Narayana's Cows and Delayed Morphisms

Jean-Paul Allouche, CNRS, LRI, F-91405 Orsay
Tom Johnson, 75 rue de la Roquette, F-75011 Paris

Abstract: Narayana's cows differ from Fibonacci's rabbits in that they begin to produce calves only in their fourth year. The sequence giving for each year the total number of calves and cows has been translated by one of us (T. J.) into a composition called Narayana's Cows. *On the other hand, morphisms have been already used in music. In this paper we define "morphisms with delays." They are morphisms that mimic the behavior of Narayana's cows: a letter remains unchanged during a fixed number of iterations of the morphism and then is transformed by the morphism. The sequences obtained can also be generated via morphism and letter replacement.*

Six or eight years ago one of us (T. J.) found a German edition of a little book on the history of mathematics by a Ukranian scholar named Andrej Grigorewitsch Konforowitsch. The book was full of curious information, but what was particularly stricking was the following, which Konforowitsch attributed to Narayana, an Indian mathematician in the 14th century.

A cow produces one calf every year. Beginning in its fourth year, each calf produces one calf at the beginning of each year. How many cows and calves are there altogether after 20 *years?*

In working this out, T. J. came to know a unique numerical sequence, and a year or so later found a way to translate this into a composition called *Narayana's Cows.* It begins with the original cow and her first calf: long-short. The second year she has another calf: long-short-short. The third year: long-short-short-short. Then in the fourth year, the first calf also becomes a mother and the herd grows from four to six: long-short-short-short-long-short. The music continues like this, though it does not go all the way to the 20th year, because by the 17th year there are already 840 cows and calves and 15 minutes of music.

Many things can be said about the mathematics of Narayana's cows, about different ways to translate them into music, about the point at which the calves begin to outnumber the cows, about the rate of population increase, the limit which this rate approaches, and so on. The essence of the problem, however, is simply the sequence resulting as the years go by: 1, 1, 1, 2, 3, 4, 6, 9, 13 ... Like the Fibonacci sequence, each number is calculated by adding earlier numbers, but instead of adding the two previous numbers, as one does for the Fibonacci series, one adds the previous number in the sequence plus the number two places before that:

$$S_n = S_{n-1} + S_{n-3}.$$

The last number above is $13 = (9 + 4)$ and the next must be $19 = (13 + 6)$.

Last year, as T. J. was writing *Self-Similar Melodies* [Johnson 1996], he decided to investigate this further and include a chapter on "Transforming with Delays". What if Narayana's cows gave birth already in their third year, instead of the fourth? What would happen to the population growth, and to the music, if they had to wait until the fifth year, or the sixth?

This can perhaps be best explained if we forget about cows and calves and work with the Thue-Morse sequence, a binary automatic sequence that has already been studied rather extensively. In the classic case, 0 goes to 01 and 1 goes to 10, there are no delays, and the population doubles with each transformation. $S_n = S_{n-1} + S_{n-1}$.

$n = 0$	0															
$n = 1$	0								1							
$n = 2$	0				1				1				0			
$n = 3$	0		1		1		0		1		0		0		1	
$n = 4$	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
	...															

This is equivalent to having calves that are born in one year and become mother cows the very next year and it is easy to see that $S_n = 2^n$. What happens to the Thue-Morse population if its digits give birth in their third year instead of their second? Well, instead of doubling with each transformation, they now follow the Fibonacci series, $S_n = S_{n-1} + S_{n-2}$.

$n = 0$	0												
$n = 1$	0												
$n = 2$	0								1				
$n = 3$	0					1			1				
$n = 4$	0			1		1			1			0	
$n = 5$	0		1	1		1		0	1		0	0	
$n = 6$	0	1	1	1	0	1	0	0	1	0	0	0	1
	...												

The values of S_n are here: 1, 1, 2, 3, 5, 8, 13 ... i.e., the celebrated Fibonacci numbers. These numbers are related to the golden ratio φ, as the limit when n goes to infinity of S_{n+1}/S_n is $\varphi = \frac{1+\sqrt{5}}{2}$. Many examples of the use of the Fibonacci numbers in art are well-known, (to quote but one example: the *Klavierstück IX* by Stockhausen).

If the ones and zeros give birth in their fourth year, the modified Thue-Morse series grows at the same rate as the Narayana series, $S_n = S_{n-1} + S_{n-3}$.

$n=0$	0												
$n=1$	0												
$n=2$	0												
$n=3$	0									1			
$n=4$	0						1			1			
$n=5$	0				1		1			1			
$n=6$	0			1	1		1			1			0
$n=7$	0		1	1	1		1		0	1		0	0
$n=8$	0	1	1	1	1	0	1	0	0	1	0	0	0
	...												

Let us take that one step further and have each digit divide into two digits in its fifth stage of existence. Now the formula is $S_n = S_{n-1} + S_{n-4}$, and the Narayana series of 1, 1, 1, 2, 3, 4, 6, 9, 13 ... becomes 1, 1, 1, 1, 2, 3, 4, 5, 7 ...

$n=0$	0									
$n=1$	0									
$n=2$	0									
$n=3$	0									
$n=4$	0							1		
$n=5$	0					1		1		
$n=6$	0				1	1		1		
$n=7$	0			1	1	1		1		
$n=8$	0		1	1	1	1		1		0
$n=9$	0	1	1	1	1	1	0	1	0	0
	...									

One day T. J. noticed that there was an easier way of calculating this: each line was actually just a repetition of the sequence one line above, followed by the inversion of the sequence four lines above. Why was the line inverted? Since he often likes to interpret such sequences as melodies that ascend with ones and descend with zeros, he also wanted to know whether the ones would continue to outnumber the zeros as they do here.
Narayana's little problem had led him to one particularly musical sequence, but now he was confronted with a whole family of sequences that seemed to have strong musical possibilities. J.-P. A. was quite interested because he had never seen explicit mathematical analysis of "delayed morphisms" of this sort. Some observations about these delayed morphisms could be useful in mathematics or theoretical computer science and could also be useful to composers working with logical sequences.

How is it possible to generate in some mechanical way the above sequences? Let us start over with the classical Thue-Morse sequence. As we have said before, it can be generated by iterating the morphism

$$0 \longrightarrow 01, \quad 1 \longrightarrow 10,$$

thus obtaining successively

0
01
0110
01101001
...

But another way of generating this sequence is to use a so-called "locally catenative" formula, (see, for example, [Shallit 1988]). Define indeed a sequence of binary words $(w_n)_n$ by

$$w_0 = 0, \quad w_n = w_{n-1}\overline{w_{n-1}} \text{ for } n \geq 1,$$

where the word $\overline{x}$ is obtained from the word x by replacing the 0's by 1's and vice-versa. Hence one has successively

$$\begin{array}{lllll} w_0 & = & 0 & & \\ w_1 & = & w_0\overline{w_0} & = & 01 \\ w_2 & = & w_1\overline{w_1} & = & 0110 \\ w_3 & = & w_2\overline{w_2} & = & 01101001 \\ \ldots & & & & \end{array}$$

The reader will notice (and show by induction) that this sequence of words tends towards the Thue-Morse sequence when n goes to infinity, i.e., each w_n is a prefix of the Thue-Morse sequence.

The situation is a bit more complicated for the next sequence (where calves become mother cows only after a year). One can generate the sequence we obtained, i.e.,

$$0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ \ldots$$

as follows. Take the four-letter alphabet $\{0, 1, 0', 1'\}$ and define a morphism on this alphabet by

$$\begin{array}{lll} 0 & \rightarrow & 0' \\ 0' & \rightarrow & 0'1 \\ 1 & \rightarrow & 1' \\ 1' & \rightarrow & 1'0. \end{array}$$

In other words $0'$ and $1'$ are "reproducing" digits, while 0 and 1 are "delaying" digits. Iterating this morphism starting from 0 gives

$$\begin{array}{l} 0 \\ 0' \\ 0'1 \\ 0'11' \\ 0'11'1'0 \\ 0'11'1'01'00' \\ \ldots \end{array}$$

Now deleting the symbols $'$, i.e., replacing $0'$ by 0 and $1'$ by 1, one obtains our infinite sequence. Of course a locally catenative approach is also possible. If we define indeed the sequence of words $(z_n)_n$ by

$$z_0 = 0, \ z_1 = 0, \ z_n = z_{n-1}\overline{z_{n-2}} \text{ for } n \geq 2,$$

then we obtain

$$\begin{array}{lllll} z_0 & = & 0 & & \\ z_1 & = & 0 & & \\ z_2 & = & z_1\overline{z_0} & = & 01 \\ z_3 & = & z_2\overline{z_1} & = & 011 \\ z_4 & = & z_3\overline{z_2} & = & 01110 \\ \ldots & & & & \end{array}$$

and the limit of the sequence of words $(z_n)_n$ is equal to our sequence

$$0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ \ldots$$

We leave to the diligent reader to obtain the six-letter alphabet (resp. the eight-letter alphabet) necessary to the three-year (resp. four-year) delay and to find out what the corresponding locally catenative formulas are. (Hint: for the "Narayana-Thue-Morse sequence" the lengths of the successive words obtained by iterating the locally catenative formula satisfy the same recursion as the number of Narayana cows.)

To end this paper we would like to come back to the Narayana cows and to study the limit of the growth rate of the population. Remember the number of animals was given by the sequence $1, 1, 1, 2, 3, 4, 6, 9, 13, \ldots$ satisfying the recurrence relation

$$S_n = S_{n-1} + S_{n-3}.$$

For the Fibonacci sequence the population growth is of the order of τ^n where τ is the largest root of the equation $X^2 - X - 1 = 0$. This number is equal to $\frac{1+\sqrt{5}}{2} = 1.61803...$, this is the famous *golden ratio*. For the Narayana cows the growth of the population is of the order of α^n where α is the largest real root of the equation $X^3 - X^2 - 1 = 0$. This quantity can be computed in close form with cubic and square roots and one can show it is equal to

$$\frac{1}{3}\left(\sqrt[3]{\frac{29+3\sqrt{93}}{2}} + \sqrt[3]{\frac{29-3\sqrt{93}}{2}} + 1\right) = 1.465571...$$

Unfortunately the limit of the growth rate with larger delays would involve solving equations of degree 4, 5, 6 ... In the case of degree four a formula of the above type could be given but it is well-known that, in general, algebraic equations of degree larger than or equal to five are not solvable by radicals, hence only approximate (i.e., non close) values can be given for the limit of the growth rate for these larger delays.

References

[1] [Allouche, Johnson 1995] J.-P. Allouche and T. Johnson, *Finite automata and morphisms in assisted musical composition*, Journal of New Music Research, **24** (1995), 97–108.

[2] [Johnson 1996] T. Johnson, *Self-Similar Melodies*, preliminary edition, Éditions 75, 1996.

[3] [Shallit 1988] J. Shallit, *A generalization of automatic sequences*, Theoretical Computer Science, **61** (1988), 1–16.

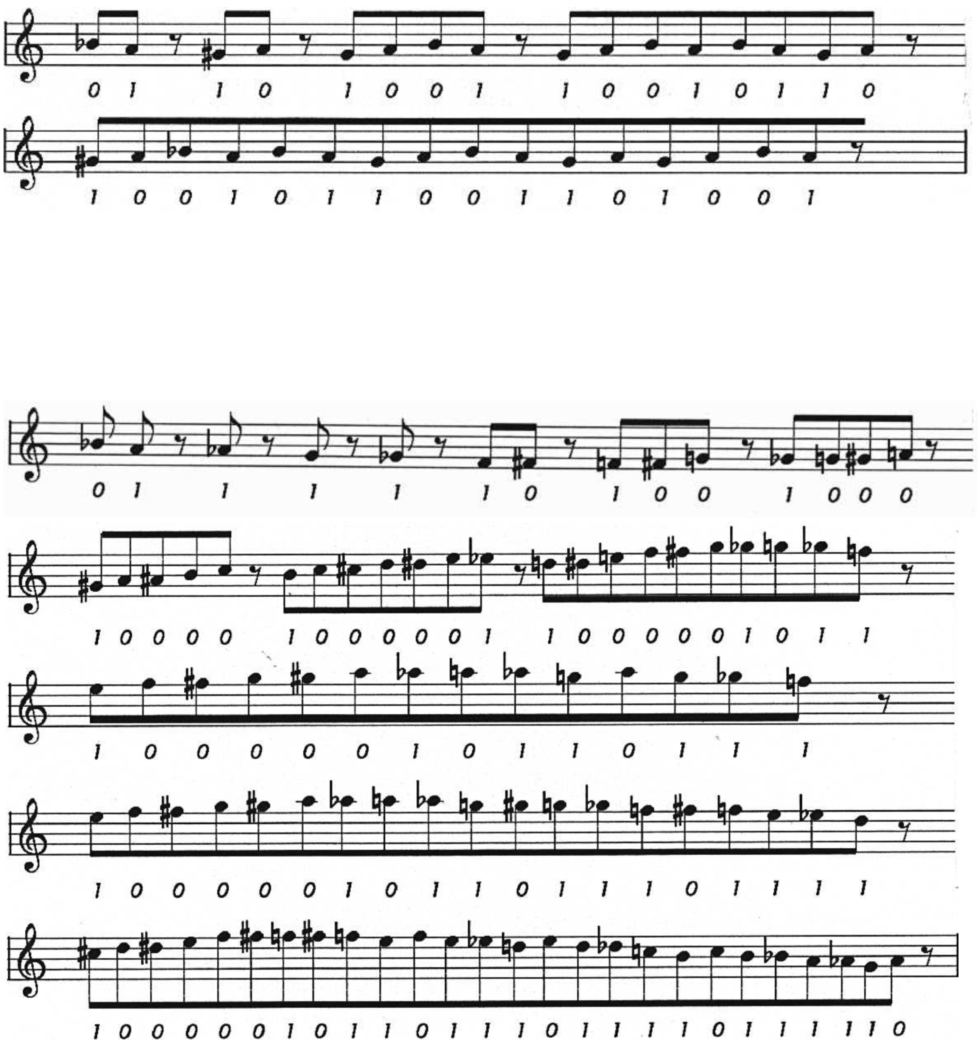

Figure 1: Melodic transcriptions of the Thue-Morse sequence and a delayed Thue-Morse sequence, where 1 is coded as a descending semi-tone, and 0 as a rising semi-tone. The first example follows the fifth iteration for the Thue-Morse sequence, with 2^5 notes and no delay. The balance of ones and zeros is always almost equal, and the melody never goes beyond a range of three notes. The second example follows the 16th iteration ($n = 16$) with a delay of three time periods, i.e., with the formula: $S_n = S_{n-1} +$ the inversion of S_{n-4}. The melody falls a bit, then rises more, as each 1 waits to be transformed into 10 and each 0 waits to be transformed into 01. In both transcriptions, pauses separate the sequences that were added at each transformation.