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Théorie des nœuds et des tresses en mathématiques et en musique

Music-Theoretical Interpretations of the Artin Relation

Thomas Noll

noll@cs.tu-berlin.de

Escola Superior de Musica de Catalunya, Barcelona
Departament de Teoria i Composició

Artin Relation QPQ = PQP

SL(2,R)

↑ exp

sl(2,R)

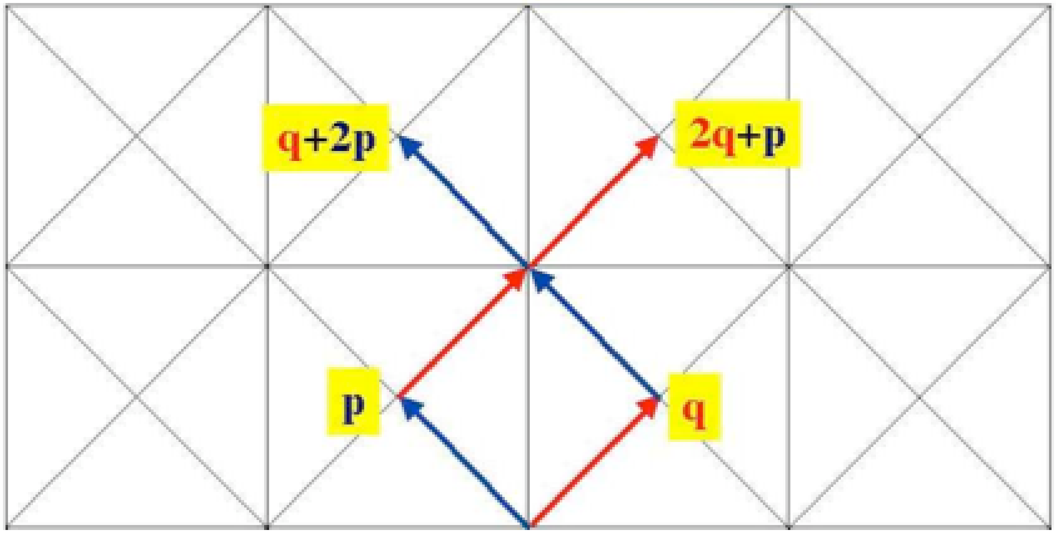
$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -b^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -b^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -b^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -b^{-1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2b \\ -b^{-1} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ -b^{-1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -b^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ -2b^{-1} & 0 \end{pmatrix}$$

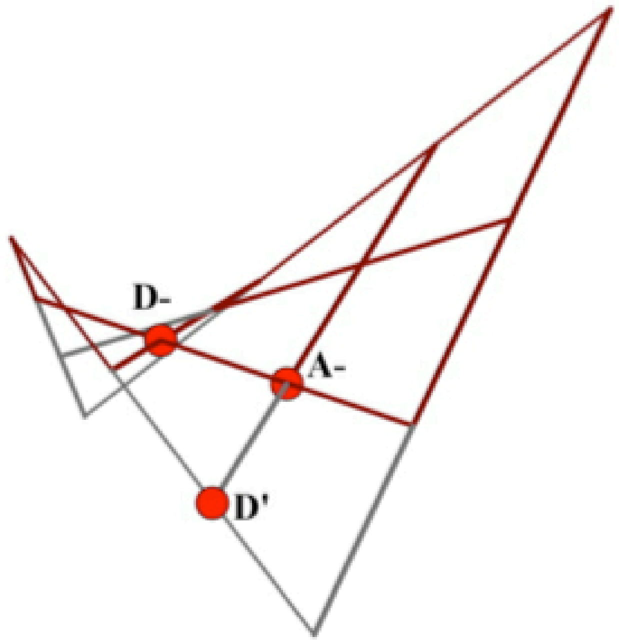
Hallucination QPQ = PQP

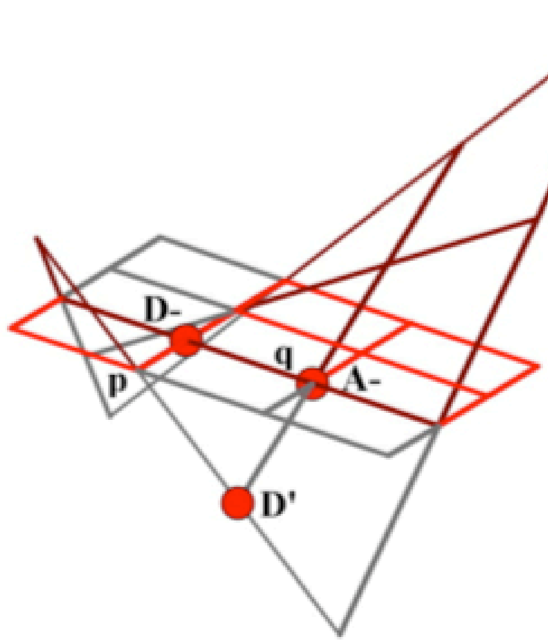


A geometric Realisation for b = 1

$$\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&0\\-1&1\end{pmatrix}\cdot\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

$$\begin{pmatrix}1&0\\-1&1\end{pmatrix}\cdot\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&0\\-1&1\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$





Added tangential plane

A curved lattice in R^4

(points = matrices)

- Multiplication with the same Matrix = prgression on a straight line
- Change of the Matrix = change of the direction (depending on the actual location)

Paraphrase of the Artin Relation

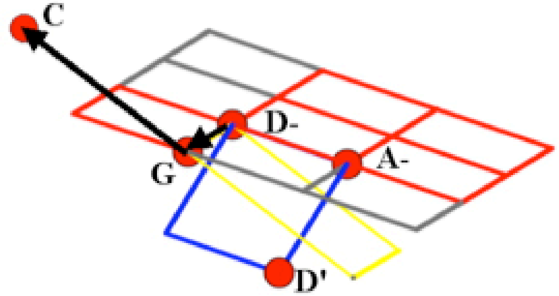
$$QPQ = PQP$$

$$Q^{-1}QPQP^{-1} = Q^{-1}PQPP^{-1}$$

$$PQP^{-1} = Q^{-1}PQ$$

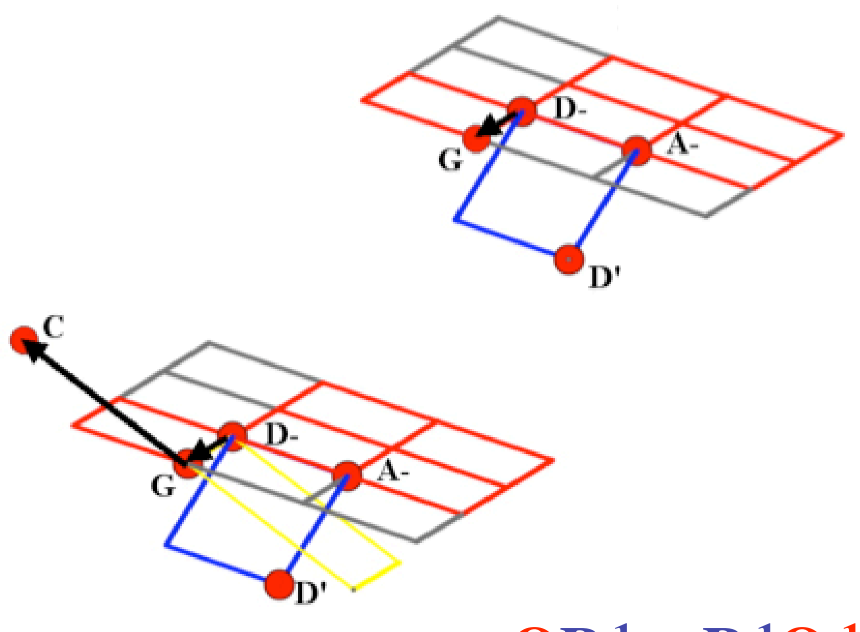
$$QP^{-1} = P^{-1}Q^{-1}PQ$$

A geometric Realisation



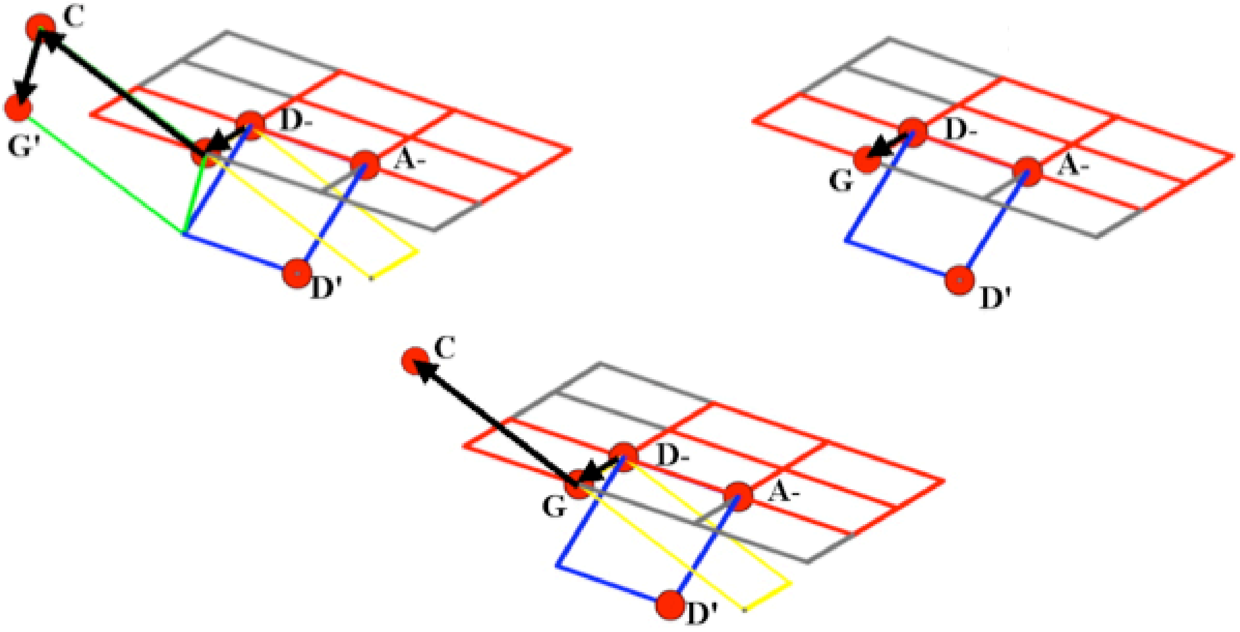
$QP^{-1} = P^{-1}Q^{-1}PQ$

A geometric Realisation



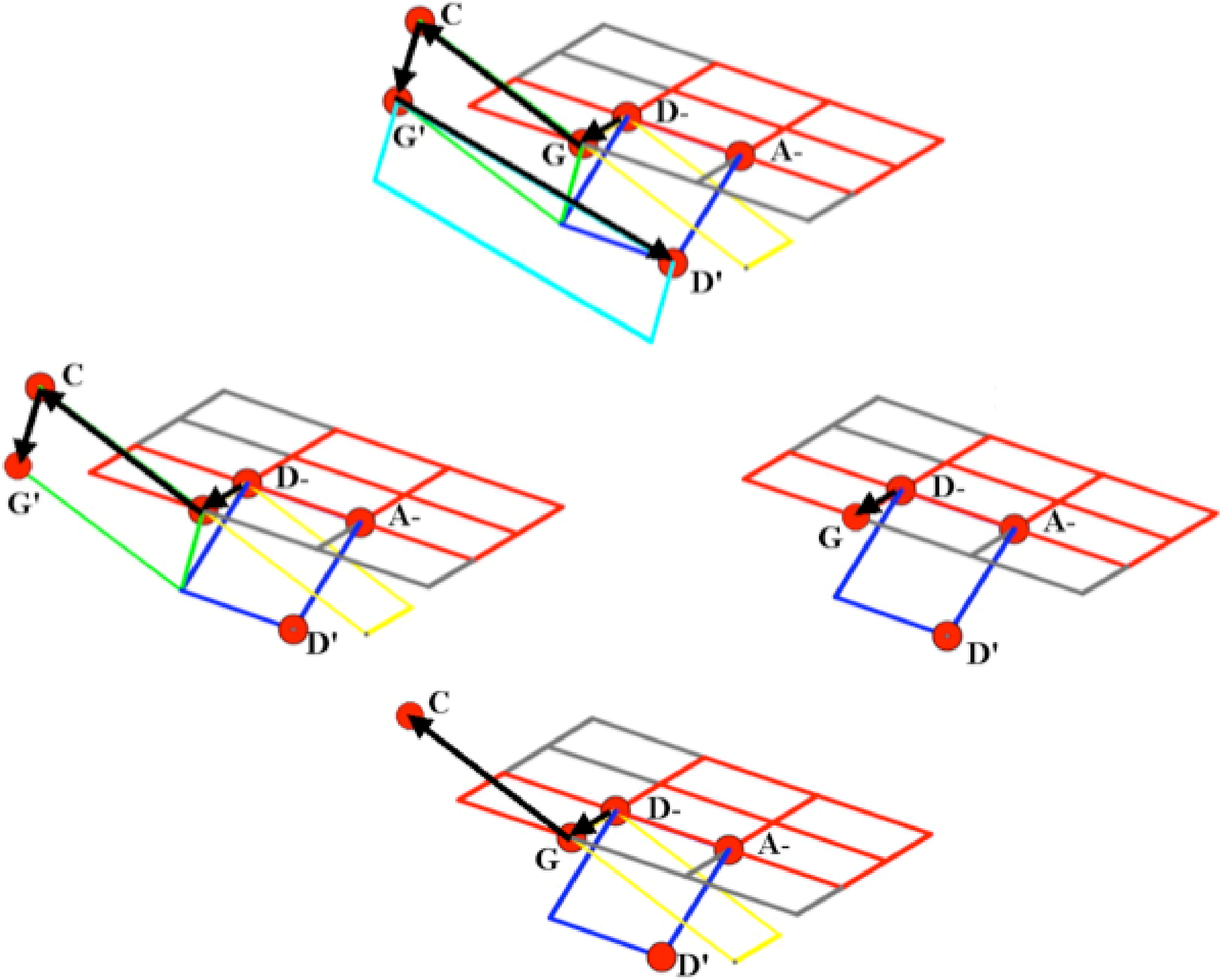
$QP^{-1} = P^{-1}Q^{-1}PQ$

A geometric Realisation



$$QP^{-1} = P^{-1}Q^{-1}PQ$$

A geometric Realisation



$$QP^{-1} = P^{-1}Q^{-1}PQ$$

Identity Paradoxes in Harmonic Tonality

- Syntonic
- Modulatory
- Synchromatic
- Octave

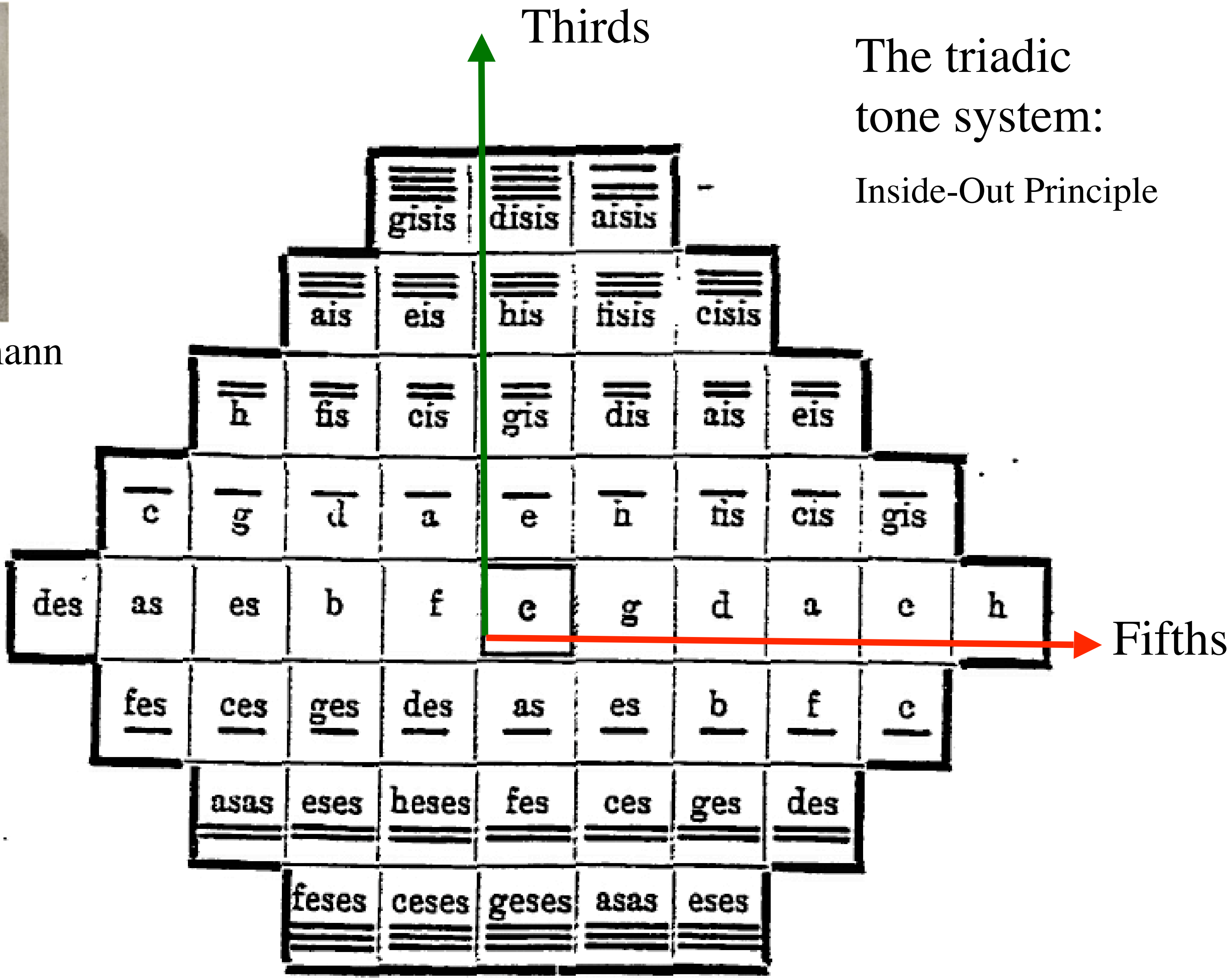
Two related Questions:

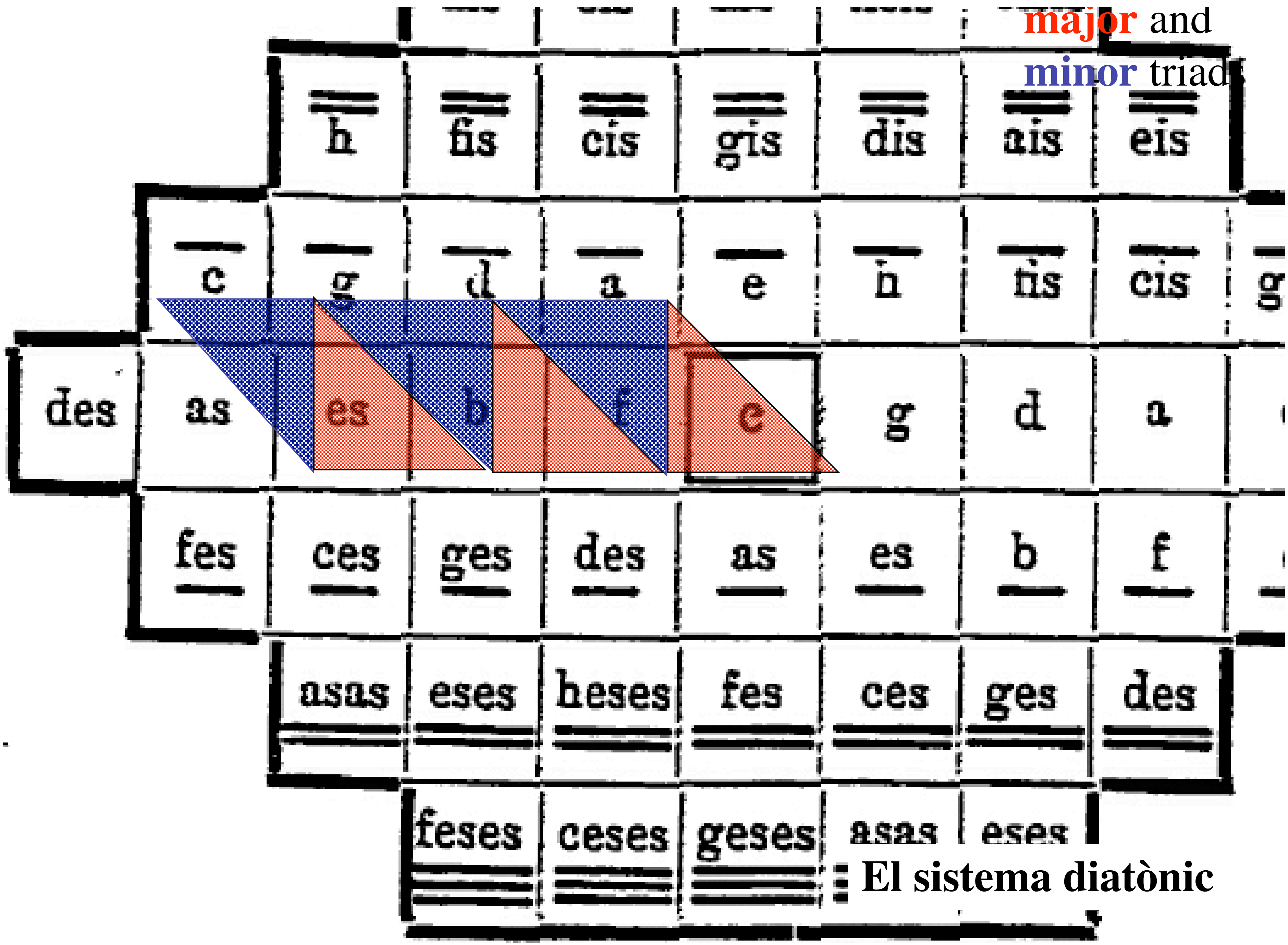
1. What is the music-theoretical status of these paradoxes?
2. Can mathematical models help to understand them?

The triadic tone system:

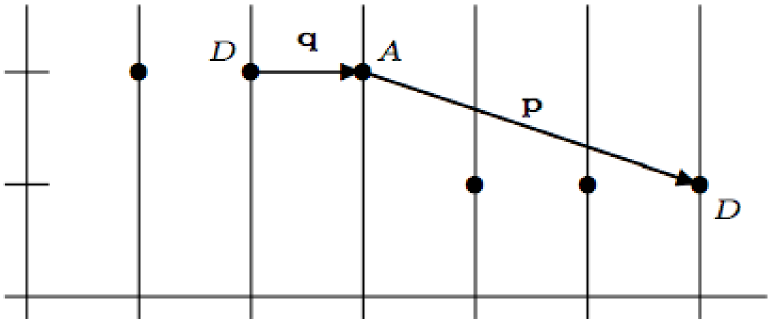
Inside-Out Principle

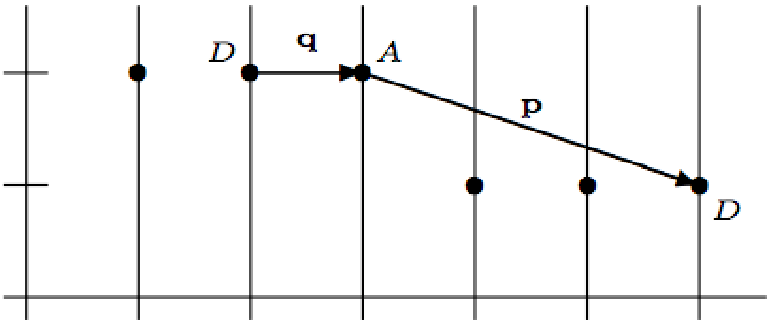
Hugo Riemann

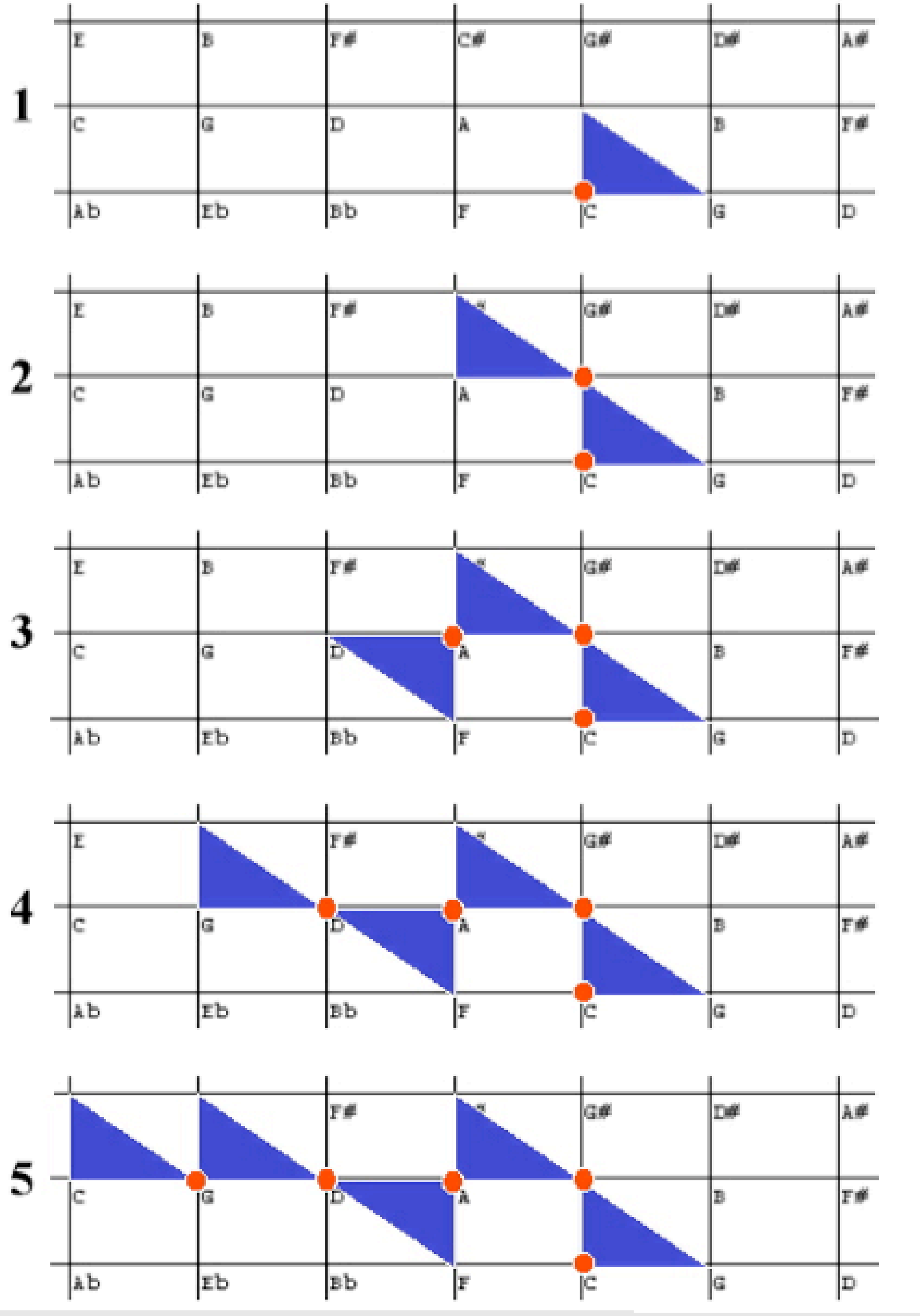



major and
minor triads
h fis cis gis dis ais eis
c g d a e h fis cis g
des as es b f e g d a
fes ces ges des as es b f
asas eses heses fes ces ges des
feses ceses geses asas eses
El sistema diatònic

Syntonic Paradox







Fifth Q   Third T $\approx \mathbb{Z}$ $\approx \mathbb{Z}_{12}$

$(Q, T;) \rightarrow (Q, T;\ Q^4 = T) \rightarrow (Q, T;\ Q^4 = T,\ Q^{12} = Id)$

$P = TQ^{-3}$

Pseudo-Fifth

$(Q, P;) \rightarrow (Q, P;\ Q = P) \rightarrow (Q, P;\ Q = P,\ Q^{12} = Id)$

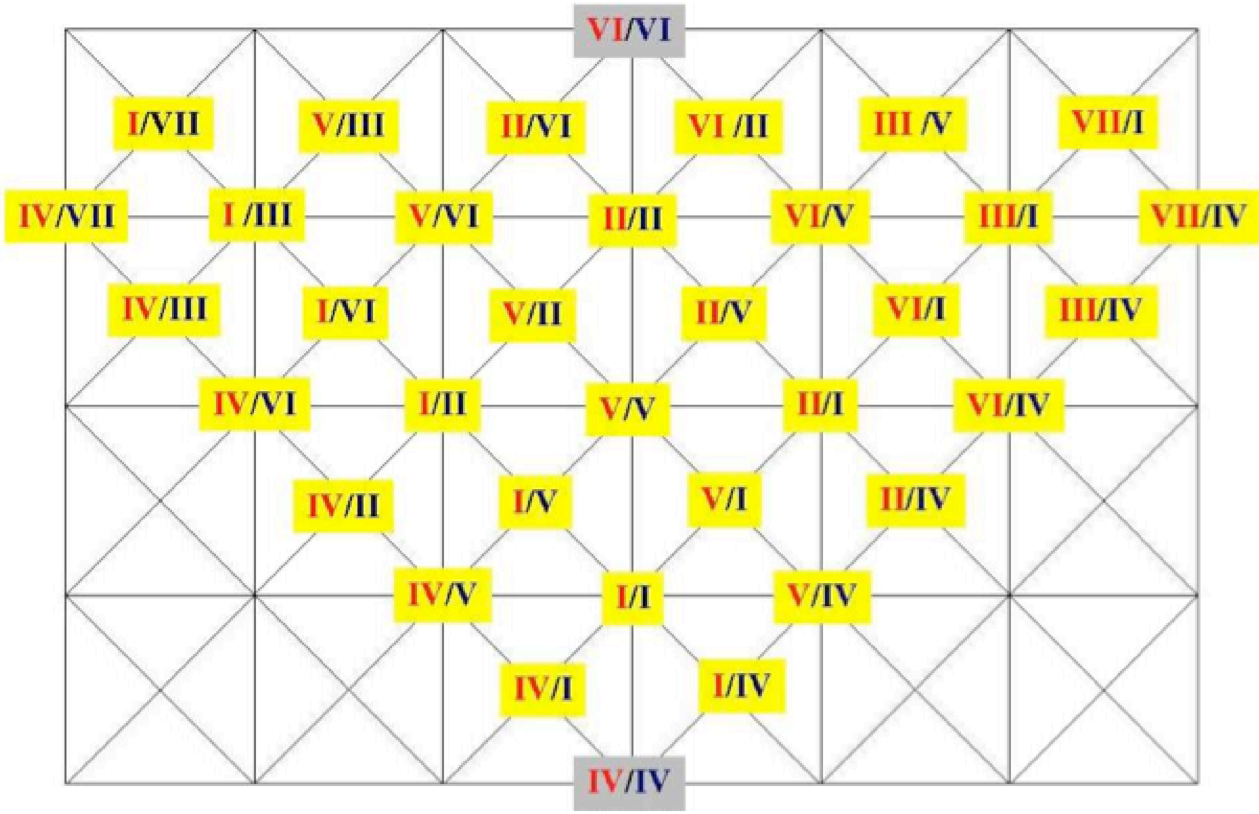
Artin Group B_3 $(Q, P;\ QPQ = PQP)$

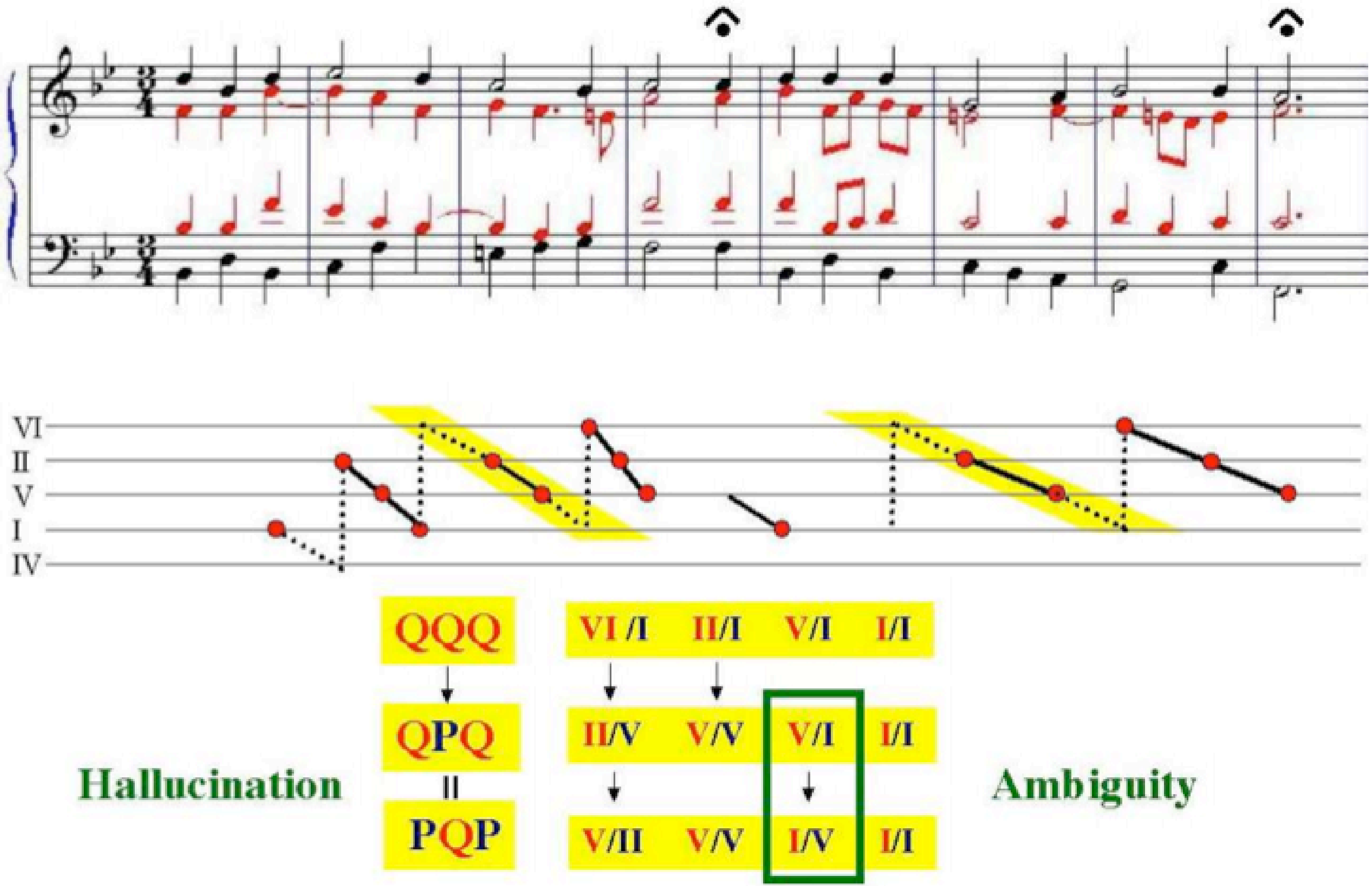
$SL_2(\mathbb{Z}) \approx (Q, P;\ QPQ = PQP,\ (QP)^6 = Id)$

$(Q, P;\ QPQ = PQP,\ (QP)^6 = Id,\ QP = PQ)$

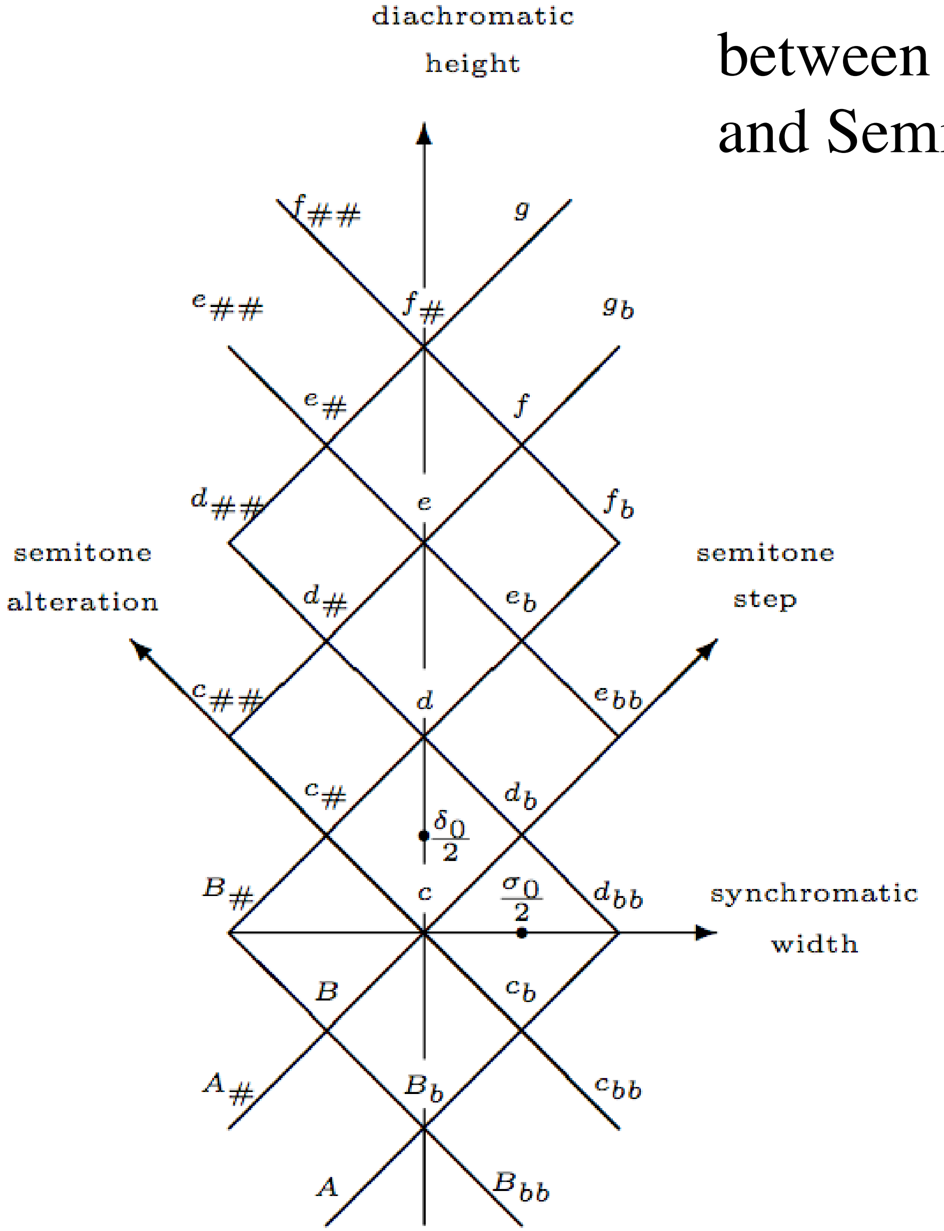
Thanks to Franck Jedrzejewski
for pointing me to his usage of B_3
in the context of scales

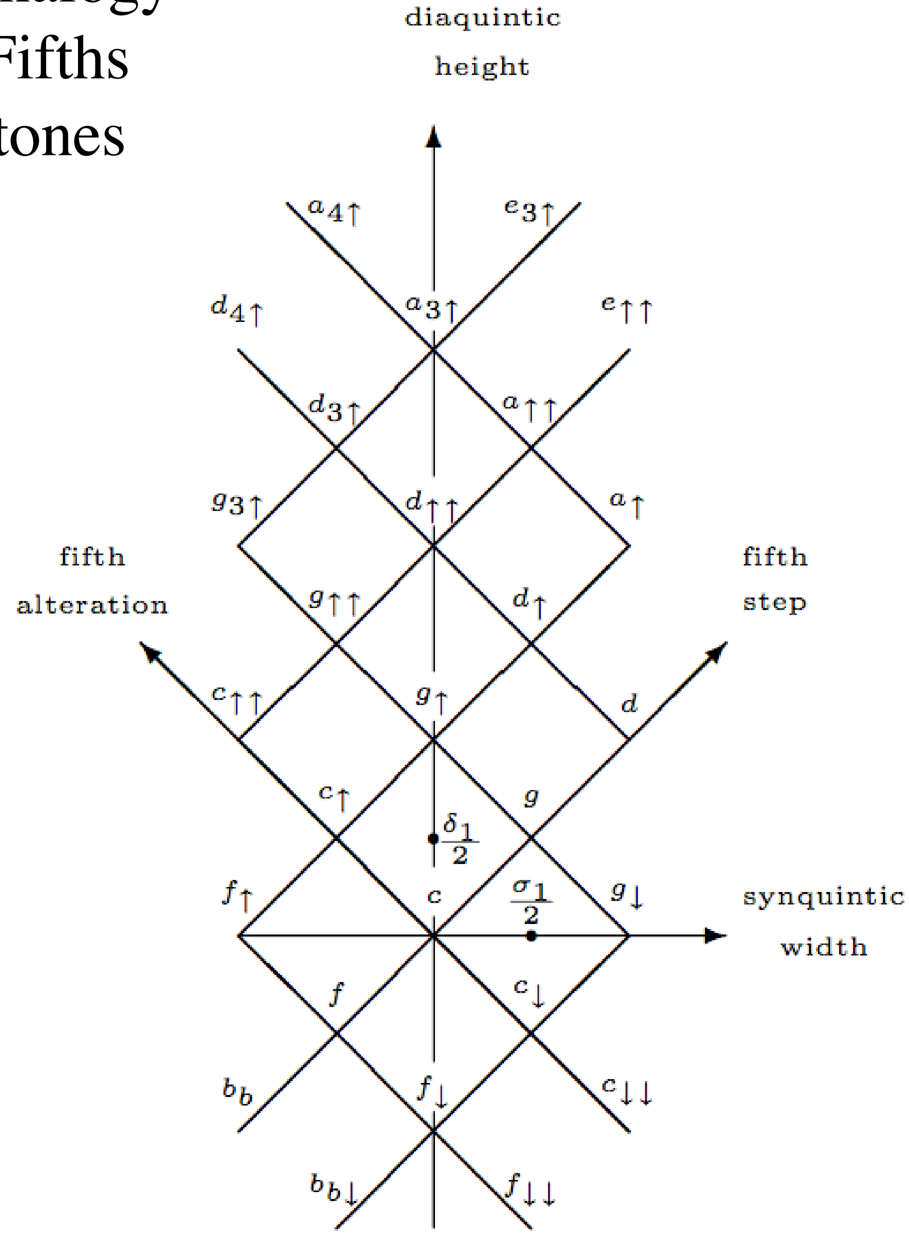
Modulatory Paradoxes V/I or I/V ?




VI
II
V
I
IV
QQQ
QPQ
=
PQP
VI /I II/I V/I I/I
II/V V/V V/I I/I
V/II V/V I/V I/I
Hallucination
Ambiguity

Formal Analogy between Fifths and Semitones





Idea:
Carey-Clampitt Theory might provide an interesting link

Jedrzewski:
Scales as B_3 - words:
ababbababab b

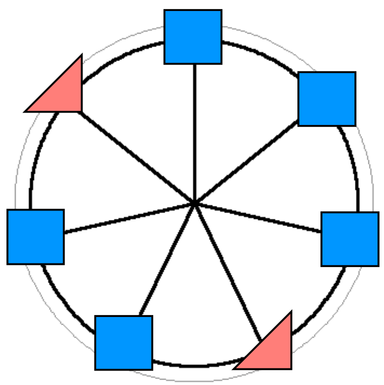
(meantone temperament)

?

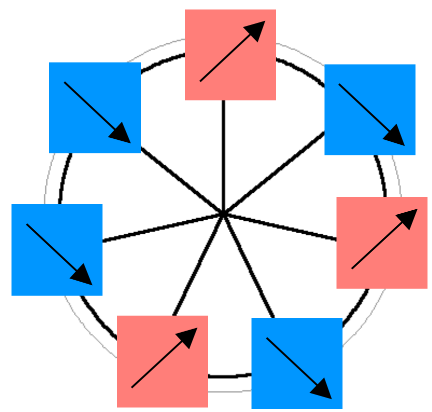
Jedrzewski:
Pythagorean Fifths/Fourths
as B_3 - words:

Carey & Clampitt:
Wellformed Scales

Carey Clampitt:
Scale Generation and Octave Indentification

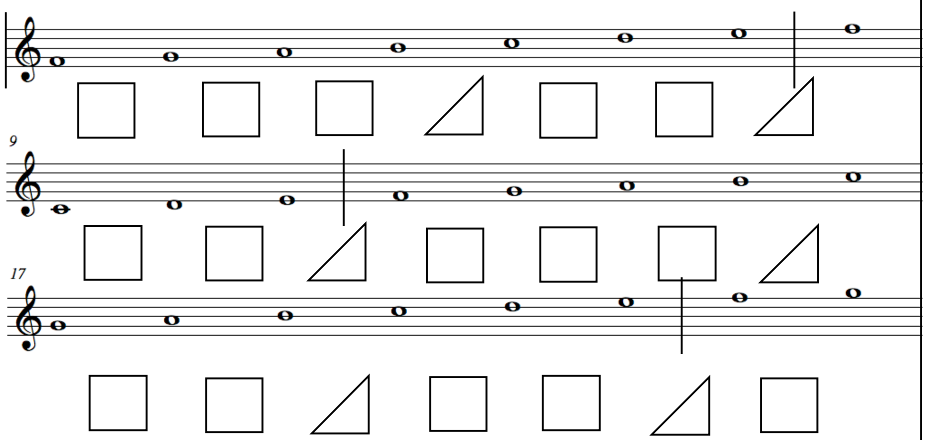


Carey Clampitt Duality

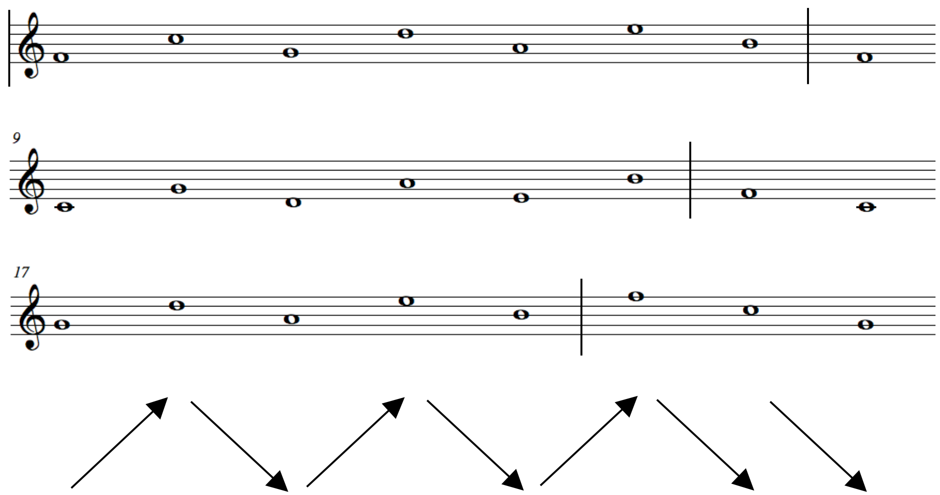


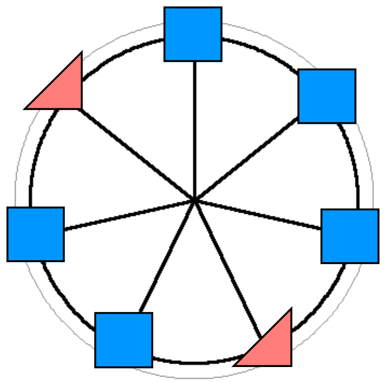
El sistema diatònic standard

La caracterització dels passos



La caracterització de la generació





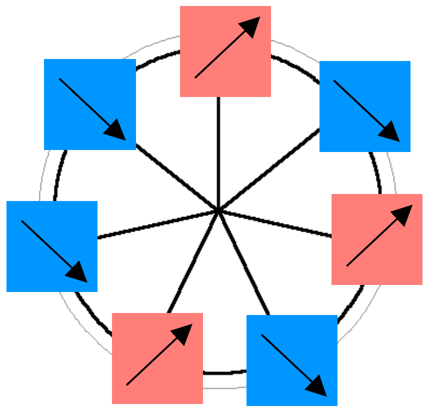
$5 \cdot 3 = 1 \bmod 7$

$5 + 2 = 0 \bmod 7$

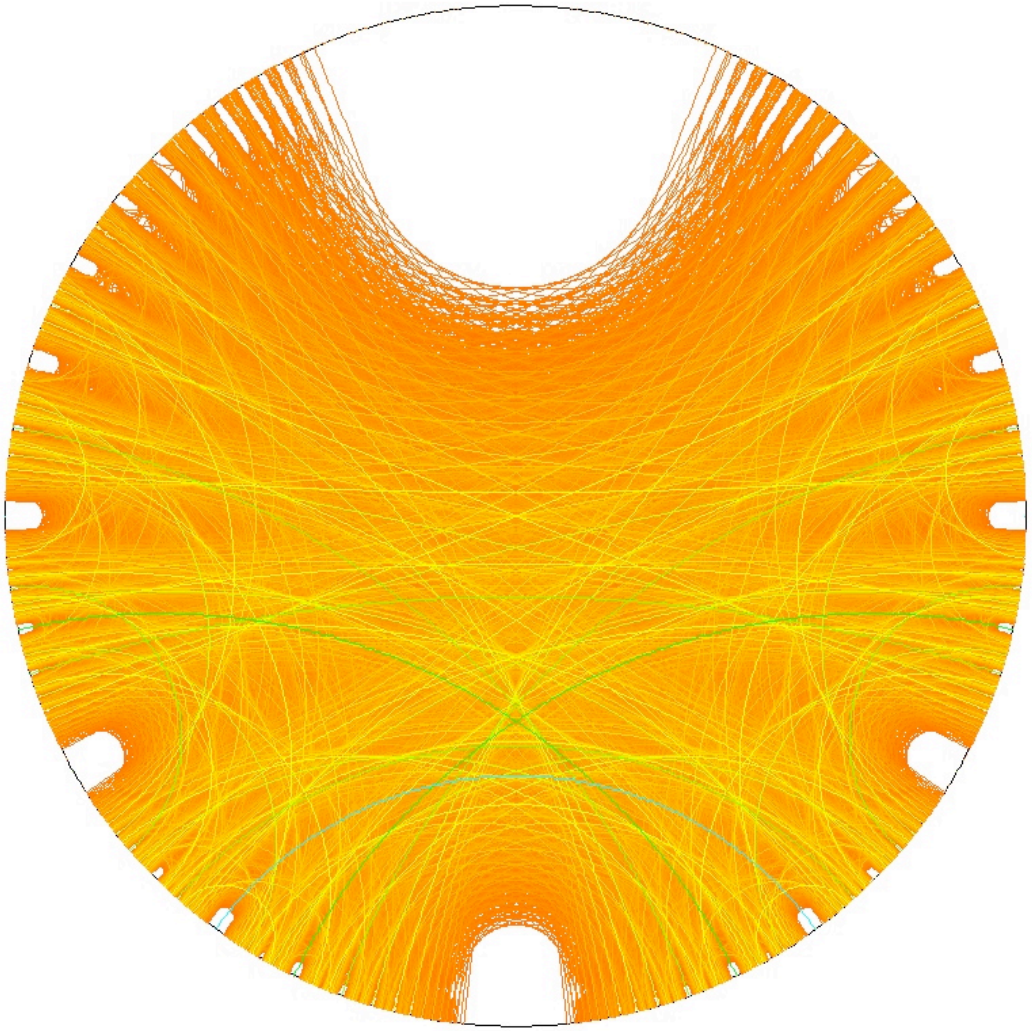
5	3
2	4

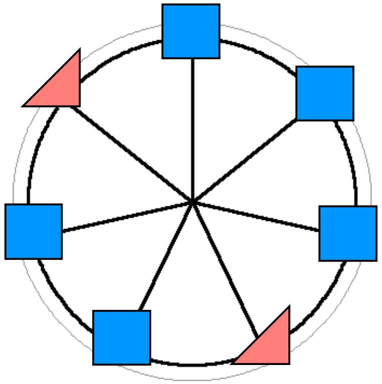
$3 + 4 = 0 \bmod 7$

$2 \cdot 4 = 1 \bmod 7$



Carey-Clampitt Duality

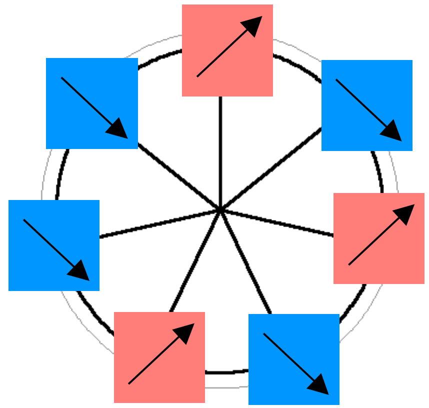




$5 \cdot 3 = 1 \bmod 7$

$5 + 2 = 0 \bmod 7$ $\quad \begin{array}{|c|c|} \hline 5 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \quad$ $3 + 4 = 0 \bmod 7$

$2 \cdot 4 = 1 \bmod 7$



$\frac{4}{7}$ $\quad \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix} \quad$ $\frac{2}{7}$

LRLL **LLLR**

La dualitat de Carey-Clampitt

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longleftrightarrow \begin{pmatrix} d-b & b \\ c+d-a-b & a+b \end{pmatrix}$$

$\frac{a + b}{c + d}$ **L WORD** **L DROW** $\frac{d}{c + d}$