Tiling by translation: Fourier analysis, number theory and algorithms

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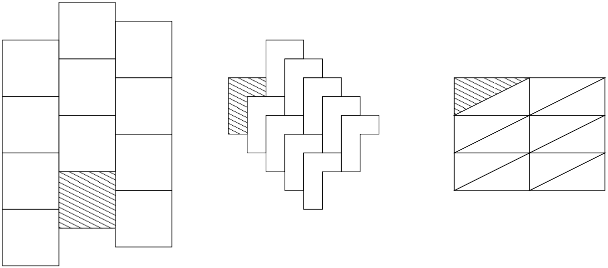
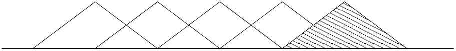
Tiling by translation only, one tile. Examples.



(a) (b) (c)

(a): has one period; (b): has a period lattice; (c): not a tiling by translation



A "soft" tiling by a function: $\sum_{\lambda\in\Lambda} f(x-\lambda) = \ell = const.$, a.e. x.
We write $f + \Lambda = \ell\mathbb{R}^d$ in this case.

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Packing and tiling in a group.

- **Setting**: Abelian group G, subset $E \subseteq G$, set of translations $T \subseteq G$.
- *Packing*: $E + t$ all disjoint, $t \in T$
- *Tiling*: Packing where all of G is covered. Denoted by $E + T = G$.
- *Multiple tiling*: Every element of G is covered ℓ times: $E + T = \ell G$.
- Packing $\Leftrightarrow$ $(E - E) \cap (T - T) = \{0\}$ (disjoint differences).
- **Remark**: E tiles a group $G \Leftrightarrow E$ tiles the subgroup it generates.
- Tiling is *intrinsic* property of set E; no dependence on G.
- When checking tiling may assume that E generates G

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Applications

- **Wavelets**: E.g., **Gröchenig and Madych, 1992** construct multiresoution analyses of $\mathbb{R}^n$ using *self-similar* tiles.
- **Orthogonal decompositions**: E.g., **Han and Wang, 2001** use $\Omega \subseteq \mathbb{R}^2$ which tiles simultaneously with two lattices L, M (with same volume) to construct a **Weyl-Heisenberg basis** with translation lattice L and modulation lattice M^*.

  These bases are of the type

$$e_{k,l}(x) = \overbrace{f(x-k)}^{\text{translation}} \cdot \overbrace{e^{2\pi l \cdot x}}^{\text{modulation}}, \quad k \in L, l \in M'.$$

- **Crystallography:** Tilings of space are used to model the shape of crystals. The discovery in nature of *quasicrystals* (non-periodic crystals) has raised interest in *aperiodic tilings*.
- **Math. music theory**: rhythm = period N, rhythmic canon = subset of $\mathbb{Z}_N$, rhythmic tiling canon = a tiling of $\mathbb{Z}_N$ by two sets, the inner rhythm and the outer rhythm.

Fourier Analysis on abelian groups

- Character is a (continuous) group homomorphism from G to the *multiplicative* group $U = \{z \in \mathbb{C} : |z| = 1\}$.
- If χ, ψ are characters then so is $\chi\psi$ (pointwise product). Write $\chi + \psi$ from now on instead of $\chi\psi$.
- Group of characters (written *additively*) $\widehat{G}$ is the dual group of G
- $G = \mathbb{Z} \Longrightarrow \widehat{G} = \mathbb{T}$: the functions $\chi_x(n) = \exp(2\pi i x n), x \in \mathbb{T}$
- $G = \mathbb{T} \Longrightarrow \widehat{G} = \mathbb{Z}$: the functions $\chi_n(x) = \exp(2\pi i n x), n \in \mathbb{Z}$
- $G = \mathbb{R} \Longrightarrow \widehat{G} = \mathbb{R}$: the functions $\chi_t(x) = \exp(2\pi i t x), t \in \mathbb{R}$
- $G = \mathbb{Z}_m \Longrightarrow \widehat{G} = \mathbb{Z}_m$: the functions $\chi_k(n) = \exp(2\pi i k n/m), k \in \mathbb{Z}_m$
- $G = A \times B \Longrightarrow \widehat{G} = \widehat{A} \times \widehat{B}$

Example

$G = \mathbb{T} \times \mathbb{R} \Longrightarrow \widehat{G} = \mathbb{Z} \times \mathbb{R}$. The characters are

$$\chi_{n,t}(x, y) = \exp(2\pi i(nx + ty)).$$

- G is compact $\Longleftrightarrow \widehat{G}$ is discrete. Pontryagin duality: $\widehat{\widehat{G}} = G$.

Tiling in Fourier space

- Suppose G finite for simplicity.
- Fourier Transform of $f : G \to \mathbb{C}$ is $\widehat{f}(\gamma) = \sum_{x \in G} f(x)\overline{\gamma(x)}$, $\gamma \in \widehat{G}$.
- $E + T = G \Leftrightarrow \sum_{t \in G} \chi_T(t)\chi_E(x-t) = 1 \Leftrightarrow \chi_E * \chi_T = 1$
- $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$ so tiling is equivalent to

$$\widehat{\chi_E} \cdot \widehat{\chi_T} = \delta_0 = \text{unit mass at } 0.$$

- Almost equivalent to $\boxed{\operatorname{supp} \widehat{\chi_T} \subseteq \{\widehat{\chi_E} = 0\} \cup \{0\}}$.

Example

$E + T = \mathbb{Z}_N \Leftrightarrow |E| \cdot |T| = N$ and for $\nu \in \mathbb{Z}_N \setminus \{0\}$

$$\sum_{e \in E} \zeta_N^{e\nu} = 0, \ \text{whenever} \ \sum_{t \in T} \zeta_N^{t\nu} \neq 0,$$

where $\zeta_N = e^{2\pi i/N}$ is a *primitive N-th root of unity.*

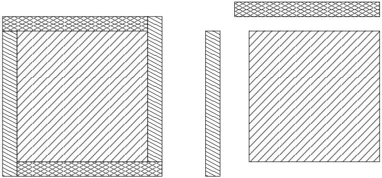
Example in Geometry: Filling a box with two types of bricks

- How to fill a rectangular box Q in $\mathbb{R}^d$ with rectangular bricks of two types A and B? No rotations allowed.

Theorem (Bower and Michael, 2004)

Only if you can cut the box in two rectangular parts and can fill one of them with A, the other with B. In any dimension $d \geq 2$.

- Fails for 3 bricks already in dimension $d = 2$



- Admits nice and simple Fourier analytic proof (**K. 2004**).

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Some successes of the Fourier method

- **Leptin and Müller, 1991, K. and Lagarias, 1995**: (structure) Each tiling of the line by a bounded set (resp. function) is periodic (resp. finite union of periodic translations).
- **K. 2000**: (structure) If a convex polygon, not a parallelogram, tiles the plane (tilings at level > 1 are acceptable) then the set of translations is a finite union of 2D lattices.
- **Beck 1989, K. 1996, K. and Wolff, 1999**: Results on the **Steinhaus** tiling problem: is there $E \subseteq \mathbb{R}^d$ which is simultaneously a tile for all rotations of the lattice $\mathbb{Z}^d$?
  For example, it has been proved that the answer is NO for $d \geq 3$.
- **K., Lagarias, Wang, Iosevich, Katz, Tao, others**: Results related to the spectral problem: when does a set $\Omega \subseteq \mathbb{R}^d$ have an orthogonal basis of exponentials?
  This is a tiling problem! More later in the talk.
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A decision problem

Input: finite set $A \subseteq \mathbb{Z}$.
Output: YES, if A tiles $\mathbb{Z}$ by translation, NO otherwise.

Not immediately obvious how to decide.

- **D. Newman, 1977**: (*Pigeonhole argument*)

If A is finite and $A + B = \mathbb{Z}$ then

$$B = B + t,$$

where the period $t \in \mathbb{Z}$ satisfies $0 \leq t \leq 2^D$, where $D = \operatorname{diam} A$.

$\Longrightarrow$ A tiles $\mathbb{Z}$ $\Leftrightarrow$ $\exists N \leq 2^D$ such that A tiles $\mathbb{Z}_N := \mathbb{Z} \bmod N$.

- **Algorithm**: For all $N \leq 2^{\operatorname{diam} A}$ try all possible complements of A in $\mathbb{Z}_N$.

Time: $O(2^{(\text{maximum period})}) = O(2^{2^D})$.

Speedup option: reduce max period

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The max period for 1-D tilings with tiles of given diameter

- Write $\mathcal{T}$ for the least period of a tiling, $\mathcal{T}(D)$ for the largest value of $\mathcal{T}$ if $\mathrm{diam}\, A \le D$.
- For long, state of the art was: $2D \le \mathcal{T}(D) \le 2^D$.
- **Ruzsa, 2002, K., 2002**: $\mathcal{T}(D) = O(\exp(C\sqrt{D\log D}))$.
- **K., 2002**: $CD^2 \le \mathcal{T}(D)$, a finite analog of *aperiodic tiling*.
- Best bounds:
  A. Biró, 2004: $\mathcal{T}(D) = O(\exp(D^{1/3+\epsilon}))$, $\forall \epsilon > 0$.
  J. Steinberger, 2005: $\mathcal{T}(D) \ge C_N D^N$, $\forall N$.
- But, it may be that every tile A admits a tiling of short period, for example $O(D)$.
- (**K. 2002**): If $A \subseteq \{0, \ldots, D\}$ tiles an <u>interval</u> longer than $2D$ then the tiling is periodic and a shorter interval is tiled.
- Deciding the tiling of an interval is easy: the first gap to the right can only be filled by the first element of the tile.

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The bound $\mathcal{T}(D) = O(\exp(C\sqrt{D}\mathrm{polylog}(D)))$

- Doing Fourier Analysis on a finite cyclic group.
- Suppose $A + B = \mathbb{Z}_M$. Write $A(x) = \sum_{a\in A} x^a$, $B(x) = \sum_{b\in B} x^b$. Then
$$A(x)B(x) = 1 + x + \cdots + x^{M-1} \bmod (x^M - 1).$$
$\Rightarrow \ x^M - 1 | A(x)B(x) - \frac{x^M-1}{x-1}$.
- All M-th roots of unity (except 1) are roots of $A(x)$ or $B(x)$.
- Assume $A \subseteq \{0, \ldots, D\}$ and M is "large".
Goal: Prove $B + t = B$ for some non-zero $t \in \mathbb{Z}_M$.
- $B = B + t$ is equivalent to $B(x) = x^t B(x) \bmod (x^M - 1)$ or
$$x^M - 1 | (x^t - 1)B(x).$$
- So, all M-th roots of unity which are not t-th roots must be roots of $B(x)$.
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The bound $\mathcal{T}(D) = O(\exp(C\sqrt{D}\text{polylog}(D)))$, contd

- $x^M - 1$ factors as $\prod_{d|M} \Phi_d(x)$.
- The irreducible *cyclotomic* polynomials $\Phi_d(x)$ partition the M-th roots of unity into algebraically conjugate classes.
- Roots of unity belonging to the same Φ_s participate as a block as roots of integer polynomials.
- $\deg \Phi_n(x) = \phi(n) = |\{k \le n : \ (k,n) = 1\}|$ (the Euler function) $\phi(n) \ge Cn \big/ \log\log n$
- Let $\Phi_{s_1}, \ldots, \Phi_{s_k}$ be the divisors of $A(x)$, $t = s_1 \cdots s_k$. It follows that $t \le e^{C\sqrt{D}\log^\eta D}$.
- If $\Phi_d(x)$ does not divide $x^t - 1$ then it does not divide $A(x)$, so it must divide $B(x)$, by the tiling condition. So if the tiling is non-periodic $M \le e^{C\sqrt{D}\log^\eta D}$.

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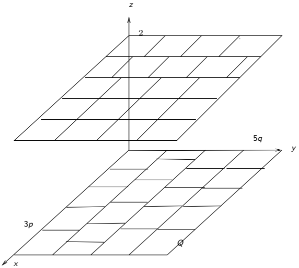
The bound $\mathcal{T}(D) \geq CD^2$

- Consider $G = \mathbb{Z}_{3p} \times \mathbb{Z}_{5q} \times \mathbb{Z}_2$, where $p, q \sim D$ are different primes.
- Consider a tiling of G as shown. This has no periods as the only period present in one level is destroyed in the other.



- The tile is the rectangle $A = \{(i, j, 0) : \ 0 \leq i < 3, \ 0 \leq j < 5\}$.
- Now flatten G and view it as $\mathbb{Z}_{2 \cdot 3p \cdot 5q}$ under the isomorphism

$$\psi(i, j, k) = i(2 \cdot 5q) + j(2 \cdot 3p) + k(3p5q) \bmod M.$$

- The tiling remains non-periodic and the tile has diameter $\sim D$, while $30pq \sim D^2$.

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Open Problem 1: Long periods for tiles of gcd=1

The tiling just shown of diameter D and period $\geq CD^2$ does not have the gcd of its elements equal to 1.

The whole idea is to find a tile with two different periods, then dilate the tile and interleave these two tilings of different periods, getting their product as the period of the whole.

Same is true with Steinberger's result.

Question: Is there a tile $0 \in A \subseteq \{0, \ldots, D\}$, with $\gcd(A) = 1$, which has a tiling of period much larger than D?

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Open Problem 2: Tiles with only long periods

Question: Are there tiles $A \subseteq \{0, \ldots, D\}$ all of whose tilings have periods much larger than D?

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When the tile has size $|A| = p^a q^b$

- Let $S_A = \{s = p^a : \ \Phi_s(x) | A(x)\}$.
- Two conditions on $A \subseteq \mathbb{Z}$:

  (T1) $|A| = A(1) = \prod_{s \in S_A} \Phi_s(1)$,

  (T2) if $s_1, \ldots, s_m \in S_A$ are powers of different primes then $\Phi_{s_1 \cdots s_m}(x) | A(x)$.

Theorem (Coven and Meyerowitz, 1998)

(a) (T1) and (T2) $\Rightarrow$ A is a tile.
(b) Tiling $\Rightarrow$ (T1)
(c) If $|A| = p^a q^b$ then tiling $\Rightarrow$ (T2) as well

The following would be very important:

Conjecture (Coven and Meyerowitz, 1998)

For any finite set A,

$$A \text{ tiles} \Leftrightarrow \ (T1)\ \&\ (T2) \text{ hold.}$$

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Deciding when (T1) & (T2) hold in polynomial time

- Given is $A \subseteq \{0, \ldots, D\}$.
- Compute all cyclotomic polynomials of degree up to D.
  Determine the cyclotomic divisors of $A(x)$.
- Test (T1).
- Determine the polynomials $\Phi_{p_i^{a_i}}(x)$ which are divisors of $A(x)$.
- Let $N_i \geq 1$, $i = 1, \ldots, k$, be the number of relevant powers of p_i.
- If (T2) is to hold then there are at least

  $$(N_1 + 1) \cdots (N_k + 1) - 1$$

  different cyclotomic divisors of $A(x)$.
- This gives the bound $\boxed{k \lesssim \log D}$, comparing degrees.
- If this inequality fails, so does (T2).
  If not, exhaustively verify that (T2) holds.

The Coven-Meyerowitz Conjecture $\Longrightarrow$

We can decide if A is a tile in time polynomial in $\operatorname{diam} A$.

Periodicity and decidability in 2D

- Analog of periodicity in 2D is to have a lattice of periods of full rank. Essentially the tiling is defined in a bounded region of space.
- There are tilings with no period lattice: e.g. columns of squares shifted arbitrarily up or down.
- Tweaking this example gives tiling with no periods at all.

The periodic tiling conjecture of Lagarias and Wang (1996)

Every tile admits a tiling with a period lattice of full rank.

False for non-translational tilings.

- **Wijshoff and van Leeuwen, 1984**:
  True for simply "connected" subsets of $\mathbb{Z}^2$.
- **Girault-Beauquier and Nivat, 1989**:
  True for topological disks in $\mathbb{R}^2$.
- **M. Szegedy, 1998**:
  Conjecture OK if $A \subseteq \mathbb{Z}^2$ has $|A|$ equal to a prime or 4. Decidable.

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Open Problem 3: Decidability in $d \geq 2$

Suppose $A \subseteq \mathbb{Z}^d$ is a finite set.

- Easy diagonal argument shows that if A can tile a region larger than any disk then it tiles the whole infinite lattice $\mathbb{Z}^d$.
- If A of diameter D does not tile then there is a disk of radius $\leq R(D)$ which cannot be covered by A in a non-overlapping way.
- **Question**: Find <u>any</u> computable bound for the function $R(D)$.
  Nothing is known.
  This would imply a decision algorithm for tiling.
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Complexity of tiling a cyclic group

- Given $0 \in A \subseteq \mathbb{Z}_N$ decide if A tiles $\mathbb{Z}_N$.
- Can assume
  - A generates $\mathbb{Z}_N$. That is $gcd(A, N) = 1$, and,
  - $|A|$ divides N, obviously.
- A is a tile $\Leftrightarrow \exists \Lambda \subseteq \mathbb{Z}_N$ s.t.

$$\overbrace{(A - A) \cap (\Lambda - \Lambda) = \{0\}}^{\text{packing condition}} \ \& \ \overbrace{|\Lambda| \cdot |A| = N}^{\text{maximality}}.$$

- Not known to be computable in time polynomial in N.

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Open Problem 4: Certificates for non-tiling?

Question: For a non-tile $A \subseteq \mathbb{Z}_N$, is there a certificate for non-tiling of size polynomial in N?

- This would follow from any characterization of tiling that is computable in polynomial time.
- Would be a very strong indication that the problem is solvable in polynomial time.

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A related problem: spectrality

$E \subseteq G$ is *spectral* if $L^2(E)$ has ortho. basis of characters (the *spectrum*).

Example

$E = (0,1)^d \subseteq \mathbb{R}^d$ is spectral. A basis of $L^2((0,1)^d)$ is

$$e_n(x) := e^{2\pi i n\cdot x}, n \in \mathbb{Z}^d, \quad \text{(Fourier series)}.$$

$\Lambda \subseteq \widehat{G}$ is a spectrum of $E \subseteq G \Leftrightarrow$

$$\sum_{\lambda\in\Lambda} \left|\widehat{\chi_E}(\xi-\lambda)\right|^2 = |E|^2,\ \xi \in \widehat{G}.$$

In other words: $\boxed{\left|\widehat{\chi_E}\right|^2 + \Lambda \text{ is a tiling at level } |E|^2}$.

The Spectral set conjecture (Fuglede 1970s)

E is spectral $\Leftrightarrow$ E is a tile.

Now dead, at least in groups like $\mathbb{R}^d$, $d \geq 3$.

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Tiling, Spectrality and large difference sets

Characters $\lambda \neq \mu \in \widehat{G}$ are orthogonal over E $\Leftrightarrow$

$$\widehat{\chi_E}(\lambda - \mu) = 0.$$

E is spectral $\Leftrightarrow$ there exists $\Lambda \subseteq \widehat{G}$ s.t.

$$|\Lambda| = |E| \;\; \& \;\; \Lambda - \Lambda \subseteq \{\widehat{\chi_E} = 0\} \cup \{0\}.$$

To decide if E tiles: find large T such that

$$T - T \subseteq \{0\} \cup (E - E)^c.$$

To decide if E is spectral: find large Λ such that

$$\Lambda - \Lambda \subseteq \{0\} \cup \{\widehat{\chi_E} = 0\}.$$

Reduce to problem:

Optimization problem DIFF

Input: Sets $A, E \subseteq \mathbb{Z}_N$

Output: Maximum size $B \subseteq E$ with $B - B \subseteq A \cup \{0\}$.

Problem DIFF is NP-complete

- Reduce decision version of max. clique in graphs to DIFF.
- Given graph G on vertices $V = \{1, \ldots, n\}$.
- For $m = O(n^3)$ we define: $\phi : V \to \mathbb{Z}_m$ & $A \subseteq \mathbb{Z}_m$ s.t.

$$i, j \text{ connected in } G \Leftrightarrow \phi(i) - \phi(j) \in A.$$

- *Greedily* set $\phi(1) = 0$ and, having defined, $\phi(1), \ldots, \phi(r)$ let $\phi(r+1)$ be min ν s.t.

$$\nu \notin \{\phi(i) + \phi(j) - \phi(l) : \ 1 \le i, j, l \le r\}.$$

- Enough to have $m \ge Cn^3$ to define ϕ up to n.
- Ensures all differences $\phi(i) - \phi(j)$ are distinct.
  $\phi(V)$ is called a *Sidon set*.
- Define $A = \{\phi(i) - \phi(j) : \ (i, j) \in G\}, \ \ E = \phi(V)$.
- A clique in G corresponds to a set $B \subseteq E$ s.t. $B - B \subseteq A$, and vice-versa.

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Negative resolution of the Spectral Set conjecture

- **Tao, 2003**: "Spectral $\Rightarrow$ tile" is false in dimension $d \ge 5$.
- **Matolcsi, 2004**: $d = 4$ as well.
- **K. and Matolcsi, 2004**: $d = 3$ as well. Direction still open in $d = 1, 2$.
- Work in finite groups. Counterexample in $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_d}$ lifts first to $\mathbb{Z}^d$ then to $\mathbb{R}^d$.
- **K. and Matolcsi, 2004**: "Tile $\Rightarrow$ spectral" is false in $d \ge 5$.
- **Farkas and Revesz, 2004**: same for $d = 4$.
  Farkas, Matolcsi, Mora, 2005: same for $d = 3$.
- Conjecture remains open in dimensions $d = 1, 2$, in both directions. Could be true in natural classes of domains, e.g. convex domains (true in $d = 2$ by **Iosevich, Katz and Tao, 2003**.
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