Of Rhythmic Canons and Mathematics

RC & Maths

- A definition
- Some formalizations
- "Aperiodicity" and cyclotomic factors
- Vuza canons
- Decompositions and Spectral Properties

In view of a book in progress, I am trying to present and organize the 'state of the art' of what we know and gathered on the subject. On the fly it enabled a few small but interesting results to emerge. And, of course, several questions…

RC & Maths
Fourier

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  - NB : same zeroes set for F_A and F_{IC_A}

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- $IC_A \star IC_B = IC(\mathbb{Z}_c)=c$ and Card A × Card B = c

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Isomorphism between several algebras: $Z[X]/(X^c-1) = Z^c$ (DFT's) because a poly with d°<c is known from its values in c places. Two algebras of complex valued maps over Z_c (* or x).

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- $A(X) = \sum X^k$, $k \in A$. A is defined mod c, $A(X)$ mod X^c-1.

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Cyclo, whose irreducibility just means that these roots come together in bunches.
NB: periodicity shows on the DFT… a smaller period means less fourier coefficients, i.e. many are zero !

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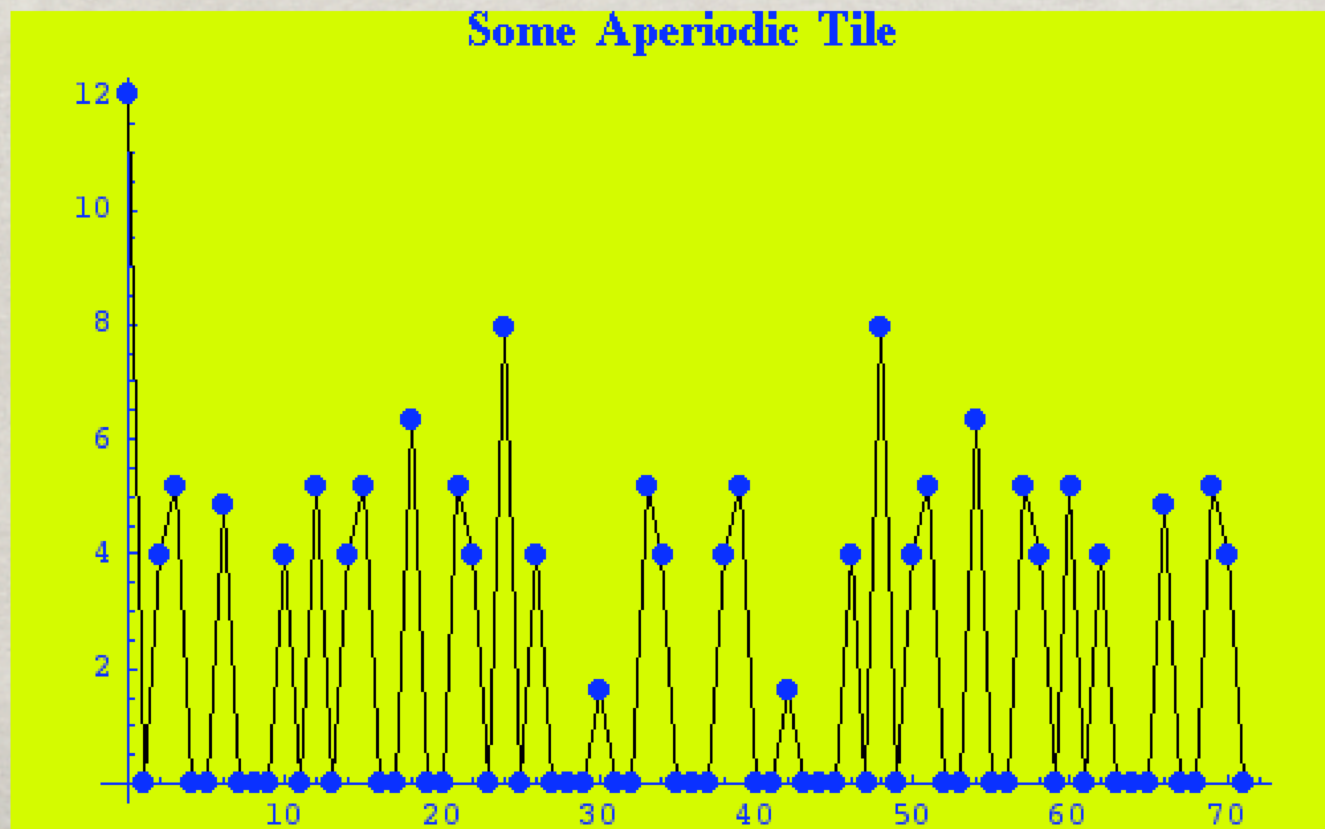

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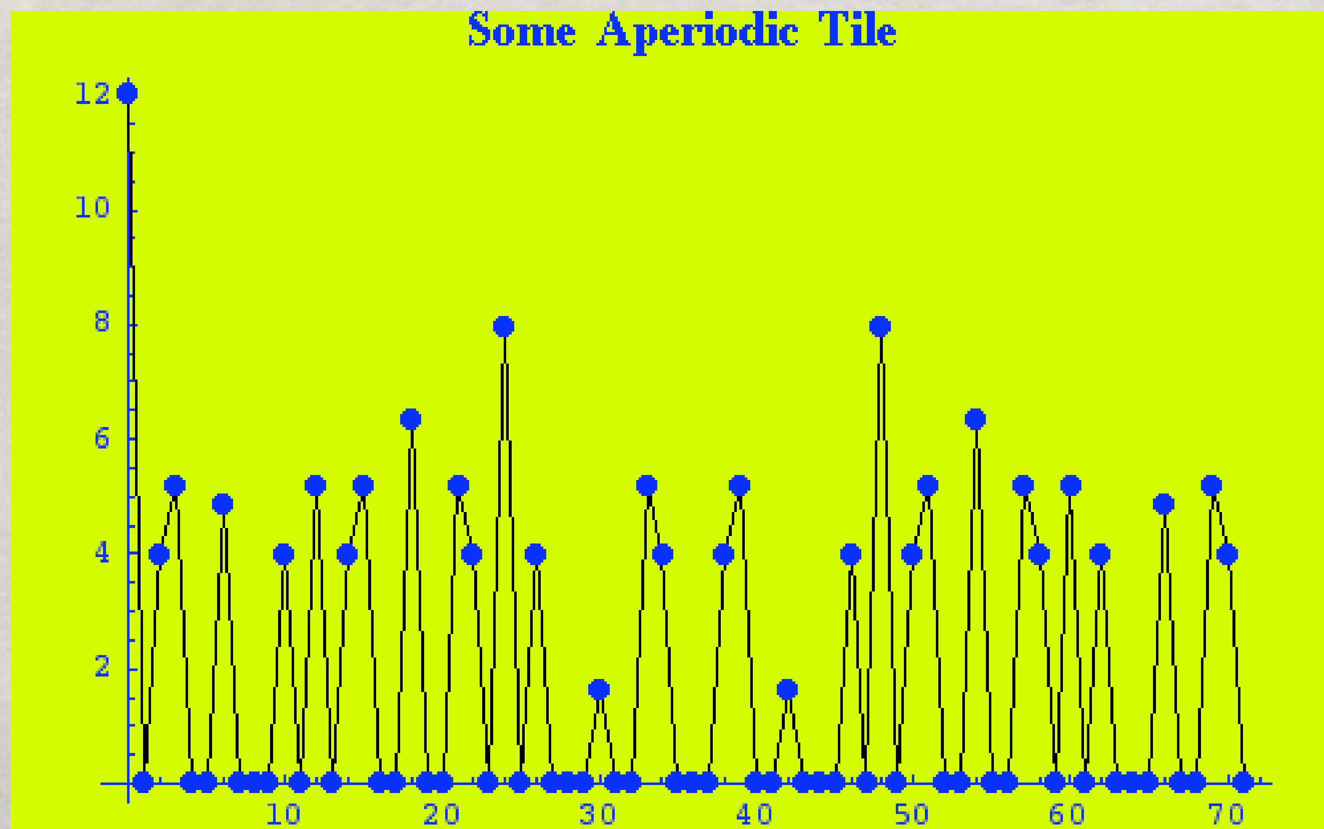

Roots:

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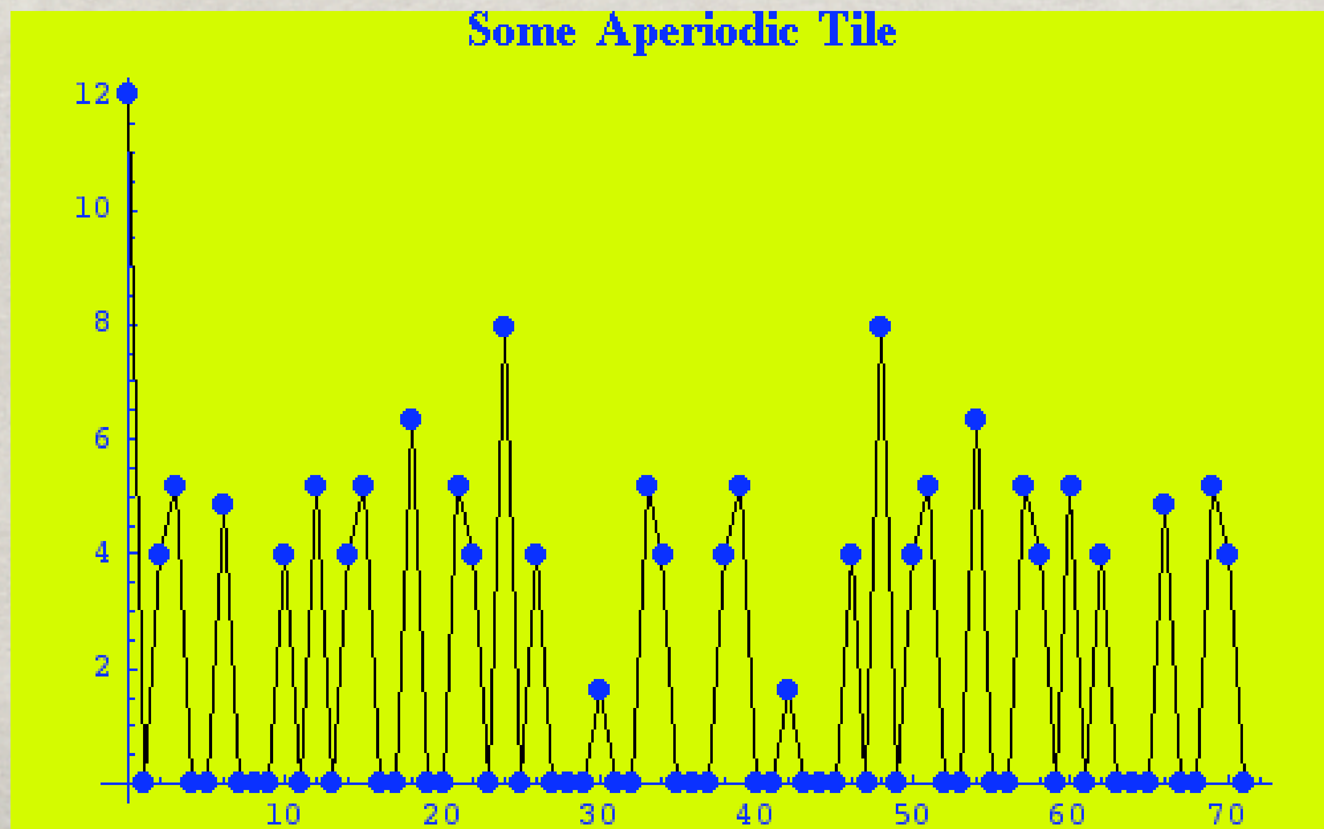

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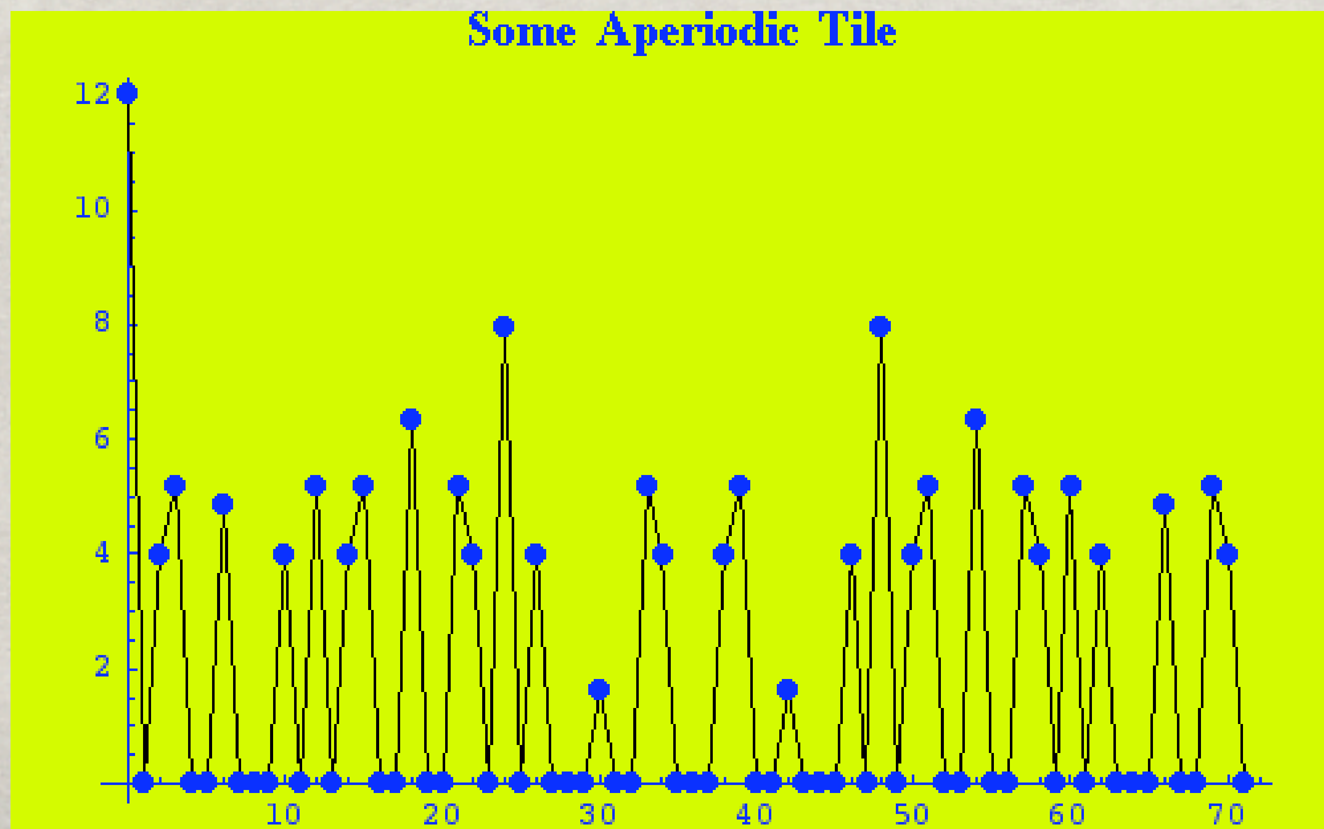

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- (T_2): $p^k, q^m \ldots \in S_A \Rightarrow p^k \times q^m \times \ldots \in R_A$

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- Example: A = (0 8 10 18 26 64), c=72,
  $R_A = \{3,4,6,12,24,36\}$, $S_A = \{3,4\}$.

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Is then (T_2) mandatory for Tiling???…

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- Say $B=B' \oplus (0\ c'\ 2c' \ldots)$: then $A \oplus B' = \mathbb{Z}_{c'}$ and canon $A \oplus B = \mathbb{Z}_c$ is a ***concatenation*** of $A \oplus B'$.

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Vuza canons

Musically most interesting for obvious reasons… Duality
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  Theorem: A is k-periodic (with $k \mid c$) $\Leftrightarrow$

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  all divisors of c not dividing k belong to R_A.

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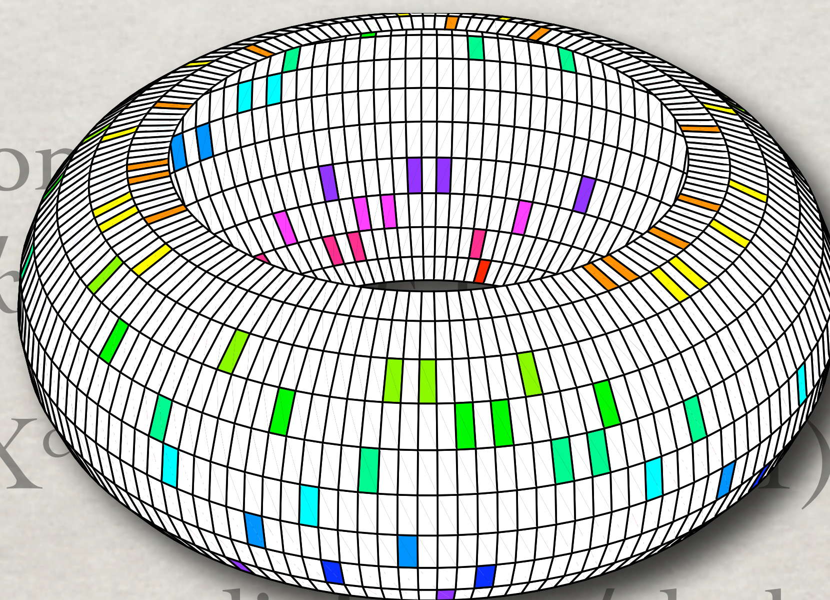

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- Let *c/p, p* prime be a maximal multiple of *k*.

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RC & Maths
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RC & Maths
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- ***Theorem***: A is aperiodic $\Leftrightarrow$ for all prime

  power factors p^a of c, there exists some multiple lying outside R_A (perforce in R_B).
  Reminiscent of condition (T_2).

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RC & Maths
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RC & Maths
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- *The condition is not sufficient.*
  Counter example:
  No canon with R_A= {2,3,6,8,18,24} and
  R_B= {4,9,12,36,72} for instance.

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RC & Maths
aside on a snide pb

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- Same pb with the Z-relation (given the interval vector, find the set): in the codomain of DFT the conditions are easy to check; pb is to find pre-images ***that are 01, i.e. sets.***

RC & Maths

aside on a snide pb

Développer: ens des facteurs tq prod = 01

Take R_A=(4 9 12 36 72), not 01: a complement would have to satisfy $R_B \supset \{2,3,6,8,24\}$. But complements of such B's do not include any A with this R_A (always 18 too and usually no 72)

RC & Maths
aside on a snide pb

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- Maybe (perhaps only when A tiles), these extraneous factors are congruent mod X^c-1 to a cyclotomic polynomial (or a product of such) ?

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RC & Maths
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RC & Maths
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RC & Maths
Vuza canons

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RC & Maths
Vuza canons

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RC & Maths
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- True for the Vuza-algo-generated Vuza canons (cf. algo F.Jz) and whenever $c=p^a q^b$.
- In all these cases, ***demultiplexing*** yields a smaller canon.

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RC & Maths
Vuza canons

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Vuza canons

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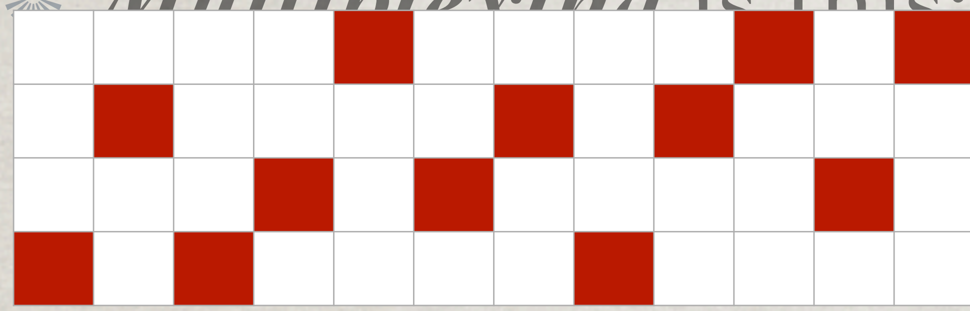
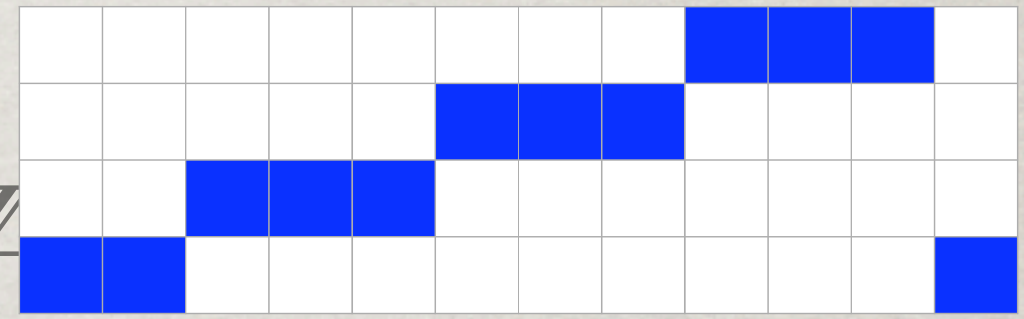


$\oplus$ B + $\mathbb{Z}$



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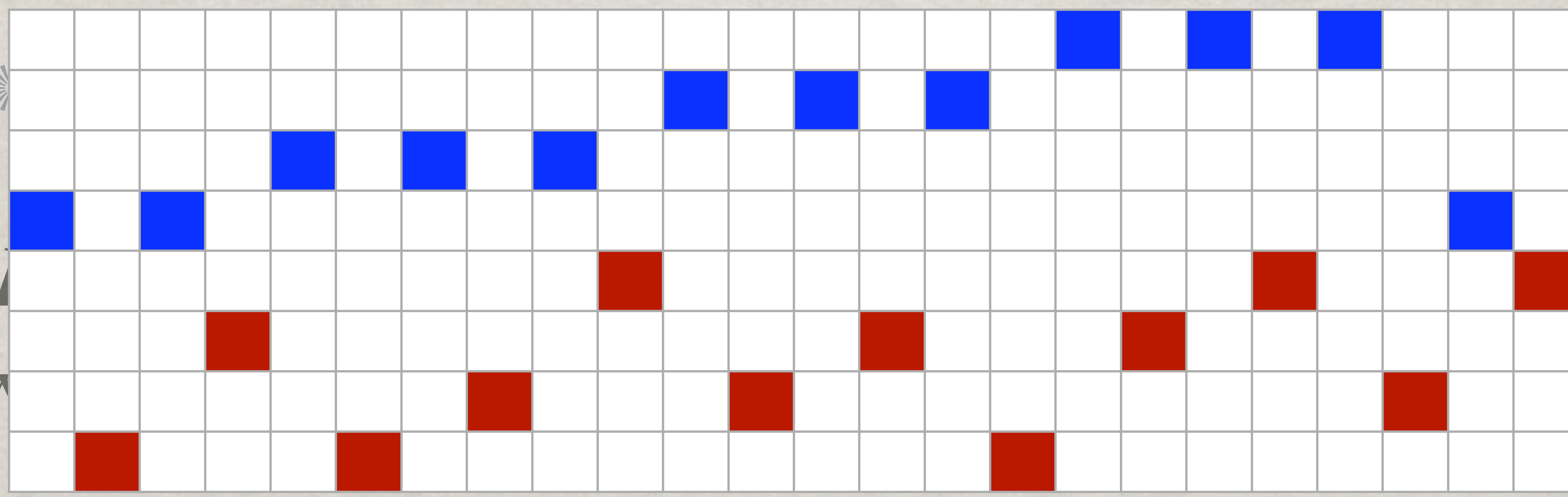

RC & Maths
multiplexing

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RC & Maths
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RC & MATHS
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- PB: how does one construct ALL Vuzas ?

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RC & Maths
Canons & Fuglede

Originally a question of a Hilbert base for functions on set A.

RC & Maths
Canons & Fuglede

The spectral conjecture in dim 1:

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RC & Maths
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The spectral conjecture in dim 1:

Definition: *A is spectral* $\Leftrightarrow$ *there is some spectrum*

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$$\text{card } \Lambda = \text{card } A \quad \text{and} \quad \Lambda - \Lambda \subset (Z_A \cup \{0\})$$

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Conjecture: Spectral ⇔ Tiles

- Funny how difference sets arise again (see about tiling: in the definition with $(A\text{-}A) \cap (B\text{-}B)$, <A-A> and also IC_A)

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RC & Maths
Vuza vs. Fuglede

Heurk-some.
(Edouard a prouvé que ces techniques de réduction peuvent servir aussi pour l'autre sens de la conjecture.)

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- All musical transformations of a canon (concatenation, multiplexing) preserve condition (T_2). (Amiot 2003, Gilbert 2007)
- Hence for any canon ***that can be reduced to*** 0+0 (trivial canon), (T_1) and (T_2) hold. Hence both A, B are spectral (Laba, 2000).

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RC & Maths
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RC & Maths
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  - More precisely: a non demultiplexable one, which proves ***again*** (T_2) when $c=p^a q^b$
- Hence (T_2) is true for (some) Vuza canons. ⇔

(tiling ⇒ (T_2)) is true unconditionally

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RC & Maths
Vuza vs. Fuglede

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RC & Maths
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- There is a link between spectrality and periodicity (Z_A).
- So… why haven't we cracked it yet ?

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