# Supplementary Sets and Regular Complementary Unending Canons (Part Two)

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## 4. Generalities about Unending Rhythmic Canons

In an unending rhythmic canon in strict style, two or more voices starting at different points in time deliver the same rhythmic pattern. Because of the strict identity between the rhythmic patterns produced by the different voices, it follows that for any couple $V_1$, $V_2$ of voices in the canon, the beats marked by $V_1$ are separated by a constant temporal interval $t_{12}$ from the beats marked by $V_2$. Moreover, the rhythmic pattern is periodically repeated by each voice. And finally, because in this study we consider only rhythmic aspects and neglect others such as pitch, we may and shall assume that there is no complete overlapping of beats between any two different voices in the canon.

Within the framework of the rhythmic model of Section 3, the above-described musical situation is abstracted in the following definition.

DEFINITION 4.1. *An unending rhythmic canon is a finite non-empty subset of a rhythmic class.*

In the spirit of the convention of Section 3, we shall omit the words "unending" and "rhythmic" and speak simply about "canons." Canons will be designated by the letter $\mathcal{C}$ affected if necessary by subscripts and/or superscripts. Thus, if $\mathcal{C} = \{R_1, \ldots, R_l\}$ is a canon, $R_i$ represents the set of time points associated with the *i*-th voice in the canon. (Hence, in order to obtain the "abstract" model of a "real" canon, one has to take the periodic infinite continuations of the sets of time points associated with each voice.) It is therefore natural to call the number $l = \#\mathcal{C}$ the *number of voices* in the canon $\mathcal{C}$. For formal reasons (see Sections 5 and 6) and in accord with Definition 4.1, we shall be obliged to consider also as canons those sets $\mathcal{C}$ consisting of a single rhythm.

DEFINITION 4.2. *Two canons* $\{R_1, \ldots, R_l\}$ *and* $\{R_1', \ldots, R_m'\}$ *are called equivalent if* $\{R_1', \ldots, R_m'\} = \{t + R_1, \ldots, t + R_l\}$ *for some* $t \in \mathbf{Q}$.

An equivalence class with respect to the above-introduced relation is called a *canon class*.

Hence equivalent canons may be viewed as obtained each from the other via a temporal translation; or, from another viewpoint, equivalent canons represent different mathematical objects describing the same musical situation, the differences arising from various choices of the referential time-point zero. In particular, the numbers of voices in equivalent canons are the same.

There are several rhythmic classes connected to a canon $\mathcal{C} = \{R_1, \ldots, R_l\}$ Firstly, there is the *ground class Grd* $\mathcal{C}$, which is by definition the rhythmic class implied by Definition 4.1; that is,

$$Grd\ \mathcal{C} = [R_1] = \ldots = [R_l]\ .$$

An equality *Grd* $\mathcal{C} = R$ will also be referred to as "$\mathcal{C}$ is built on *R*." Secondly there is the *resultant class Res* $\mathcal{C}$, defined as

$$Res\ \mathcal{C} = \left[\ \bigcup_{i=1}^{l} R_i \right] ;$$

it is the class of the rhythm perceived by hearing the canon as a whole, that is, by superposing the beats from all voices involved in it.

To assign other rhythmic classes to the canon $\mathcal{C}$, we have to consider the problem of meters in canons. The rhythmic model presented in Section 3 takes into account only the intervallic structure of rhythms and neglects metric aspects. In this study we shall be concerned only with those meters which emphasize the periodic character of a periodic rhythm; that is, given a periodic rhythm R, we shall consider only metric accents occuring on some beats in R so that the interval between any two successive accents is constant and equal to an *integer multiple* of Per R. Here is the formal definition:

DEFINITION 4.3. *Let k be any integer* $\geq 1$. *A meter of order k on a rhythm* R *is a regular rhythm* S *satisfying the relations* $S \subset R$ *and* Per S = *k*Per R.

The rhythm S should be viewed as marking the "strong" beats in R.

Now suppose that a meter $S_i$ in the sense of Definition 4.3 is imposed on each rhythm $R_i$ in a canon $\mathcal{C}$. In order that strict identity between the rhythmic patterns delivered by the voices in the canon be observed, we have to impose the condition that for any couple $i, j$, the meter $S_i$ bears to $R_i$ the same relation as $S_j$ does to $R_j$. This requirement is formally expressed by the following definition.

DEFINITION 4.4. *Let* $\mathcal{C} = \{R_1, \ldots, R_l\}$ *be a canon on l voices (so that* $R_i \neq R_j$ *for* $i \neq j$*). A meter of order* $k \geq 1$ *on* $\mathcal{C}$ *is a set* $\{S_1, \ldots, S_l\}$ *of regular rhythms satisfying the conditions below:*

(M1) *For any* $i \in \{1, \ldots, l\}$, $S_i$ *is a meter of order k on* $R_i$;

(M2) *For any couple* $i, j \in \{1, \ldots, l\}$, *there is* $t_{ij} \in \mathbf{Q}$ *such that* $R_i = t_{ij} + R_j$ *and* $S_i = t_{ij} + S_j$.

Remark that the existence of $t_{ij}$ such that $R_i = t_{ij} + R_j$ is implied by the very definition of a canon, while the existence of $t_{ij}'$ such that $S_i = t_{ij}' + S_j$ is a consequence of the equality of the periods of $S_i$ and $S_j$. The identity between the relations "$S_i$ to $R_i$" and "$S_j$ to $R_j$" is expressed in Definition 4.4 precisely by the identity $t_{ij} = t_{ij}'$.

In addition to hearing the voices in a canon as embedded into a resultant structure, we are also interested in hearing the metric structure as a whole; in other words, if $\{S_1, \ldots, S_l\}$ is a meter on $\mathcal{C}$, we are interested in the resultant of the meter, that is, the superposition

$$\bigcup_{i=1}^{l} S_i$$

of the metric accents from all voices. We say that $\mathcal{C}$ admits a rhythmic class $S$ as a *metric class of order k* if there is a meter $\{S_1, \ldots, S_l\}$ of order $k$ on $\mathcal{C}$ such that

$$S = \left[ \bigcup_{i=1}^{l} S_i \right].$$

The mathematical description of all metric classes admitted by $\mathcal{C}$ is given by the next proposition.

PROPOSITION 4.1. *The canon $\mathcal{C}$ on $l$ voices admits $S$ as a metric class of order $k$ iff there are $t_1, \ldots, t_l \in \mathbf{Q}$ and a rhythm R satisfying the conditions below:*

(i) $S = \left[ \bigcup_{i=1}^{l} (t_i + (k\text{Per R})\mathbf{Z}) \right]$;

(ii) $\{t_1 + \text{R}, \ldots, t_l + \text{R}\}$ *is a canon equivalent to* $\mathcal{C}$.

The proof follows immediately from the definitions and we leave it to the reader.

It follows in particular from Proposition 4.1 that there is a unique rhythmic class admitted by $\mathcal{C}$ as a metric class of order 1; we shall call it the ***primary metric class*** of $\mathcal{C}$ and denote it by $Met\ \mathcal{C}$. On the contrary, if $k \geq 2$ then $\mathcal{C}$ admits in general several metric classes of order $k$; any such class will be called a ***secondary metric class*** of $\mathcal{C}$.

The following proposition establishes the fundamental relation between ground classes, metric classes, and resultant classes. This relation involves composition of rhythmic classes. Before stating the proposition, we describe a general procedure for constructing canons in the sense of Definition 4.1. Let $R, S \in Rhyt$ and let $\text{R} \in R$, $\text{S} \in S$. The collection of all sets of the form $s + \text{R}$, as $s$ ranges over S, forms a canon $\mathcal{C}$; the fact that this collection is finite follows from the equality

$$\{s + \text{R} \mid s \in \text{S}\} = \{s + \text{R} \mid s \in \text{S} \cap [0, \text{Per R} \vee \text{Per S})\} \qquad (1)$$

(the verification of which is also elementary; we assume that neither R nor S is empty). The canon class of $\mathcal{C}$ does not depend on the particular choices of R in $R$ and S in $S$; we denote this canon class by $Can(R,S)$.

PROPOSITION 4.2. *Let the canon $\mathcal{C}$ admit S as a metric class (of any order). Then the following are true:*

(i) *The canon class of $\mathcal{C}$ equals Can(Grd $\mathcal{C}$, S);*
(ii) *Res $\mathcal{C}$ = Grd $\mathcal{C}$ + S.*

*In particular,*

$$Res\ \mathcal{C} = Grd\ \mathcal{C} + Met\ \mathcal{C}$$

*for any canon $\mathcal{C}$.*

*Proof.* Let $t_1, \ldots, t_l$ and R be given by Proposition 4.1 applied to $\mathcal{C}$ and S. If we let

$$S = \bigcup_{i=1}^{l} (t_i + (k\mathrm{Per}\ R)\mathbf{Z}),$$

then it is easy to see that $\{s + R \mid s \in S\}$ is a canon equivalent to $\mathcal{C}$, so that the first assertion is proved. The second assertion follows from

$$Res\ \mathcal{C} = [\bigcup_{s \in S} (s + R)] = [S + R] = Grd\ \mathcal{C} + S.$$

There are some relations connecting the order, the period, and the number of beats per period of a metric class to the other numerical entities associated to a canon, as is shown by the next proposition.

PROPOSITION 4.3. *Let the canon $\mathcal{C}$ on $l$ voices admit S as a metric class of order $k$. Then*

$$k = (\mathrm{Per}\ Grd\ \mathcal{C} \vee \mathrm{Per}\ S)/\mathrm{Per}\ Grd\ \mathcal{C},$$

$$l = (\mathrm{Nrp}\ S)(\mathrm{Per}\ Grd\ \mathcal{C} \vee \mathrm{Per}\ S)/\mathrm{Per}\ S.$$

*In particular,*

$$\mathrm{Per}\ Met\ \mathcal{C} \mid \mathrm{Per}\ Grd\ \mathcal{C}$$

*and*

$$l = (\mathrm{Nrp}\ Met\ \mathcal{C})(\mathrm{Per}\ Grd\ \mathcal{C})/\mathrm{Per}\ Met\ \mathcal{C}.$$

*Proof.* Let $s$ = Per *Grd* $\mathcal{C}$ **v** Per $S$. Let also $t_1, \ldots, t_l$ and R be given by Proposition 4.1 applied to $\mathcal{C}$ and $S$. Set

$$\mathrm{S} = \bigcup_{i=1}^{l} (t_i + (k\mathrm{Per\ R})\mathbf{Z}).$$

As [S] = $S$ and Per $S \mid s$ by the definition of $s$, we must have $s$ + S = S, that is

$$\bigcup_{i=1}^{l} (s + t_i + (k\mathrm{Per\ R})\mathbf{Z}) = \bigcup_{i=1}^{l} (t_i + (k\mathrm{Per\ R})\mathbf{Z}).$$

Both sides of this equality are unions of cosets of $\mathbf{Q}$ modulo ($k$Per R)$\mathbf{Z}$; hence the coset $s + t_1 + (k\mathrm{Per\ R})\mathbf{Z}$ in the left side must equal one coset in the right side, say $t_j + (k\mathrm{Per\ R})\mathbf{Z}$. Hence $k$Per R $\mid s + t_1 - t_j$; as Per R $\mid s$ we infer that Per R $\mid t_1 - t_j$. This means that $t_1$ + R = $t_j$ + R, which necessarily implies $j = 1$, as otherwise the set $\{t_1 + \mathrm{R}, \ldots, t_l + \mathrm{R}\}$ would have less than $l$ elements and could not be equivalent to $\mathcal{C}$. In conclusion, $s + t_1 - t_j = s$ and $k$Per R $\mid s$. On the other hand, it follows from the definition of S that Per $S$ = Per S $\mid$ $k$Per R; hence $s \mid k$Per R by the definition of $s$ and finally $s = k$Per R, a relation equivalent to the first equality in the statement of the proposition.

To prove the second equality, we count #S ∩ $[0,s)$ in two different ways. Firstly, $[0,s)$ is divided into $s$/Per S intervals of length Per S, each of them containing Nrp S beats in S. Hence

$$\#\,\mathrm{S} \cap [0,s) = s\mathrm{Nrp\ S/Per\ S}\,.$$

Secondly, we shall see that $l$ = #S ∩ $[0,s)$ and this will conclude the proof. To this purpose, we remark that conditions (i) and (ii) in Proposition 4.1 are not altered when some integral multiples of $k$Per R$=s$ are added or substracted from the $t_i$'s. Hence we may assume that $t_i \in [0,s)$ for $1 \leq i \leq l$. Consequently, $t_i \in \mathrm{S} \cap [0,s)$ for $1 \leq i \leq l$ by the definition of S. If $t \in \mathrm{S} \cap [0,s)$ then $s = k$Per R $\mid t - t_i$ for some $i \in \{1, \ldots, l\}$ ; but, by virtue of our assumption on the $t_i$'s, this is possible only if $t = t_i$. Hence $\mathrm{S} \cap [0,s) = \{t_i \mid 1 \leq i \leq l\}$ and our assertion is proved.

The set of all secondary metric classes is determined by the primary metric class and by the period of the ground class. More precisely, we have

PROPOSITION 4.4. *A canon $\mathcal{C}$ admits S as a metric class iff S satisfies the relations*

$$Met\ \mathcal{C} = S + [\text{Per } Grd\ \mathcal{C}]$$

*and*

$$S \perp [\text{Per } Grd\ \mathcal{C}].$$

*Proof.* Let $\mathcal{C}$ admit $S$ as a metric class of order $k$, let $t_1, \ldots, t_l$ and R be given by Proposition 4.1 applied to $\mathcal{C}$ and $S$ and let

$$\text{S} = \bigcup_{i=1}^{l} (t_i + (k\text{Per R})\mathbf{Z}).$$

The first relation in the statement of the proposition is a consequence of conditions (i) and (ii) verified by the $t_i$'s and R (Proposition 4.1). To prove the second relation, suppose that $s, s' \in \text{S}$ are such that $s - s' \in (\text{Per R})\mathbf{Z}$. According to Definition 3.4, we have to show that Per S $|\ s - s'$. By the definition of S we have $k$Per R $|\ s - t_i$ and $k$Per R $|\ s' - t_j$ for some $i$ and $j$. In particular, Per R $|\ (s - t_i) - (s' - t_j)$; as we have assumed that Per R $|\ s - s'$, we infer that Per R $|\ t_i - t_j$. But this is possible only if $i = j$ (see the proof of the preceding proposition). Hence $k$Per R $|\ (s - t_i) - (s' - t_j) = s - s'$, while Per S $|\ k$Per R by Proposition 4.3 ; in conclusion, Per S $|\ s - s'$ and the proof of the second relation is complete.

Conversely, assume that $S$ satisfies the two relations in the statement of the proposition. Choose R $\in Grd\ \mathcal{C}$ and S $\in S$ and set $k = (\text{Per R} \vee \text{Per S})/\text{Per R}$. As Per S $|\ k$Per R it follows that $k\text{Per R} + \text{S} = \text{S}$; hence (Proposition 3.1) S is a finite union of cosets of $\mathbf{Q}$ modulo $(k\text{Per R})\mathbf{Z}$, that is,

$$\text{S} = \bigcup_{j=1}^{m} (s_j + (k\text{Per R})\mathbf{Z})$$

with $k$Per R non $|\ s_j - s_{j'}$ for $j \neq j'$. The first relation imposed to $S$ yields

$$Met\ \mathcal{C} = [\text{S} + (\text{Per R})\mathbf{Z}] = \left[\bigcup_{j=1}^{m} (s_j + (k\text{Per R})\mathbf{Z} + (\text{Per R})\mathbf{Z})\right]$$

$$= \left[\bigcup_{j=1}^{m} (s_j + (\text{Per R})\mathbf{Z})\right] \quad . \qquad (2)$$

On the other hand, by Proposition 4.1 we have

$$Met\ \mathcal{C} = \left[ \bigcup_{i=1}^{l} (t_i + (\text{Per R})\mathbf{Z}) \right] \tag{3}$$

where the $t_i$'s are such that $\{t_1 + \text{R}, \ldots, t_l + \text{R}\}$ is a canon equivalent to $\mathcal{C}$. Comparing (2) and (3) we obtain

$$\bigcup_{j=1}^{m} (s_j + (\text{Per R})\mathbf{Z}) = \bigcup_{i=1}^{l} (t + t_i + (\text{Per R})\mathbf{Z}) \tag{4}$$

for some $t \in \mathbf{Q}$. Both sides of (4) are unions of distinct cosets of $\mathbf{Q}$ modulo (Per R)$\mathbf{Z}$. This is obvious for the right side; for the left side, if $s_j$ + (Per R)$\mathbf{Z} = s_{j'}$ + (Per R)$\mathbf{Z}$ then $s_j - s_{j'} \in$ (Per R)$\mathbf{Z}$ and hence Per S$|\ s_j - s_{j'}$ as $S \perp$ [Per R] by hypothesis. We obtain thus Per R $|\ s_j - s_{j'}$ and Per S $|\ s_j - s_{j'}$ which implies $k$Per R = Per R $\vee$ Per S $|\ s_j - s_{j'}$; the latter is not possible unless $j = j'$.

Consequently, we infer from (4) that $l = m$ and, after relabelling the $t_i$'s, that $s_i = t + t_i$ for $1 \leq i \leq l$. The $s_i$'s and R are thus satisfying the conditions (i) and (ii) in Proposition 4.1, showing that $\mathcal{C}$ admits $S$ as a metric class of order $k$.

It is now time to draw the reader's attention to the following aspect. The theory to be subsequently developed is mainly concerned with the intervallic structure of canons, ***regardless of any particular metric***; this was the reason for presenting canons in Definition 4.1 as ametric phenomena. We could in fact dispense with the secondary metric classes and with the metric interpretation of the primary metric class *Met* $\mathcal{C}$; we could introduce *Met* $\mathcal{C}$ just formally (namely, as being the class of any rhythm satisfying condition (i) for $k = 1$ and condition (ii) in Proposition 4.1) and motivate its introduction by saying that it is a mathematical object which controls the relative positions of the voices in the canon $\mathcal{C}$. However, insofar as primary metric classes, together with ground classes, play an essential role in the study (even ametric) of regular complementary unending canons, I have decided to give to the former a concrete musical interpretation, this being the reason for choosing the above metric approach. I thought that having the relative positions of the voices controlled by a periodic rhythm embedded in the "vertical" structure of the canon (hence by a concrete audible phenomenon) would be more suggestive than having them controlled by a set of relative distances (which is an abstract concept). As concerns the secondary metric classes, their importance will be apparent in the next section, where a certain procedure of transformation of canons will be discussed. Besides, such classes are useful in the following situations:

(*a*) When the "real" canon begins in such a manner that the distance between the entrances of certain voices exceeds the period of the ground class. In other words, the primary metric class together with the secondary metric classes admitted by some canon $\mathcal{C}$ describe all possible ways of "dictating" the entrances of voices in a concrete musical realization of the mathematical object $\mathcal{C}$. See Examples 4.3–4.4, below.

(*b*) When the study of an unending canon in strict style also takes into account other musical characteristics apart from rhythmic organization. In this situation, it might happen that the complete periodic pattern in the canon (including all characteristics in question, such as pitch, for instance) could spread over several repetitions of the periodic rhythmic pattern of the canon, so that secondary metric accents are used to delimit the complete period.

The foregoing discussion showed how to associate several rhythmic classes to a given canon; clearly all these classes do not change when the canon in question is replaced by an equivalent one. We shall consider now the converse problem: given two rhythmic classes $R$ and $S$, under what conditions there is a canon $\mathcal{C}$ built on $R$ and admitting $S$ as a metric class?

PROPOSITION 4.5. *Let the rhythmic classes R and S be given.*

(i) *In order that there is a canon $\mathcal{C}$ built on R and admitting S as a metric class, it is necessary and sufficient that* $S \perp [\text{Per } R]$. *The canon $\mathcal{C}$, if it exists, is determined up to equivalence; its canon class equals Can (R,S).*

(ii) *In order that there is a (necessarily unique up to equivalence) canon $\mathcal{C}$ built on R and admitting S as its primary metric class, it is necessary and sufficient that* Per $S$ | Per $R$.

*Proof.* We have to prove only (i), as (ii) is a consequence of (i) and of Proposition 4.3. The necessity in (i) follows from Proposition 4.4; the uniqueness assertion is a consequence of Proposition 4.2(i). It remains to prove the existence assertion, under the assumption $S \perp [\text{Per R}]$. Choose $\mathrm{R} \in R$ and $\mathrm{S} \in S$ so that $0 \in \mathrm{R}$ and let $s_1, \ldots, s_l$ be the distinct elements in $\mathrm{S} \cap [0, \text{Per R} \vee \text{Per S})$. *Can* $(R,S)$ is the canon class of the canon

$$\mathcal{C} = \{s_i + \mathrm{R} \mid 1 \leq i \leq l\}$$

(see the equality (1)). Set $k$ = (Per R $\vee$ Per S)/Per R. We claim that $\{s_i + (k\text{Per R})\mathbf{Z} \mid 1 \leq i \leq l\}$ is a meter of order $k$ on $\mathcal{C}$; this will be proved as soon as we show that $s_i + R \neq s_j + R$ for $i \neq j$. Indeed, if $s_i + R = s_j + R$, then $s_i - s_j \in (\text{Per R})\mathbf{Z}$. As $S \perp [\text{Per R}]$ by assumption, we must have Per S $\mid s_i - s_j$ and hence, Per R $\vee$ Per S $\mid s_i - s_j$. But this is possible only if $i = j$, as $s_i$ and $s_j$ both lie in [0, Per R $\vee$ Per S).

By a ***normal pair*** we mean an ordered pair $(R,S)$ of rhythmic classes such that Per $S \mid$ Per $R$. Proposition 4.5 shows in particular that *the map* $\mathcal{C} \mapsto$ *(Grd* $\mathcal{C}$*, Met* $\mathcal{C}$*) induces a bijective correspondence between canon classes and normal pairs.*

COROLLARY 4.1. *Let R and S be two rhythmic classes. Then for any canon* $\mathcal{C}$ *in Can(R,S) we have Grd* $\mathcal{C} = R$, *Res* $\mathcal{C} = R + S$ *and Met* $\mathcal{C} = S + [\text{Per } R]$.

*Proof.* The assertions concerning the ground class and the resultant class are clear from the definition of *Can*$(R,S)$. This definition also shows that *Can*$(R,S)$ = *Can*$(R, S + [\text{Per } R])$. As $(R, S + [\text{Per } R])$ is a normal pair, Proposition 4.5 yields *Met* $\mathcal{C} = S + [\text{Per } R]$.

We list now the integers which, as announced in the Introduction, serve as numerical measures of the complexity of a canon $\mathcal{C}$:

(*a*) the ***ground number*** of $\mathcal{C}$, defined as Nrp *Grd* $\mathcal{C}$;

(*b*) the ***category*** of $\mathcal{C}$, defined as Nrp *Met* $\mathcal{C}$;

(*c*) the ***modulus*** of $\mathcal{C}$, defined as Per *Grd* $\mathcal{C}$/Div *Res* $\mathcal{C}$.

All these are absolute invariants, that is, they do not change when one makes changes in the choices of the referential time-point zero and the referential time unit.

The fact that the modulus of $\mathcal{C}$ is an integer follows from the relation

$$\text{Div } \textit{Res } \mathcal{C} = \text{Div } \textit{Grd } \mathcal{C} \wedge \text{Div } \textit{Met } \mathcal{C}$$

(Propositions 4.2 and 3.2). The reason for calling it in this manner is clear from Section 3: a canon of modulus $n$ involves translation classes of $\mathbf{Z}_n$ (namely, $H_{a,b}$ (*Grd* $\mathcal{C}$) and $H_{a,b}$ (*Met* $\mathcal{C}$), with $a$ = Div *Res* $\mathcal{C}$ and $b$ = Per *Grd* $\mathcal{C}$).

Concerning the category, this terminology is inspired from Grigoriev and Müller 1961. In §48 of that book, a canon $\mathcal{C}$ on two voices is said to be of first category if Nrp *Met* $\mathcal{C} = 1$, of second category if Nrp Met $\mathcal{C} = 2$ (we have rephrased the original definitions from the book in terms of metric classes, the original definitions being phrased in terms of relative distances in time between voices).

By Proposition 4.3, the category divides the number $l$ of voices and hence can take values only in the range from 1 to $l$. This remark motivates the following definition.

DEFINITION 4.5. *A canon of maximal category is a canon whose category equals its number of voices.*

By Proposition 4.3, a canon $\mathcal{C}$ is of maximal category iff Per *Grd* $\mathcal{C}$ = Per *Met* $\mathcal{C}$.

We conclude this section with some examples which illustrate the concepts introduced above. The presentation of all examples of unending rhythmic canons in this study will comply with the following rules:

(*a*) A canon $\mathcal{C} = \{R_1, \ldots, R_l\}$ endowed with a meter $\{S_1, \ldots, S_l\}$ will be displayed in a diagram formed by several horizontal lines.

(*b*) On the $i$-th line of the diagram we shall represent a segment from the infinite set $R_i$ of length at least Per $R_i$; the elements (beats) in that segment are indicated by large dots (○). The first dot on the $i$-th line corresponds to the entering of the voice $V_i$, the successive enterings being dictated according to the metric class

$$\left[\bigcup_{i=1}^{l} S_i\right].$$

(*c*) Accents (ˇ) over some of the large dots on the $i$-th line indicate the "strong" beats, in accord with the meter $S_i$ on $R_i$.

(*d*) On the $(l+1)$-th, and the $(l+2)$-th line, we display a segment from the set

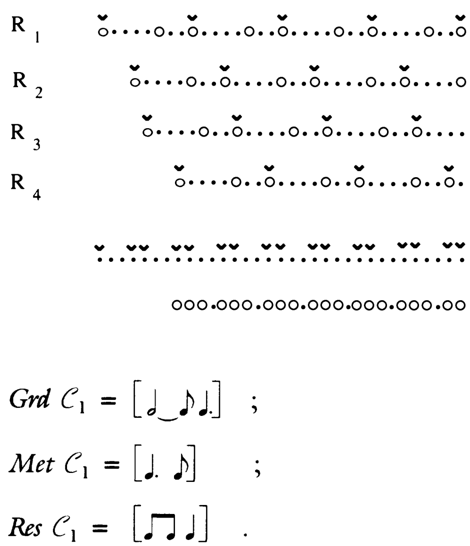
$$\bigcup_{i=1}^{l} S_i$$

(by using accents), and the set

$$\bigcup_{i=1}^{l} R_i$$

(by using dots), respectively; the $(l+2)$-th line may be omitted in some situations.

(*e*) In order to help the reader in deciphering the intervallic structure of $\mathcal{C}$, small dots (·) are inserted between the large dots. The distance between two consecutive dots (large or small) has to be considered as equal to Div *Res* $\mathcal{C}$. A similar convention applies to the $(l+1)$-th line. The diagram in Example 4.1 displays a primary meter on $\mathcal{C}_1$.

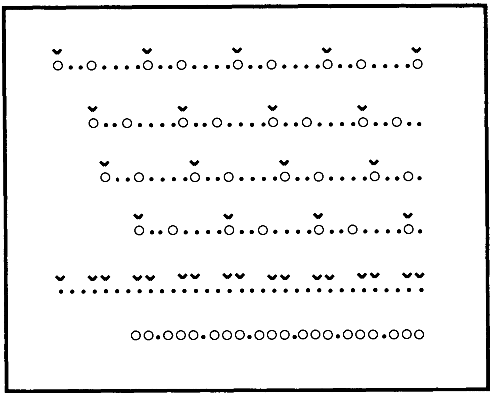



Ground number of $\mathcal{C}_1$ = 2 ;

Category of $\mathcal{C}_1$ = 2 ;

Modulus of $\mathcal{C}_1$ = 8.

EXAMPLE 4.1: A CANON $\mathcal{C}_1$ ON FOUR VOICES

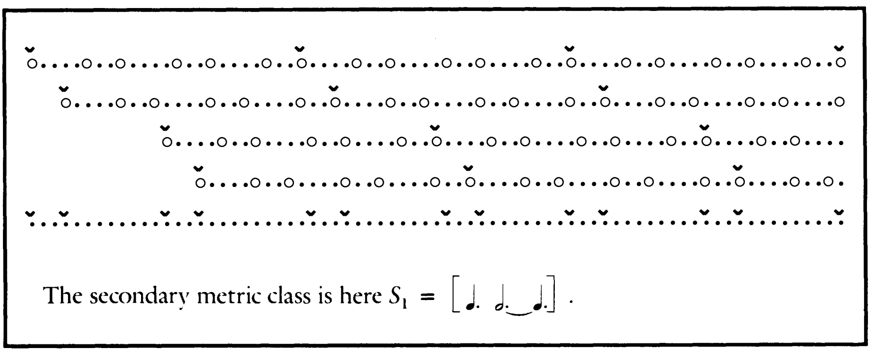


EXAMPLE 4.2: ANOTHER PRIMARY METER ON $\mathcal{C}_1$.

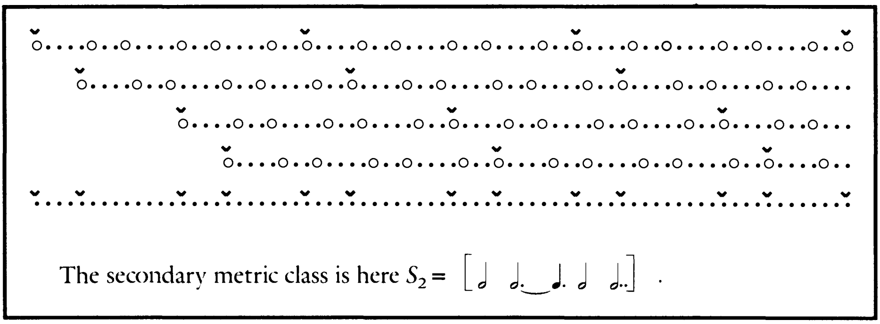
Observe that the resultants of the primary meters in Examples 4.1–4.2 belong to the same rhythmic class, namely *Met* $\mathcal{C}_1$. As can be seen on the diagram,

$$\bigcup_{i=1}^{4} S_i' = 1/8 + \bigcup_{i=1}^{4} S_i$$

($\{S_1, \ldots, S_4\}$ and $\{S_1', \ldots, S_4'\}$ are the meters in Examples 4.1 and 4.2, respectively).




EXAMPLE 4.3: A SECONDARY METER OF ORDER 3 ON $\mathcal{C}_1$




EXAMPLE 4.4: ANOTHER SECONDARY METER OF ORDER 3 ON $\mathcal{C}_1$

Examples 4.3–4.4 show that secondary metric classes of equal order admitted by a given canon need not have equal periods: Per $S_1 = 3/2$ while Per $S_2 = 3$.

The reader should note how the rhythms $R_1, \ldots, R_l$ in a canon are permuted when the entering of voices is dictated according to a secondary metric class (Example 4.4).

## 5. The Inversion of Canons

We start with the remark that in the symbol *Can*$(R,S)$, the places of $R$ and $S$ can be inverted. The procedure of passing from a canon in the class *Can* $(R,S)$ to a canon in the class *Can*$(S,R)$ will be therefore called *inversion*; starting with a single class one obtains via inversions several classes, as the representation of a canon class as *Can*$(R,S)$ is not unique.

Given a canon class *Can*$(R,S)$, what can be said about *Can*$(S,R)$? Firstly, the canons in that class are built on $S$. Secondly, the commutativity of composition of rhythmic classes implies that the canons in the class *Can*$(S,R)$ have the same resultant class as the canons in *Can*$(R,S)$, namely the class $R + S$.

As by inversion metric classes become ground classes, one might expect that inversion also transforms ground classes into metric classes. However, the situation is not so simple as might be guessed: even if $S$ is the primary metric class of the canons in *Can*$(R,S)$, $R$ need not be admitted at all as a metric class (even a secondary one) by the canons in *Can*$(S,R)$. Thus inversion, although always possible as a formal procedure, does not always have a rhythmic meaning, at least from the point of view developed in the preceding section. We say that a canon $\mathcal{C}$ in *Can*$(R,S)$ and a canon $\mathcal{C}'$ in *Can*$(S,R)$ are related by a *rhythmically meaningful inversion* if $\mathcal{C}$ admits $S$ as a metric class and $\mathcal{C}'$ admits $R$ as a metric class. It follows from Proposition 4.4 that this situation occurs iff $S \perp$ [Per $R$] and $R \perp$ [Per $S$].

Let us consider first the case of a normal pair $(R,S)$, that is, $R = Grd\ \mathcal{C}$ and $S = Met\ \mathcal{C}$ for some canon $\mathcal{C}$. If Per $R$ = Per $S$, then the roles played by $R$ and $S$ as "bricks" for the construction of a canon are perfectly symmetrical: one can build a canon $\mathcal{C}$ on $R$ which displays $S$ as the class of the resultant of a primary meter, and then one can invert $\mathcal{C}$ into a canon built on $S$ which displays this time $R$ as the class of the resultant of a primary meter.

The situation becomes more involved if Per $R \neq$ Per $S$. If $\mathcal{C}'$ is a canon in $Can(S,R)$, all we can expect is to hear $R$ embedded in the structure of $\mathcal{C}'$ only as a secondary metric class, the order $k$ of which is given by Proposition 4.3: $k$ = Per $R$/Per $S$. And besides, if $R$ is not intervallically disjoint from [Per $S$], we run into another difficulty: $R$ cannot be heard at all as the class of the resultant of some meter on $\mathcal{C}'$ in the sense of Definition 4.4.[1]

We summarize the above considerations into a definition.

DEFINITION 5.1. *A canon $\mathcal{C}$ is called invertible if* $Grd\ \mathcal{C} \perp$ [Per $Met\ \mathcal{C}$]. *$\mathcal{C}$ is called primarily invertible if* Per $Grd\ \mathcal{C}$ = Per $Met\ \mathcal{C}$.

Clearly, every primarily invertible canon is invertible.

The reason for considering only primary metric classes in the definition of primary invertibility is that, by virtue of Proposition 4.3, Per $S \neq$ Per $Grd\ \mathcal{C}$ for *every* secondary metric class $S$ admitted by $\mathcal{C}$.

If $\mathcal{C}$ is an invertible canon, then any canon $\mathcal{C}'$ in the class $Can(Met\ \mathcal{C}, Grd\ \mathcal{C})$ will be referred to as *an inverse* of $\mathcal{C}$. According to Proposition 4.3, the number of voices in $\mathcal{C}'$ equals the ground number of $\mathcal{C}$. The class $Can(Met\ \mathcal{C}, Grd\ \mathcal{C})$ itself will be called *the inverse* of the class $Can(Grd\ \mathcal{C}, Met\ \mathcal{C})$.

From the formal viewpoint, primary invertibility is equivalent to maximal category. We do not, however, identify these concepts, as in some concrete situations the value of the information that a canon $\mathcal{C}$ of maximal category is primarily invertible might remain only theoretical, the musical realization of the inversion not being possible because too large a number of voices is required by an inverse of $\mathcal{C}$.

Suppose now that $\mathcal{C}$ is not invertible. In this situation, a rhythmically meaningful inversion can still be performed on $\mathcal{C}$, but to this end we have to replace the primary metric class by a secondary one; that is, we have to represent the class of $\mathcal{C}$ as $Can(Grd\ \mathcal{C}, S_1)$ where $(Grd\ \mathcal{C}, S_1)$ is now a not-normal pair submitted to the conditions

$$S_1 + [\text{Per } Grd\ \mathcal{C}] = Met\ \mathcal{C}, \qquad (1)$$

$$S_1 \perp [\text{Per } Grd\ \mathcal{C}], \tag{2}$$

$$Grd\ \mathcal{C} \perp [\text{Per } S_1]. \tag{3}$$

If $S_1$ satisfies (1), (2), and

$$\text{Per } Grd\ \mathcal{C} \mid \text{Per } S_1, \tag{4}$$

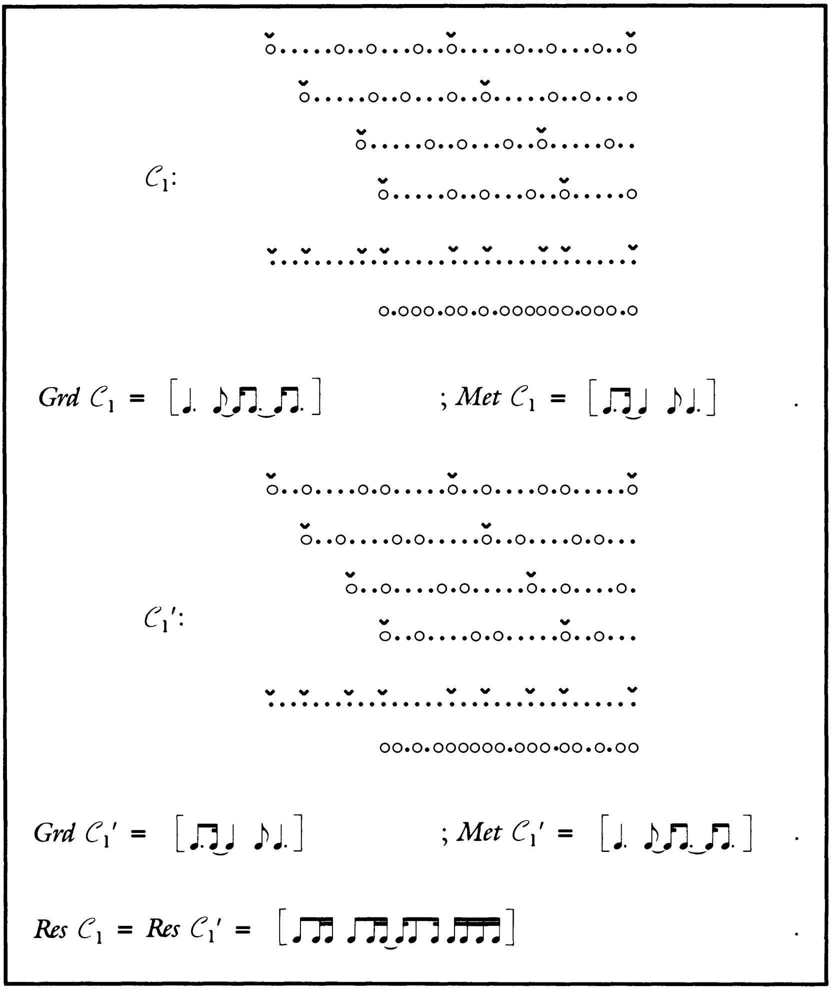
then (3) is automatically satisfied and Grd $\mathcal{C}$ becomes the primary metric class of the canons in $Can(S_1,\ Grd\ \mathcal{C})$. However, unlike the case of invertible canons, the canons obtained by the inversion procedure we have just described are not built on the metric class of $\mathcal{C}$, but on an extension of that class.

The fact that a class $S_1$ satisfying the above requirements always exists is a consequence of the following proposition.

PROPOSITION 5.1. *For every canon $\mathcal{C}$ there is $S_1 \in Rhyt$ satisfying the conditions* (1), (2), *and* (4) *above.*

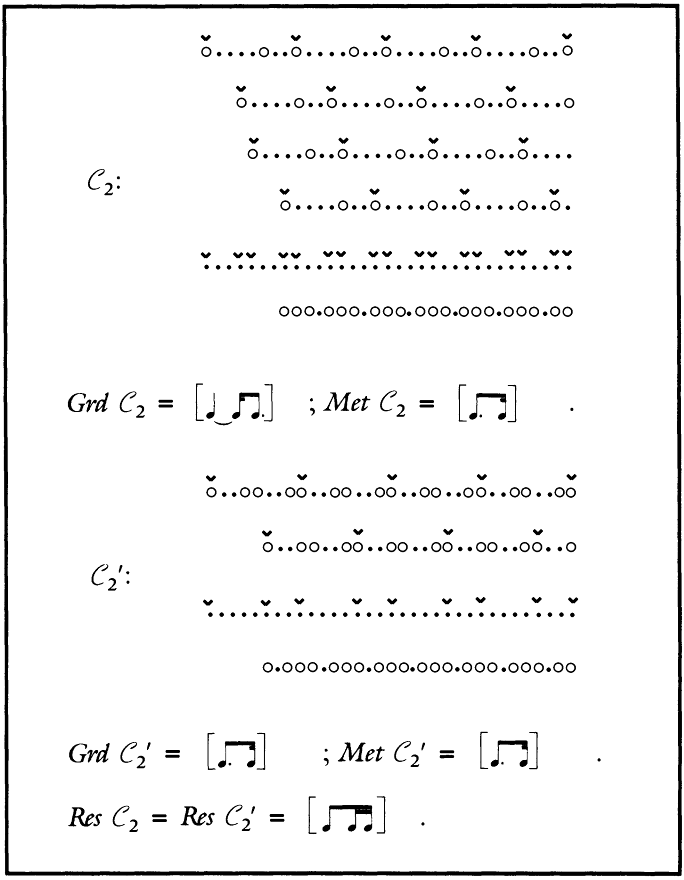
*Proof.* Set $a$ = Div $Res\ \mathcal{C}$, $b$ = Per $Grd\ \mathcal{C}$, $n = b/a$. Choose a prime $p$ and consider the homomorphism $\phi_{pn,n}:\mathbf{Z}_{pn} \to \mathbf{Z}_n$. It suffices to construct a nonperiodic subset M of $\mathbf{Z}_{pn}$ which verifies the equality $[\phi_{pn,n}(\mathrm{M})] = H_{a,b}\,(Met\ \mathcal{C})$ and has the property that the restriction of $\phi_{pn,n}$ to M is one-to-one; indeed, Propositions 3.4 and 3.8 show that for such an M, the class $S_1 = H^{-1}_{a,pb}([\mathrm{M}])$ satisfies all the requirements. To this end, choose a set N in $H_{a,b}$ $(Met\ \mathcal{C})$ and let G be the stability subgroup of N. As N is a union of cosets of $\mathbf{Z}_n$ modulo G, it can be represented as $\mathrm{N}_1 + \mathrm{G}$ where $\mathrm{N}_1 = \{x_1, \ldots, x_k\}$ is a subset of a set of representants of $\mathbf{Z}_n$ modulo G. For every $x_i \in \mathrm{N}_1$ choose $y_i \in \mathbf{Z}_{pn}$ such that $\phi_{pn,n}\,(y_i) = x_i$. Choose also a nonperiodic set $\mathrm{M}_1$ of representants of the group $\phi^{-1}_{pn,n}$ (G) modulo its subgroup Ker $\phi_{pn,n}$. The latter choice can always be done by virtue of Lemma 2.1. Finally set $\mathrm{M} = \mathrm{M}_1 + \{y_1, \ldots, y_k\}$. It is clear that $\phi_{pn,n}\,(\mathrm{M}) = \mathrm{N}$. If $\mathrm{N} = x_1 + \mathrm{G}$, then $\mathrm{M} = y_1 + \mathrm{M}_1$ is nonperiodic. Suppose now that $\mathrm{N} \neq x_1 + \mathrm{G}$ and let $y \in \mathbf{Z}_{pn}$ be such that $y + \mathrm{M} = \mathrm{M}$. Applying $\phi_{pn,n}$ to both sides of this equality we get $\phi_{pn,n}\,(y) \in \mathrm{G}$. Consequently, $(y + y_1 + \mathrm{M}_1) \cap (y_i + \mathrm{M}_1) = \emptyset$ for $2 \leq i \leq k$ as $\phi_{pn,n}\,(y + y_1 + \mathrm{M}_1) \subset x_1 + \mathrm{G}$ while $\phi_{pn,n}\,(y_i + \mathrm{M}_1) \subset x_i + \mathrm{G} \subset \mathbf{Z}_n \backslash \mathrm{G}$ for $2 \leq i \leq k$. We must therefore have $y + \mathrm{M}_1 = \mathrm{M}_1$ ; as $\mathrm{M}_1$ is nonperiodic the latter implies $y = 0$. Hence M is nonperiodic and the proof is complete.

The following examples will serve as an illustration of the foregoing discussion.




EXAMPLE 5.1: A PRIMARILY INVERTIBLE CANON $\mathcal{C}_1$ TOGETHER WITH AN INVERSE $\mathcal{C}_1'$ OF IT

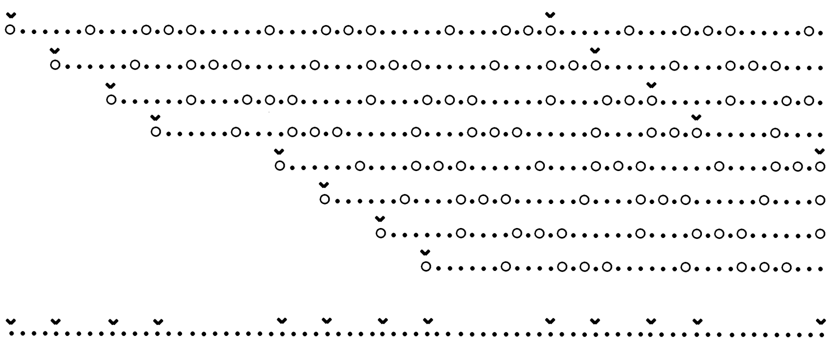
Observe that the numbers of voices in $\mathcal{C}_1$ and $\mathcal{C}_1'$ are the same; this happens because the ground number of $\mathcal{C}_1$ equals its number of voices.

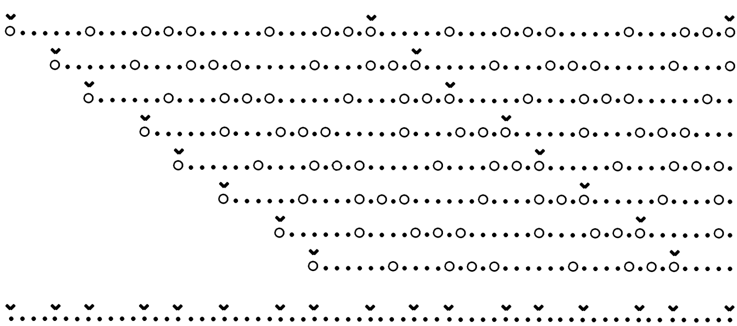



EXAMPLE 5.2: AN INVERTIBLE CANON $\mathcal{C}_2$
WHICH IS NOT PRIMARILY INVERTIBLE
TOGETHER WITH AN INVERSE $\mathcal{C}_2'$ OF IT

Example 5.2 exhibits a secondary meter of order 2 on $\mathcal{C}_2'$ whose resultant belongs to the class Grd $\mathcal{C}_2$.

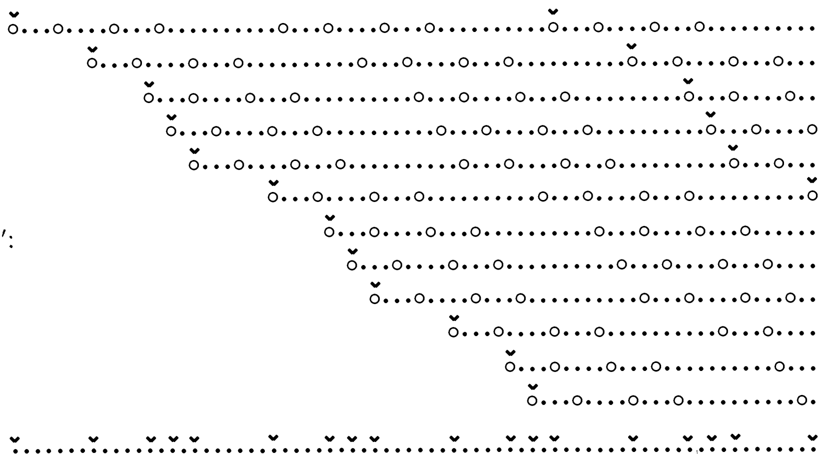
Let $\mathcal{C}_3$ be a canon of class *Can* (  ). $\mathcal{C}_3$ is not invertible, because *Grd* $\mathcal{C}_3$ is not intervallically disjoint from [Per *Met* $\mathcal{C}_3$] = [1/4]. Indeed, 1/4 $\in$ Int *Grd* $\mathcal{C}_3$ $\cap$ Int [Per *Met* $\mathcal{C}_3$] while 1 = Per *Grd* $\mathcal{C}_3$ non | 1/4. To perform rhythmically meaningful inversions on $\mathcal{C}_3$ we have therefore to replace *Met* $\mathcal{C}_3$ by secondary metric classes satisfying conditions (1)–(3). Here are two of them: a class $S_1$ =  of order 3 and a class $S_2$ =  of order 2. The next two diagrams present $\mathcal{C}_3$ endowed with meters whose resultants belong respectively to $S_1$ and to $S_2$.



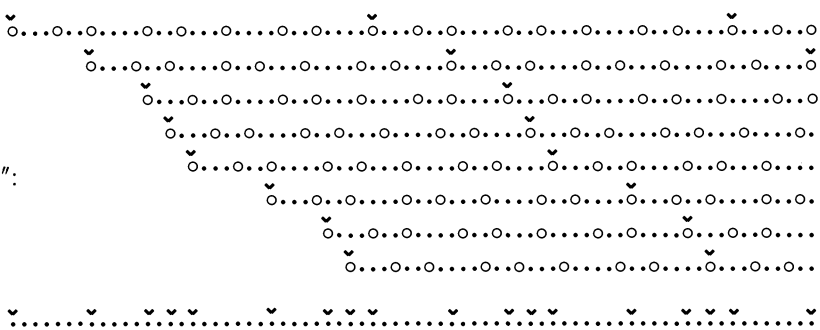


Two rhythmically meaningful inversions are now practicable on $\mathcal{C}_3$. They lead respectively to a canon $\mathcal{C}_3'$ of class *Can*($S_1$, *Grd* $\mathcal{C}_3$) and to a canon $\mathcal{C}_3''$ of class *Can*($S_2$, *Grd* $\mathcal{C}_3$). $\mathcal{C}_3'$ admits *Grd* $\mathcal{C}_3$ as a secondary metric class of order 2 while $\mathcal{C}_3''$ admits *Grd* $\mathcal{C}_3$ as its primary metric class. We present below the canons $\mathcal{C}_3'$ and $\mathcal{C}_3''$ endowed with the corresponding meters of orders 2 and 1, respectively.

$\mathcal{C}_3'$:



$\mathcal{C}_3''$:



Observe that $S_2$ satisfies condition (4) (Per $S_2 = 2$) while $S_1$ does not satisfy it (Per $S_1 = 3/2$). Hence condition (4) is not a consequence of (1)–(3).

EXAMPLE 5.3: A NONINVERTIBLE CANON

We describe now a procedure of successive inversions and condensations which applied to any canon leads in a finite number of steps to a canon of maximal category. Suppose we start with a canon $\mathcal{C}$ whose category is not maximal. By performing an inversion (at least formally, if $\mathcal{C}$ is not invertible) we arrive at the class *Can*(*Met* $\mathcal{C}$, *Grd* $\mathcal{C}$). Now

$$Can(Met\ \mathcal{C}, Grd\ \mathcal{C}) = Can(Met\ \mathcal{C}, Grd\ \mathcal{C} + [\mathrm{Per}\ Met\ \mathcal{C}])$$

and $(R_1, S_1) = (Met\ \mathcal{C}, Grd\ \mathcal{C} + [\text{Per}\ Met\ \mathcal{C}])$ is a normal pair. If the canons in $Can(R_1, S_1)$ are of maximal category, we stop. If not, we invert $Can(R_1, S_1)$ to $Can(S_1, R_1)$ and then normalize $(S_1, R_1)$ to obtain $(R_2, S_2) = (S_1, R_1 + [\text{Per}\ S_1])$ so that $Can(S_1, R_1) = Can(R_2, S_2)$. If the canons in $Can(R_2, S_2)$ are of maximal category, we stop; if not, we continue. This procedure is formally described below as an inductive process.

Let $(R,S)$ be a normal pair. Define inductively the sequence $(R_n, S_n)_{n \geq 0}$ of normal pairs via the formulas

$$R_0 = R, S_0 = S, \tag{5}$$

$$R_{n+1} = S_n, S_{n+1} = R_n + [\text{Per}\ S_n]. \tag{6}$$

If for some $n_0$ we get Per $R_{n0}$ = Per $S_{n0}$ then $(R_n, S_n)$ equals $(R_{n0}, S_{n0})$ or $(S_{n0}, R_{n0})$ for all $n \geq n_0$ and hence the process can be stopped, since after the $n_0$-th step nothing new is obtained. That this is always the case follows from the next proposition.

PROPOSITION 5.2. *For any normal pair (R,S) the above-constructed sequence has the property that there exists an even integer $n_0$ (depending on the pair (R,S)) such that* Per $R_{n0}$ = Per $S_{n0}$.

*Proof.* We have

$$R_{n+1} + S_{n+1} = S_n + (R_n + [\text{Per}\ S_n])$$
$$= R_n + (S_n + [\text{Per}\ S_n]) = R_n + S_n$$

so that $R_n + S_n = R + S$ for all $n \geq 0$. Set $a$ = Div $R \wedge$ Div $S$ = Div $R_n \wedge$ Div $S_n$ (Proposition 3.2). Because

$$R_{n+2} = S_{n+1} = R_n + [\text{Per}\ S_n]$$

it follows that Per $R_{n+2}$ | Per $R_n$ so that the integer Per $R_{n+2}/a$ divides the integer Per $R_n/a$ for every $n \geq 0$. From this fact we infer that Per $R_{n0+2}$ = Per $R_{n0}$ for some even integer $n_0$. The relation $R_{n0+2} = R_{n0} + [\text{Per}\ S_{n0}]$ implies Per $R_{n0}$ = Per $R_{n0+2}$ | Per $S_{n0}$, while Per $S_{n0}$ | Per $R_{n0}$ as $(R_{n0}, S_{n0})$ is a normal pair. We conclude that Per $R_{n0}$ = Per $S_{n0}$.

The canon class $Can(R_{n0}, S_{n0})$, where $n_0$ is the integer whose existence is asserted by Proposition 5.2, will be called the *minimal condensation of maximal category* (in short: the minmax condensation) of $Can(R,S)$. The terminology is justified by the fact that $Can(R_{n0}, S_{n0})$ is the "less condensed" form of $Can(R,S)$ whose category is maximal. This rather vague statement will be made precise by the following considerations.

Let $Can(R,S)$ and $Can(R',S')$ be two canon classes represented with the aid of the normal pairs $(R,S)$ and $(R',S')$. We say that $Can(R,S)$ is a *condensation* of $Can(R',S')$ if $R$ is a condensation of $R'$ and $S$ is a condensation of $S'$. In symbols: $Can(R',S') \to Can(R,S)$. The relation so defined turns the set of all canon classes into a partially ordered set (see Proposition 3.3). With this preparation we can state:

PROPOSITION 5.3. *The minmax condensation of $Can(R,S)$ is the least element (for the above-introduced order relation) in the set of all condensations of $Can(R,S)$ whose categories are maximal.*

*Proof.* Let $Can(R_{n_0},S_{n_0})$ be the minmax condensation of $Can(R,S)$. According to the definition of the sequence $(R_n, S_n)$, $R_n \to R_{n+2}$ and $S_n \to S_{n+2}$ for every $n \geq 0$; hence, as $n_0$ is even, $Can(R_{n_0}, S_{n_0})$ is indeed a condensation of $Can(R,S)$. Let $Can(R',S')$ be any condensation of $Can(R,S)$ whose category is maximal ($(R',S')$ being a normal pair). The proof will be concluded if we show that $R_n \to R'$ and $S_n \to S'$ for every even integer $n$. We argue by induction. For $n = 0$, these relations are true by hypothesis. Suppose them true for $n$ and let us prove them for $n + 2$. We have

$$R_{n+1} = S_n,\ S_{n+1} = R_n + [\mathrm{Per}\ S_n],$$

$$R_{n+2} = R_n + [\mathrm{Per}\ S_n],\ S_{n+2} = S_n + [\mathrm{Per}\ S_{n+1}].$$

As $S_n \to S'$ it follows that $\mathrm{Per}\ R' = \mathrm{Per}\ S' \mid \mathrm{Per}\ S_n$; hence $S_{n+1} \to R' + [\mathrm{Per}\ S_n] = R'$, which in turn implies $\mathrm{Per}\ S' = \mathrm{Per}\ R' \mid \mathrm{Per}\ S_{n+1}$. Finally, $R_{n+2} = S_{n+1} \to R'$ and $S_{n+2} \to S' + [\mathrm{Per}\ S_{n+1}] = S'$. The proof is complete.

As a corollary to the preceding result, we obtain the fact that the minmax condensations of the classes of two canons related by an inversion are inverse each to the other.

COROLLARY 5.1. *For any $R,S \in Rhyt$, the minmax condensation of $Can(R,S)$ equals the inverse of the minmax condensation of $Can(S,R)$.*

*Proof.* Let $Can(\bar{R},\bar{S})$ and $Can(\hat{S},\hat{R})$ be the minmax condensation of $Can(R,S)$ and $Can(S,R)$, respectively; by definition, $\mathrm{Per}\ \bar{R} = \mathrm{Per}\ \bar{S}$ and $\mathrm{Per}\ \hat{S} = \mathrm{Per}\ \hat{R}$. We have

$$Can(R,S) = Can(R,S + [\mathrm{Per}\ R]),$$

$$Can(S,R) = Can(S, R + [\mathrm{Per}\ S]),$$

the pairs in the right side being normal. The definition of condensation of canons implies the relations

$$R \to \bar{R},$$
$$S \to S + [\text{Per } R] \to \bar{S},$$
$$S \to \hat{S},$$
$$R \to R + [\text{Per } S] \to \hat{R}.$$

From these it follows that $R \to \hat{R}$, $[\text{Per } R] \to [\text{Per } \hat{R}]$ and

$$S + [\text{Per } R] \to \hat{S} + [\text{Per } \hat{R}] = \hat{S} + [\text{Per } \hat{S}] = \hat{S};$$

therefore $Can(R,S) \to Can(\hat{R},\hat{S})$ and from Proposition 5.3 we get $Can(\bar{R},\bar{S}) \to Can(\hat{R},\hat{S})$. By a similar argument $Can(\hat{S},\hat{R}) \to Can(\bar{S},\bar{R})$ and finally $\bar{R} = \hat{R}$, $\bar{S} = \hat{S}$.

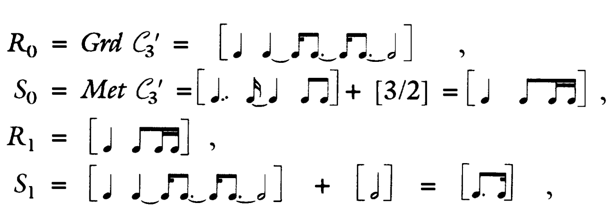
PROPOSITION 5.4. *Let $\bar{C}$ be a canon in the minmax condensation of the class of a canon $C$. Then Res $\bar{C}$ = Res $C$ and the modulus of $\bar{C}$ divides the modulus of $C$.*

*Proof.* Let $Can(R,S)$ be the class of $C$, with $(R,S)$ a normal pair. Consider the sequence $(R_n,S_n)_{n\geq 0}$ associated to $(R,S)$ via formulas (5)–(6). The relation $R_n + S_n = R + S$ for all $n \geq 0$ was observed during the proof of Proposition 5.2. As the class of $\bar{C}$ equals $Can(R_{n_0},S_{n_0})$ for some even integer $n_0$, it follows that *Res* $\bar{C} = R_{n_0} + S_{n_0} = R + S =$ *Res* $C$.

By definition, the modulus of $C$ equals Per $R$/Div *Res* $C$ while the modulus of $\bar{C}$ equals Per $R_{n_0}$ /Div *Res* $\bar{C}$. As $R \to R_{n_0}$ we have Per $R_{n_0} \mid$ Per $R$ and hence the modulus of $\bar{C}$ divides the modulus of $C$.

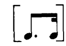
It should be noted that the minmax condensation of a canon class may have the form $Can(R, [\text{Per } R])$, that is, may be the class of a canon consisting of a single rhythm (see the next section). This was the reason we spoke of in Section 4 for also considering as canons the sets consisting of a single rhythm.

We conclude the section with an illustration of the method for finding the minmax condensation applied to the canon $C_3'$ in Example 5.3. We have



$R_2$ =  ,

$S_2$ =  +  =  .

Hence the minmax condensation of the class of $\mathcal{C}_3'$ equals *Can*(  ,  ).

## Notes

1. This difficulty could be avoided only if we allowed a complete overlapping of beats between two or more voices, the latter being distinguished one from another only by different meters. We shall not consider such a situation, as it runs contrary to our initial assumption concerning the existence of no complete overlapping of beats between any pair of voices.

## References

Grigor'ev, Stepan Stepanovich, and Teodor Müller. 1961. *Uchebnik polifonii* (A textbook on polyphony). Moscow: Gos. muzykal'noe izd-vo.