 0.5\cos(2\pi\frac{n}{N-1}) + 0.08\cos(4\pi\frac{n}{N-1}) & \text{for} \quad n \geq 0 \quad \& \quad n < N \\ 0 & \text{else.} \end{cases} \tag{44}$$

- Gaussian window (cut)

$$w(n) = \begin{cases} e^{-\frac{(n-\frac{N-1}{2})^2}{\sigma^2}} & \text{for} \quad n \geq 0 \quad \& \quad n < N \\ 0 & \text{else.} \end{cases} \tag{45}$$

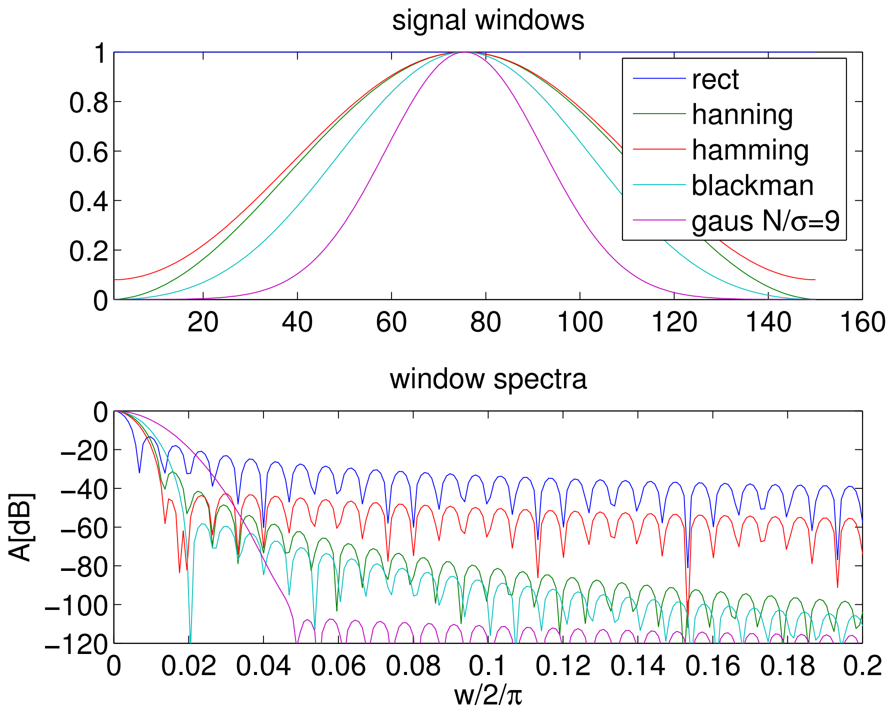

Figure 5: The common analysis windows

criteria for selection of the window form is the compromise between width of mainlobe and attenuation of sidelobes.

- smaller **mainlobe** yields better **frequency resolution**.
  better frequency resolution means that nearby sinusoids can be distinguished as separated spectral peaks.
- more **sidelobe attenuation** yields less **crosstalk** between distant sinusoids. (Note, audio spectra may require dynamic range of 60dB or more!)

4.3 Frequency resolution

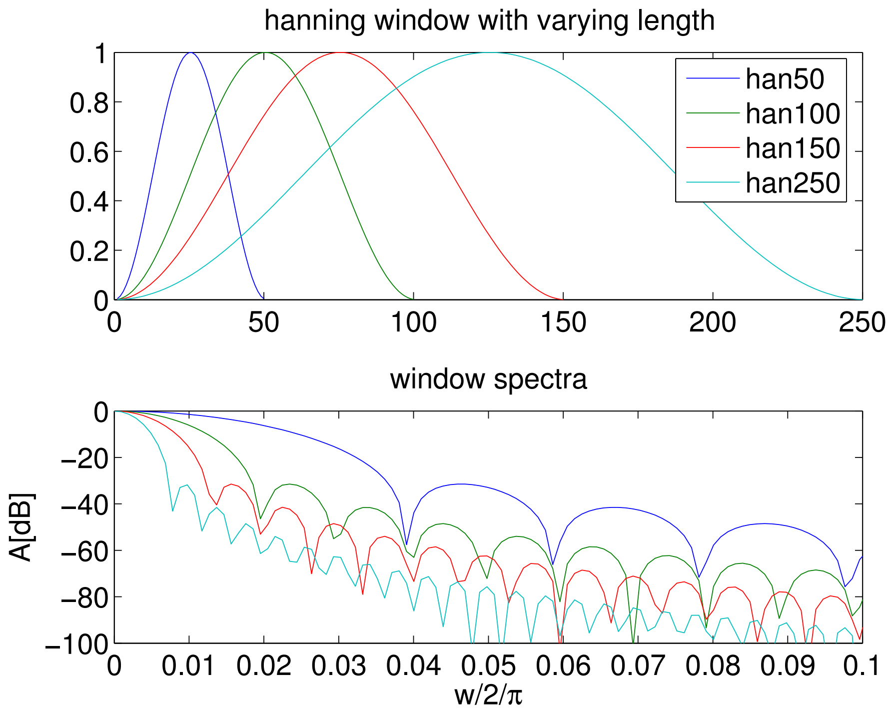

Figure 6: DFT analysis with Hanning window of different length

- The main factor that determines the frequency resolution is not the form of the window, but, it is the length of the window.
- if the form of the window does not change bandwidth B and length T of the window are inversely related such that

$$BT = \text{const}$$

4.4 The optimal window

According to the uncertainty principle the time duration T and the bandwidth of the window function B are always larger than 1.

- In the following we use the second central moment of the instantaneous amplitude distribution

$$T = \frac{\sum_{n=0}^{N-1}(n-\overline{n})^2|x(n)|}{\sum_{n=0}^{N-1}|x(n)|}, \quad (46)$$

$$B = \frac{1}{K^2}\frac{\sum_{k=0}^{K-1}(k-\overline{k})^2|X(k)|}{\sum_{k=0}^{K-1}|X(k)|}. \quad (47)$$

  $\overline{n}$ and $\overline{k}$ represent mean time and mean frequency of the signal and its spectrum.
- The optimal window is the one that has the smallest duration - bandwidth product
- best combination of time and frequency resolution.

the duration - bandwidth products of the different analysis windows presented in section **4.2** are listed in table (2) :

Window	T	B [rad]	TB	width of mainlobe [rad]
Rectangular	43.3	1.14	49.5	$\frac{M}{2} = 0.098$
Hamming	29.9	0.55	16.6	$\frac{M}{4} = 0.184$
Hanning	27.3	0.049	1.34	$\frac{M}{4} = 0.184$
Blackman	23.8	0.043	1.04	$\frac{M}{6} = 0.27$
Gauss	16.6	0.061	1.02	$\frac{M}{15} = 0.62$

Table 2: Time duration T, bandwidth B, and duration - bandwidth TB as well as the width of the mainlobe for the the different analysis windows all having window length of N=150 samples.

- From the results we can find the known fact the Gaussian window, which has the smallest TB product is optimal in terms of time frequency resolution as defined above.
- there are other definitions of T and B

- particularly important is the width of the mainlobe B_M (resolution of nearby sinusoids) and the total length of the window N (calculation time).
- from table (2) we find that the mainlobe of the Hanning and Hamming window for the same window length N is considerably smaller than the width of the mainlobe of the Gaussian window.
- The small value of TB for the Gaussian window is achieved by means of sacrificing the narrow mainlobe in favor of sidelobe attenuation.
- Perceptually for most applications, a sidelobe attenuation of more than $300dB$ is useless,
- the Gaussian window is optimal as long as we consider practically less relevant B and T definitions.

5 Appendix

5.1 Analysis/Synthesis with discrete time Fourier transform

We want to prove that the eq. (4) reverses the operation of the Fourier transform equation eq. (2).

We write the overall operation of analysis and synthesis as

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega \quad (48)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}(\sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k})e^{j\omega n}d\omega \quad (49)$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} x(k)\int_{-\pi}^{\pi} e^{j\omega(n-k)}d\omega \quad (50)$$

We investigate $\int_{-\pi}^{\pi} e^{j\omega(n-k)}d\omega$ first for the case $n = k$

$$I_{k=n} = \int_{-\pi}^{\pi} e^{j\omega 0} = \int_{-\pi}^{\pi} e^{0} = 2\pi \tag{51}$$

and for the case $n - k = l \neq 0$ we obtain

$$I_{k\neq n} = \int_{-\pi}^{\pi} e^{j\omega(n-k)} = [\frac{e^{j\omega(n-k)}}{j(n-k)}]_{-\pi}^{\pi} \tag{52}$$

$$= \frac{e^{j\pi l}}{jl} - \frac{e^{-j\pi l}}{jl} \tag{53}$$

$$= \frac{e^{j\pi l}}{jl} - \frac{e^{j\pi l}}{jl} = 0 \tag{54}$$

where the last transformation follows from the fact that clockwise and anti clockwise rotation of the complex pointer by an integer multiple of π will produce always the same result.

Combining the results we get

$$\int_{-\pi}^{\pi} e^{j\omega(n-k)} d\omega = \delta(n-k) \tag{55}$$

such that

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) = x(n) \tag{56}$$

which is what we wanted to show.

5.2 Analysis/Synthesis with Discrete Fourier transform (DFT)

We want to prove that the eq. (23) reverses the operation of the Fourier transform equation eq. (22).

As for the Fourier transform we write the combined analysis synthesis equation

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k\Omega_N)e^{j\Omega_N kn} \quad (57)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}(\sum_{m=0}^{N-1} x(m)e^{-j\Omega_N mk})e^{j\Omega_N kn} \quad (58)$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x(m)\sum_{k=0}^{N-1} e^{j\Omega_N k(n-m)} \quad (59)$$

Again we study $\sum_{k=0}^{N-1} e^{j\Omega_N k(n-m)}$ first for the case $n = k$

$$I_{k=n} = \sum_{k=0}^{N-1} e^{j\Omega_N k0} = \sum_{k=0}^{N-1} 1 = N \tag{60}$$

and for the case $n - k = l \neq 0$ we obtain using the geometric series summation formula[1]

$$\begin{aligned} I_{k\neq n} &= \sum_{k=0}^{N-1} e^{j\Omega_N kl} = \frac{1 - e^{j\Omega_N lN}}{1 - e^{j\Omega_N l}} & (61) \\ &= \frac{1 - e^{j2\pi l}}{1 - e^{j\Omega_N l}} & (62) \\ &= \frac{1 - 1}{1 - e^{j\Omega_N l}} = 0. & (63) \end{aligned}$$

Which leads us to the same conclusion as for the Fourier transform.

[1] Note, that $l = \pm\{1, \ldots, N-1\}$.

References

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