$\mathbb{Z}_{12}$-Story

A MatheMusical Tutorial

Thomas Noll

Escola Superior de Musica de Catalunya, Barcelona
Departament de Teoria i Composició

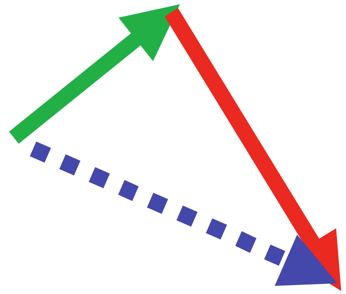
Basic Concepts

Semigroup, Monoid, Group

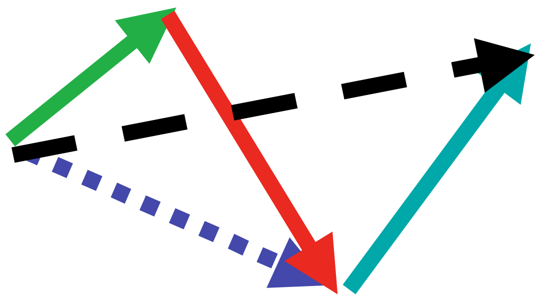
Group Action, Monoid Action

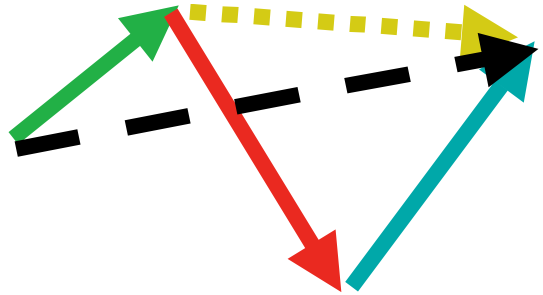
Semigroup

A family of abstract „**acts**“.
Any two acts can be concatenated
and yield another act of this family.



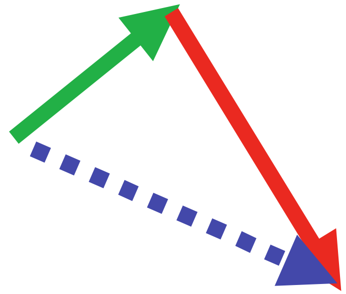
Concatenation
is associative:



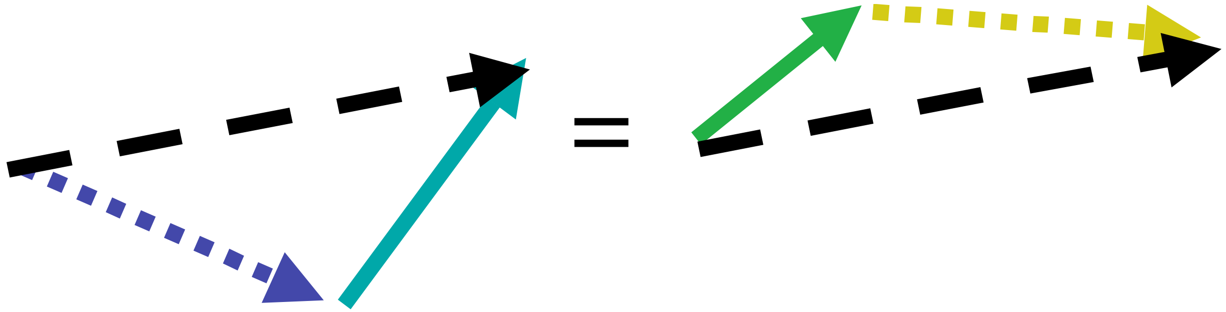


Semigroup

A family of abstract „**acts**“.
Any two acts can be concatenated
and yield another act of this family.

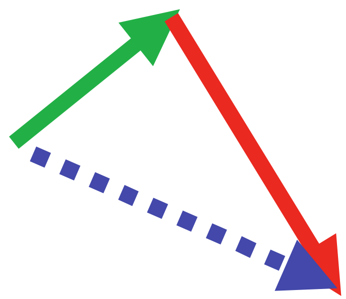


Concatenation
is associative:

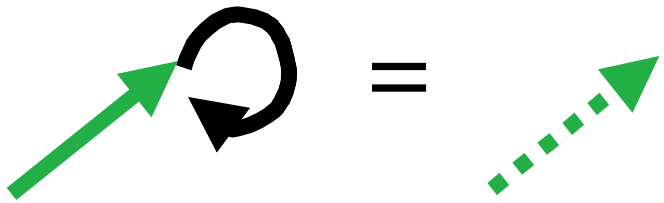


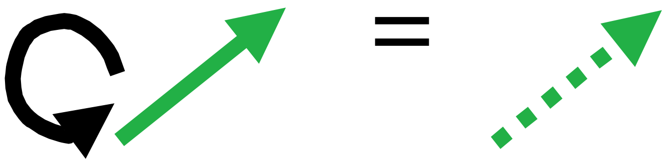
Monoid

A Semigroup.



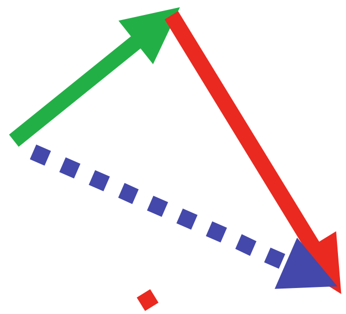
Neutral Element:



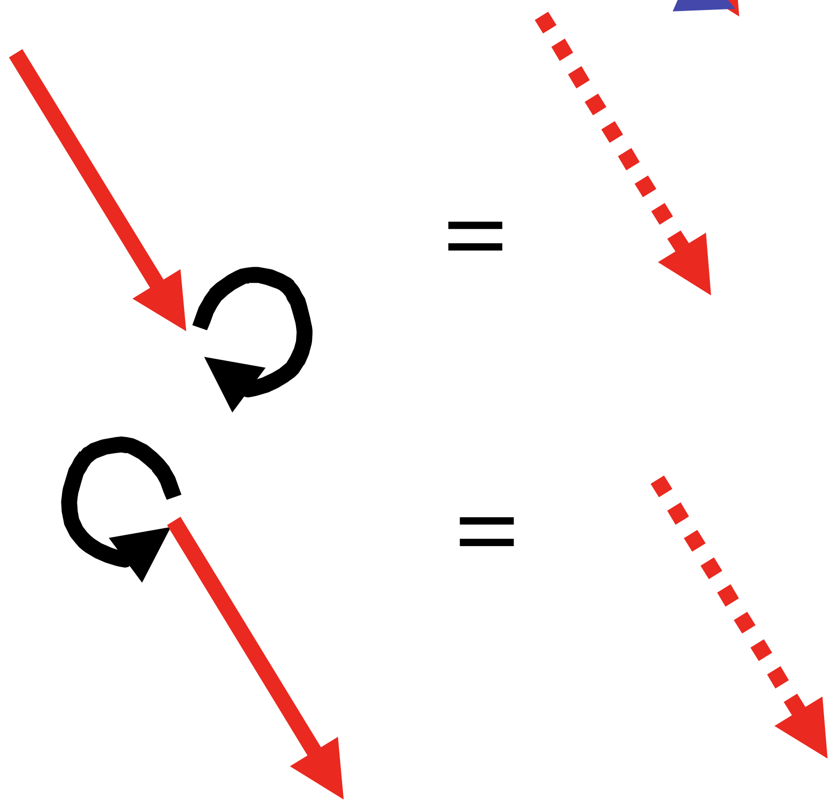


Monoid

A Semigroup.

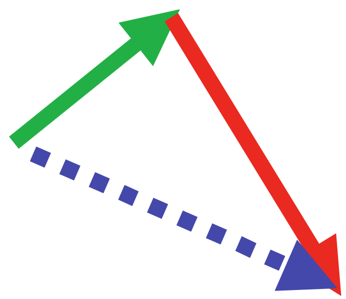


Neutral Element:

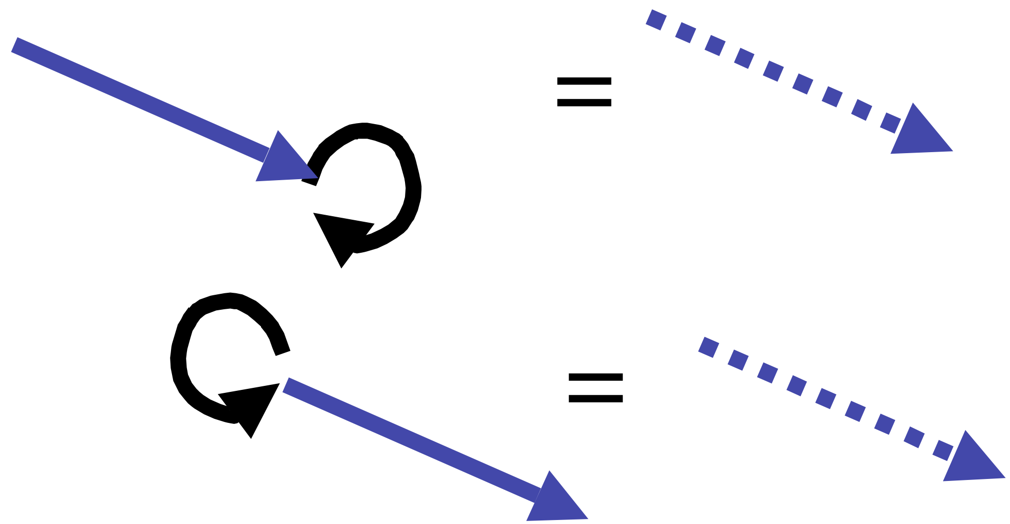


Monoid

A Semigroup.

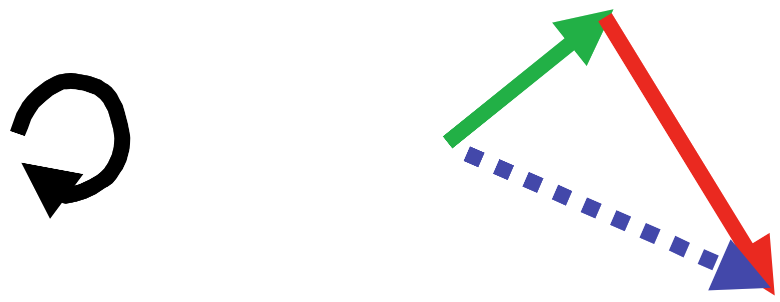


Neutral Element:

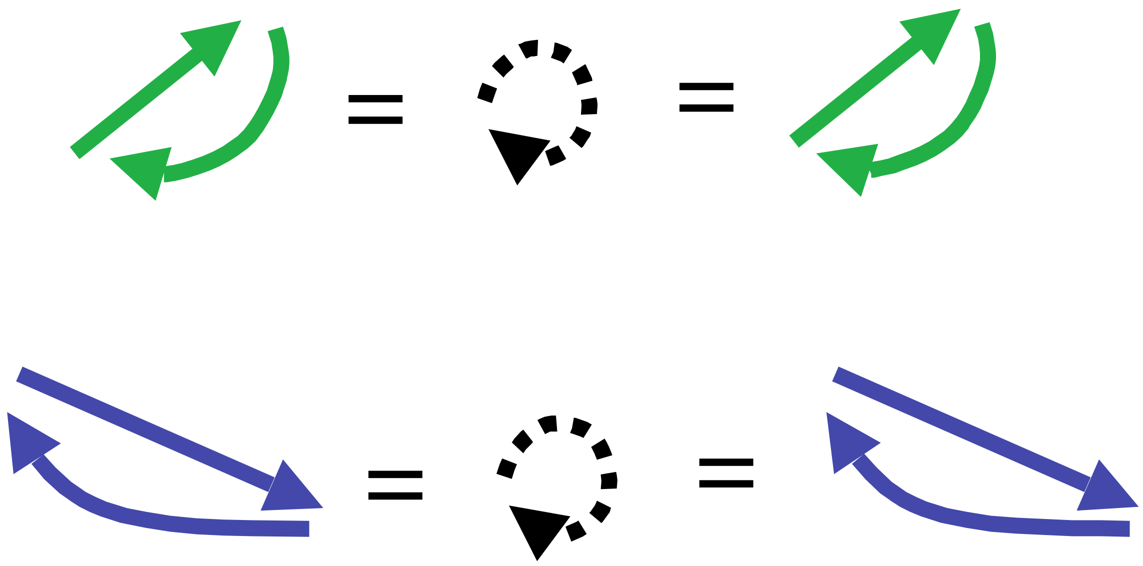


Group

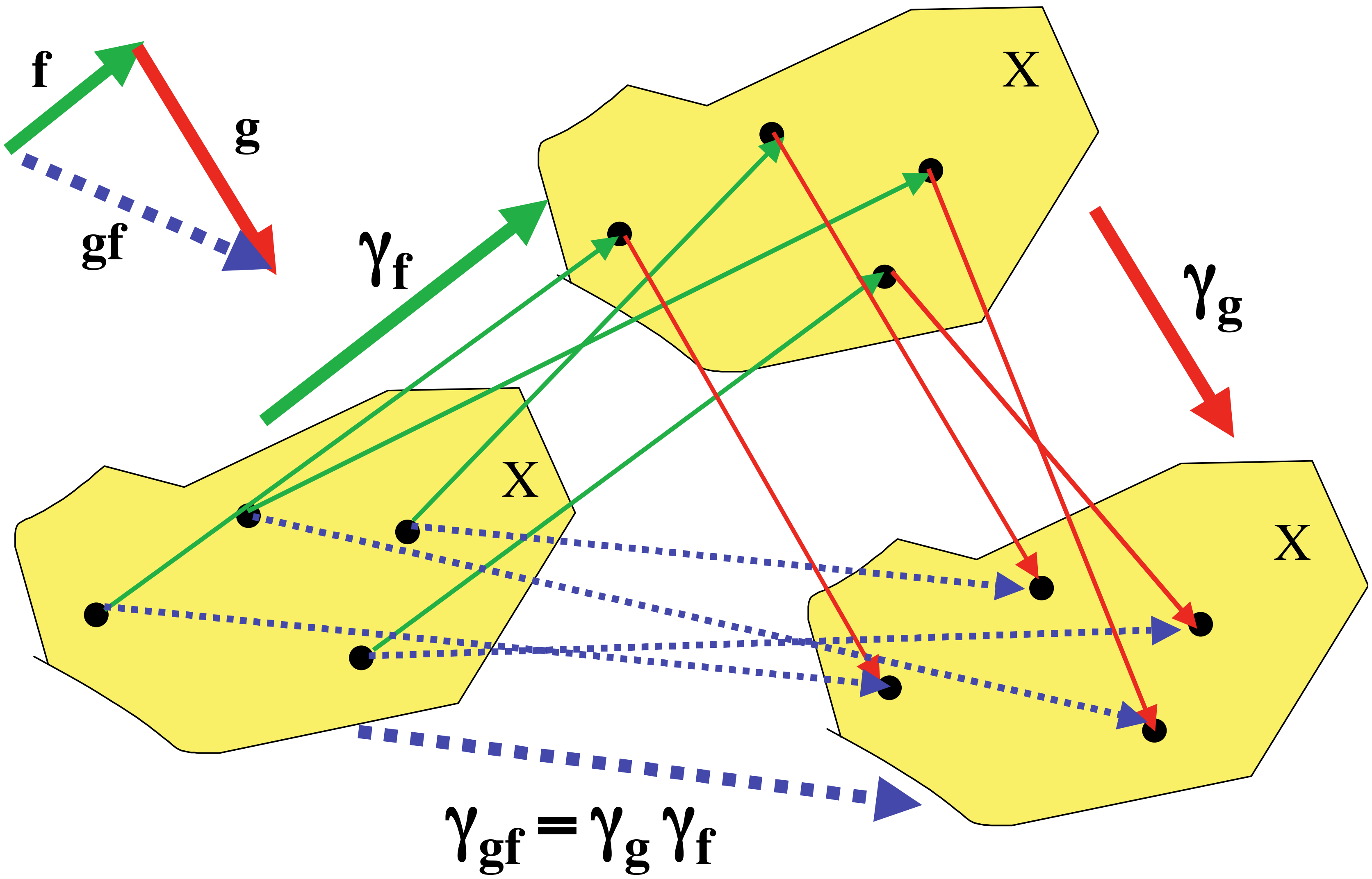
A Monoid.



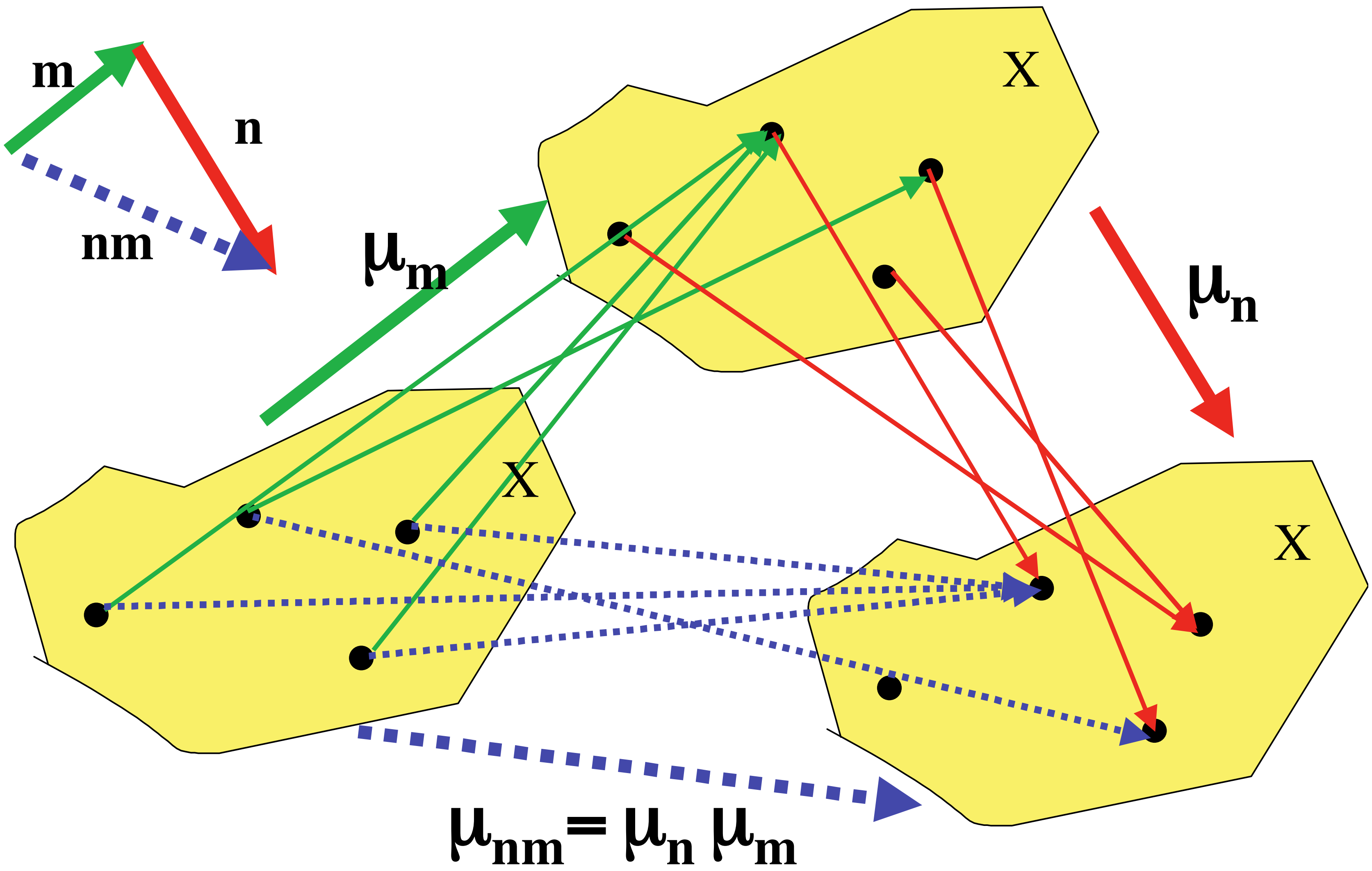
Every act
has an inverse:



Group Action $\gamma: \mathbf{G} \times \mathbf{X} \longrightarrow \mathbf{X}$ on a Set X

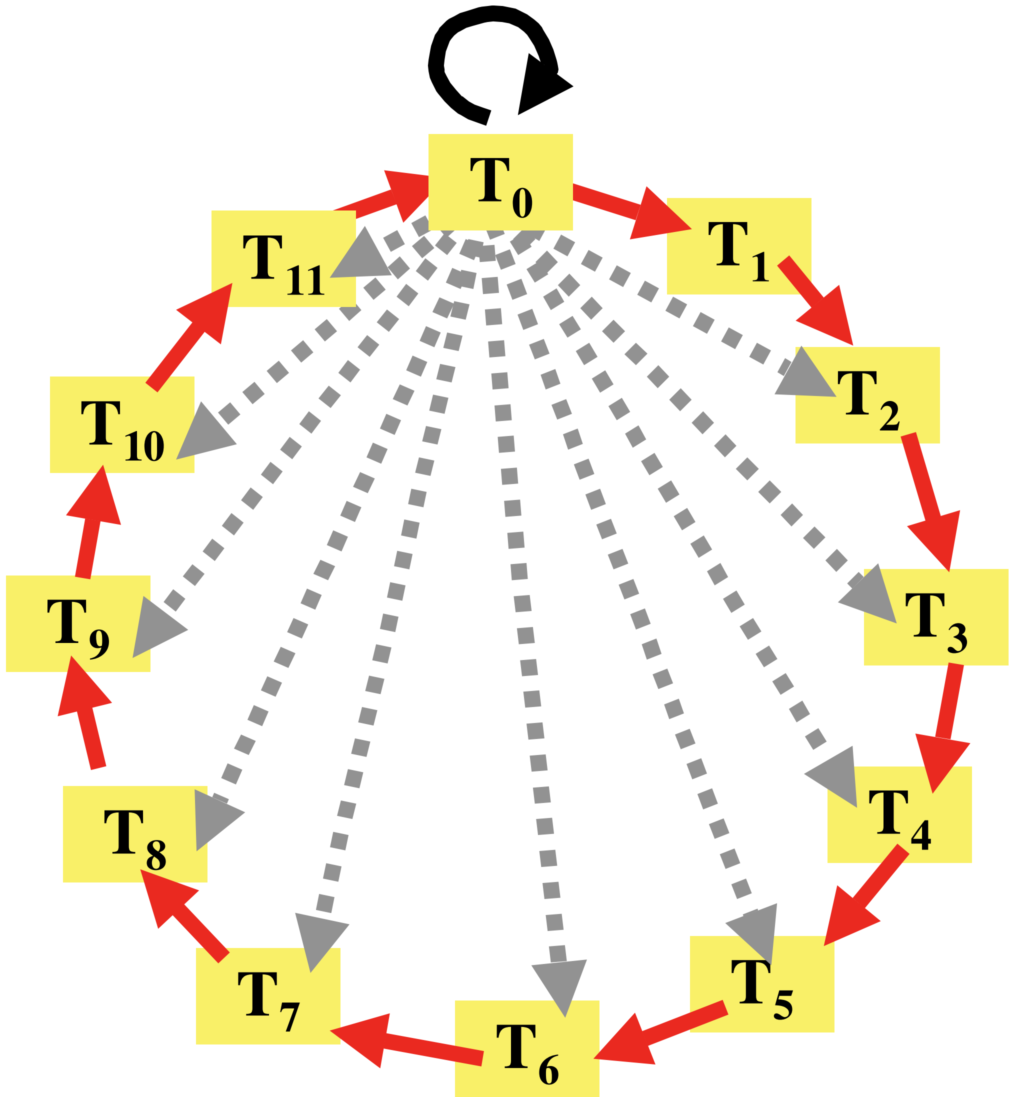


Monoid Action $\mu: \mathbf{M} \times \mathbf{X} \longrightarrow \mathbf{X}$ on a Set X

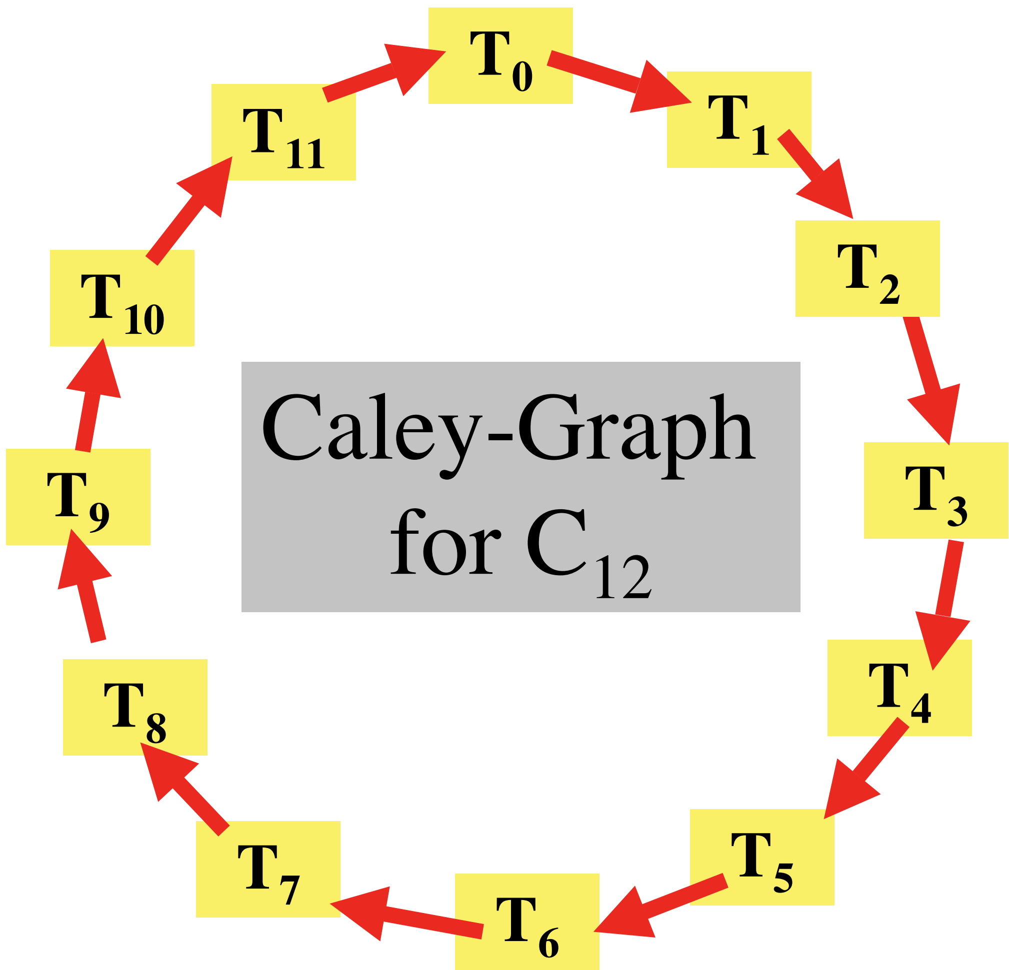


Groups and Group Actions

Musical Examples



Example: Cyclic group of order 12

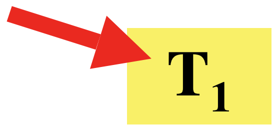


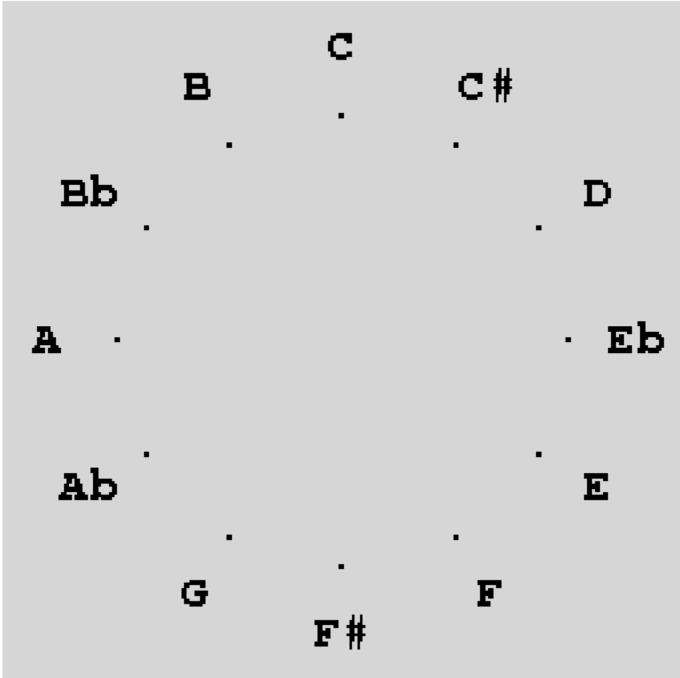
T0
T1
T2
T3
T4
T5
T6
T7
T8
T9
T10
T11
Caley-Graph for C12

Action of C_{12} on Z_{12}

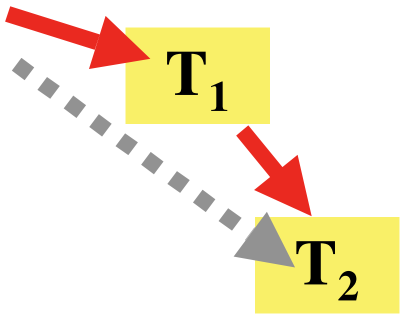
Abstract Acts

Concrete Acts of
Tone Transposition

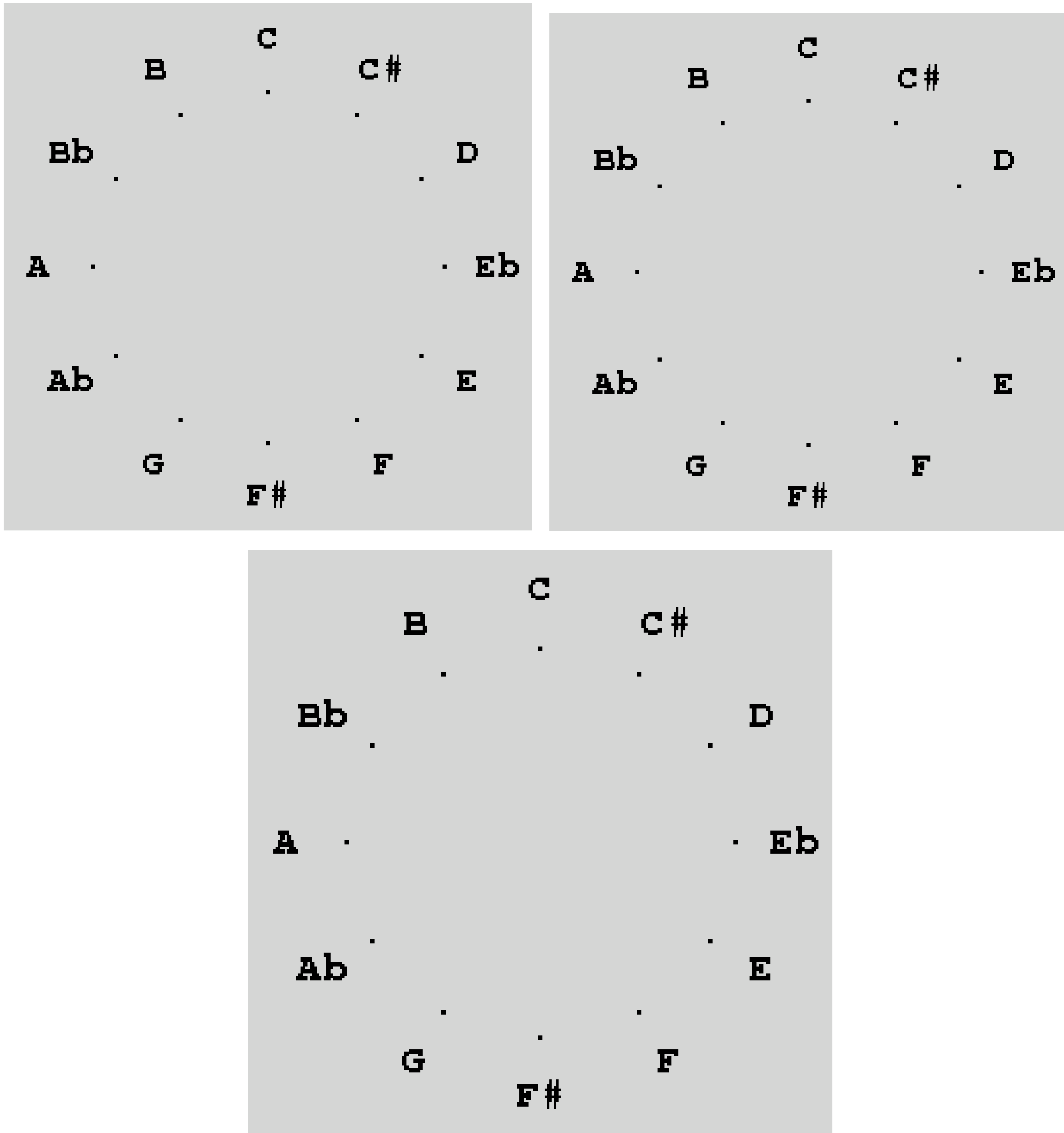




Abstract Concatenation

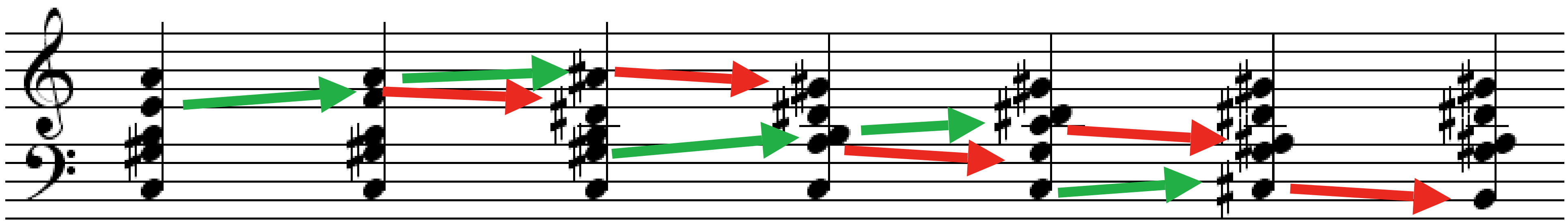


Concrete Concatenation

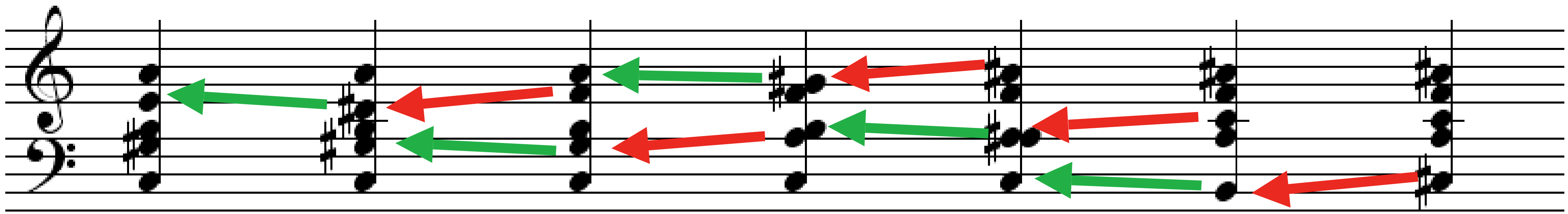


Example: Melodic Progressions as Acts of Transposition
A. Schönberg: *Five Pieces for Orchestra* Op. 16/III: Farben

Bars 1- 9



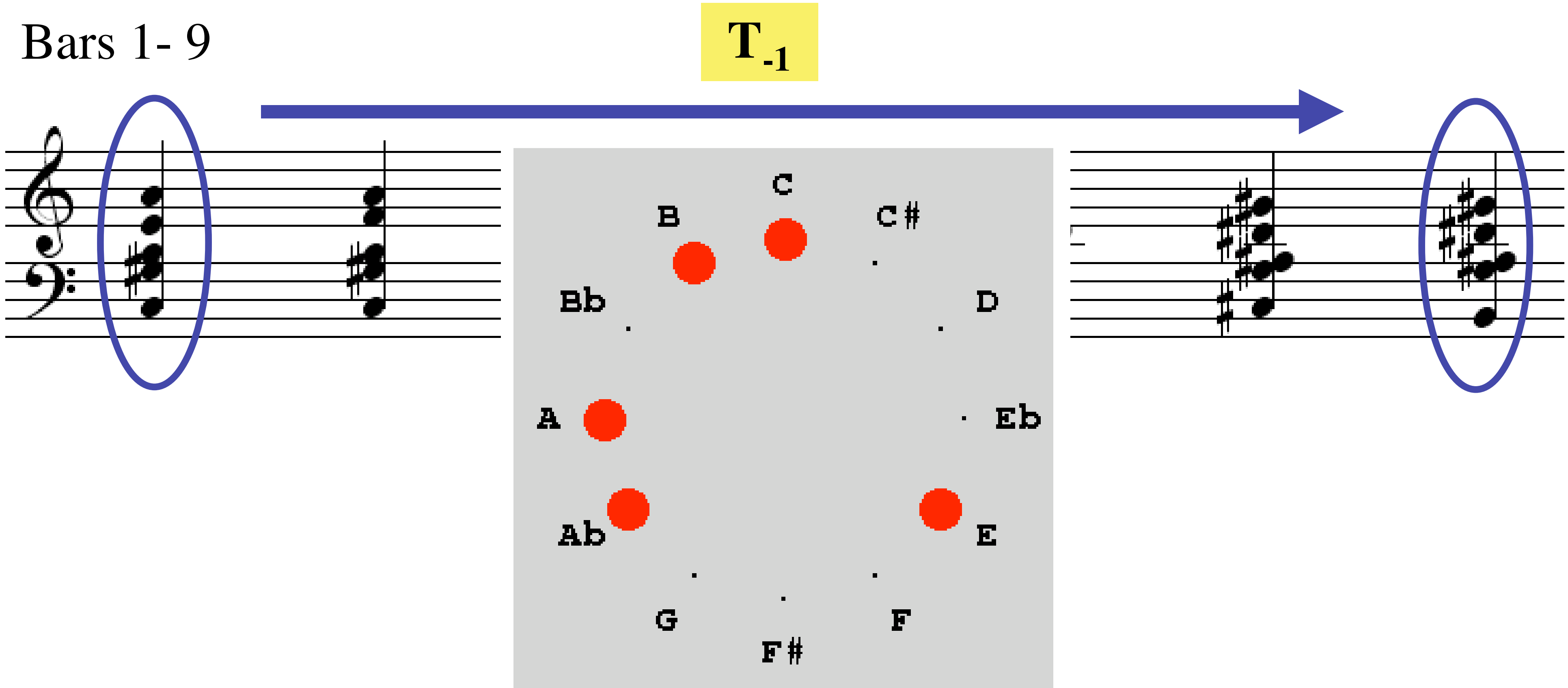
Bars 32- 38



Example: Chord Relations as Acts of Transposition

A. Schönberg: *Five Pieces for Orchestra* Op. 16/III: Farben

Bars 1- 9



Two Actions of C_{12} („T - Actions“)

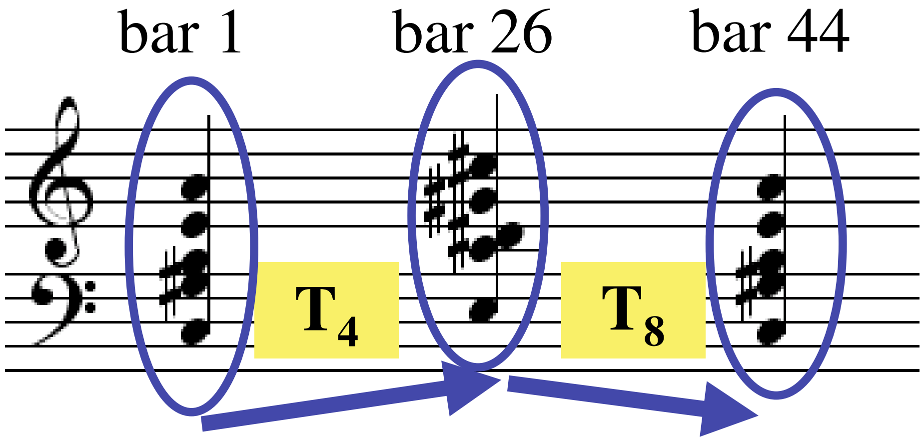
Transposition of Tones („pitch classes“)

$$\mathbf{C_{12} \times Z_{12} \longrightarrow Z_{12}}$$

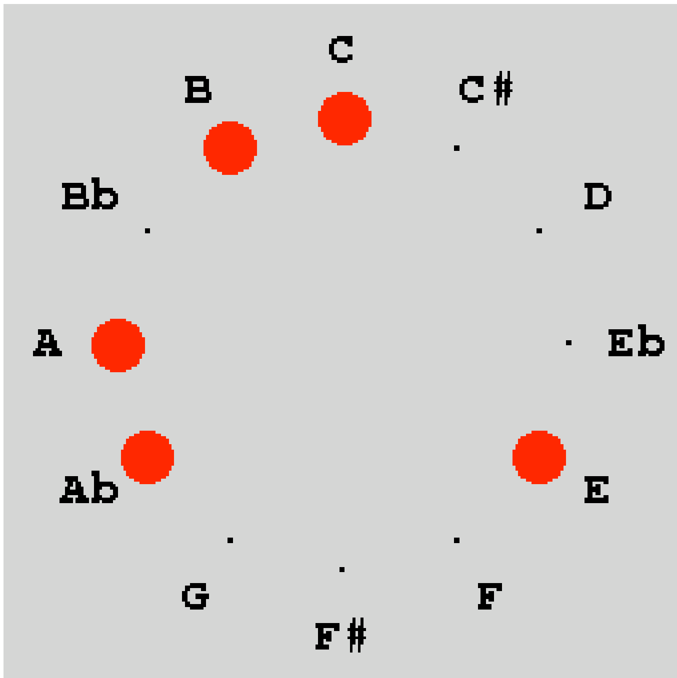
Transposition of Chords („pitch class sets“)

$$\mathbf{C_{12} \times Subsets(Z_{12}) \longrightarrow Subsets(Z_{12})}$$

Musically Interesting: Common Tone Relations

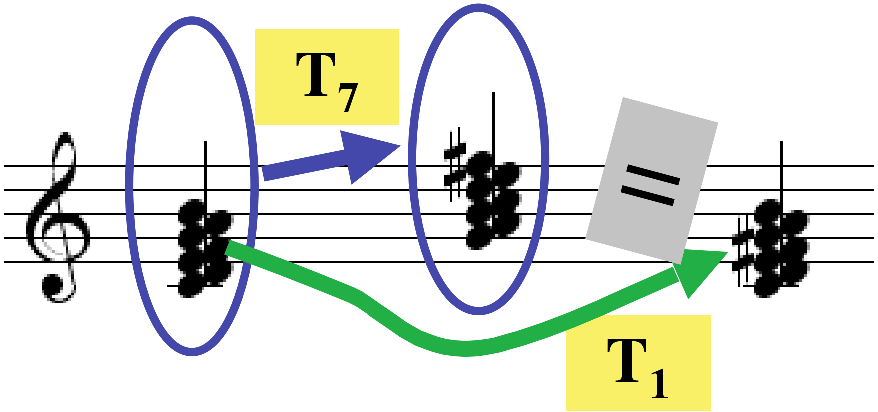


3 common tones



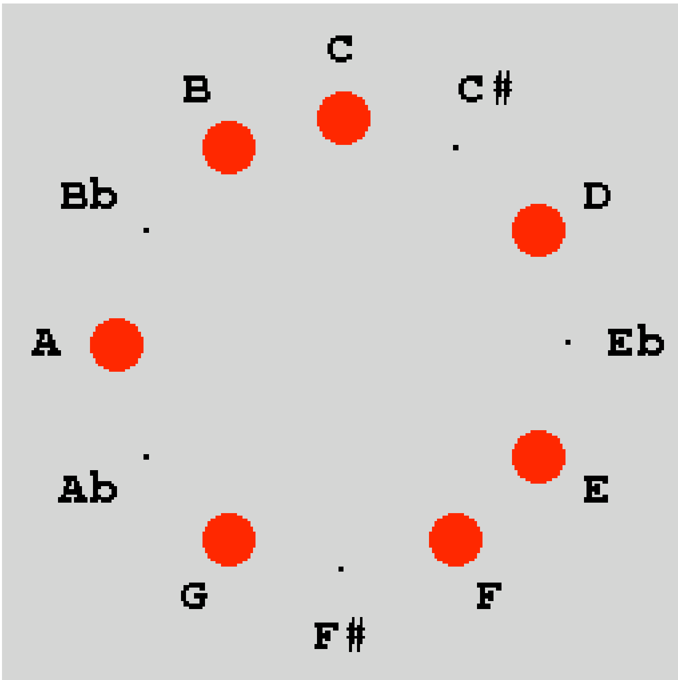
Diatonic Scale (Balzano 1980):

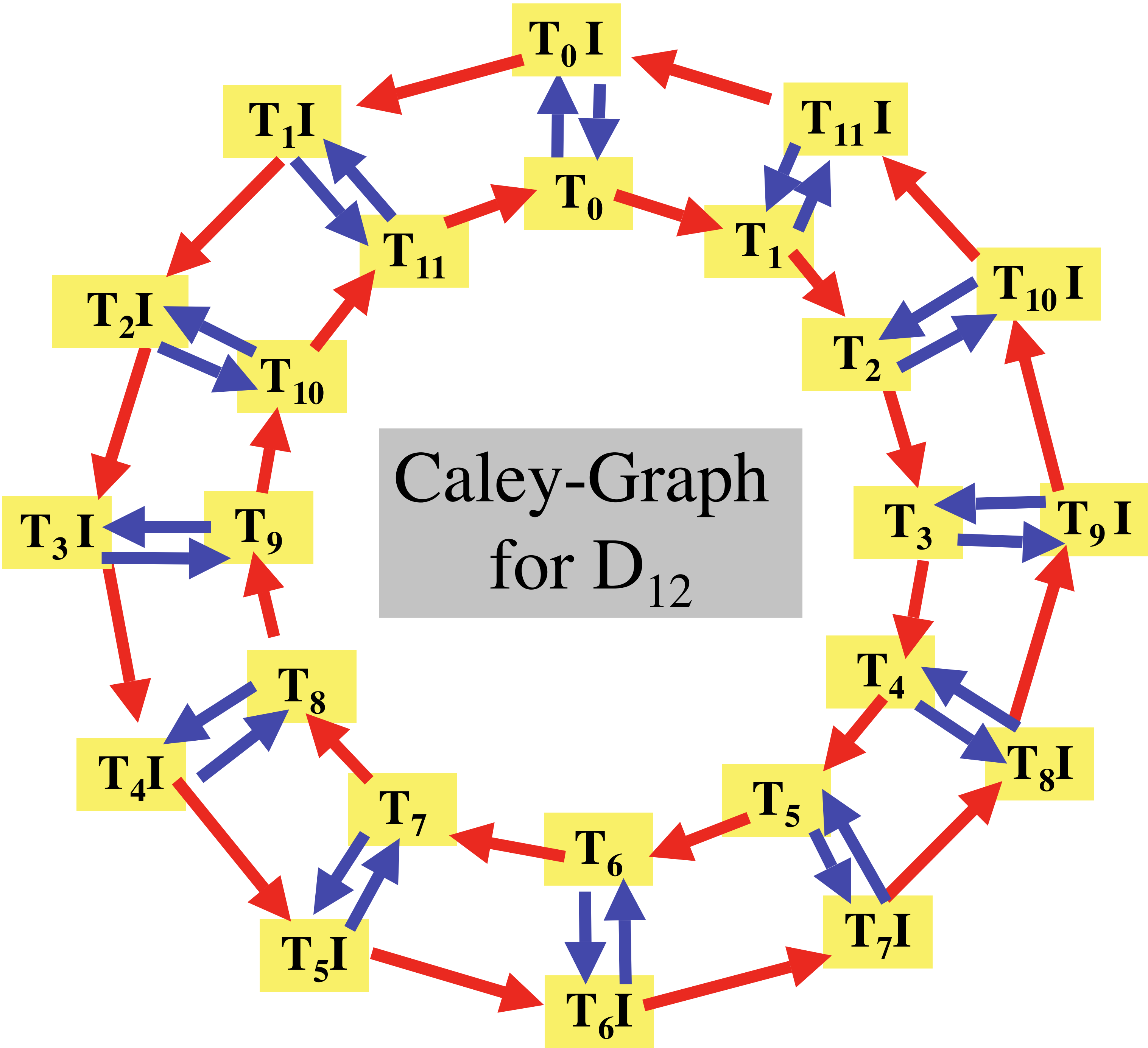
Solidarity between Fifth Transposition and Chromatic Alteration



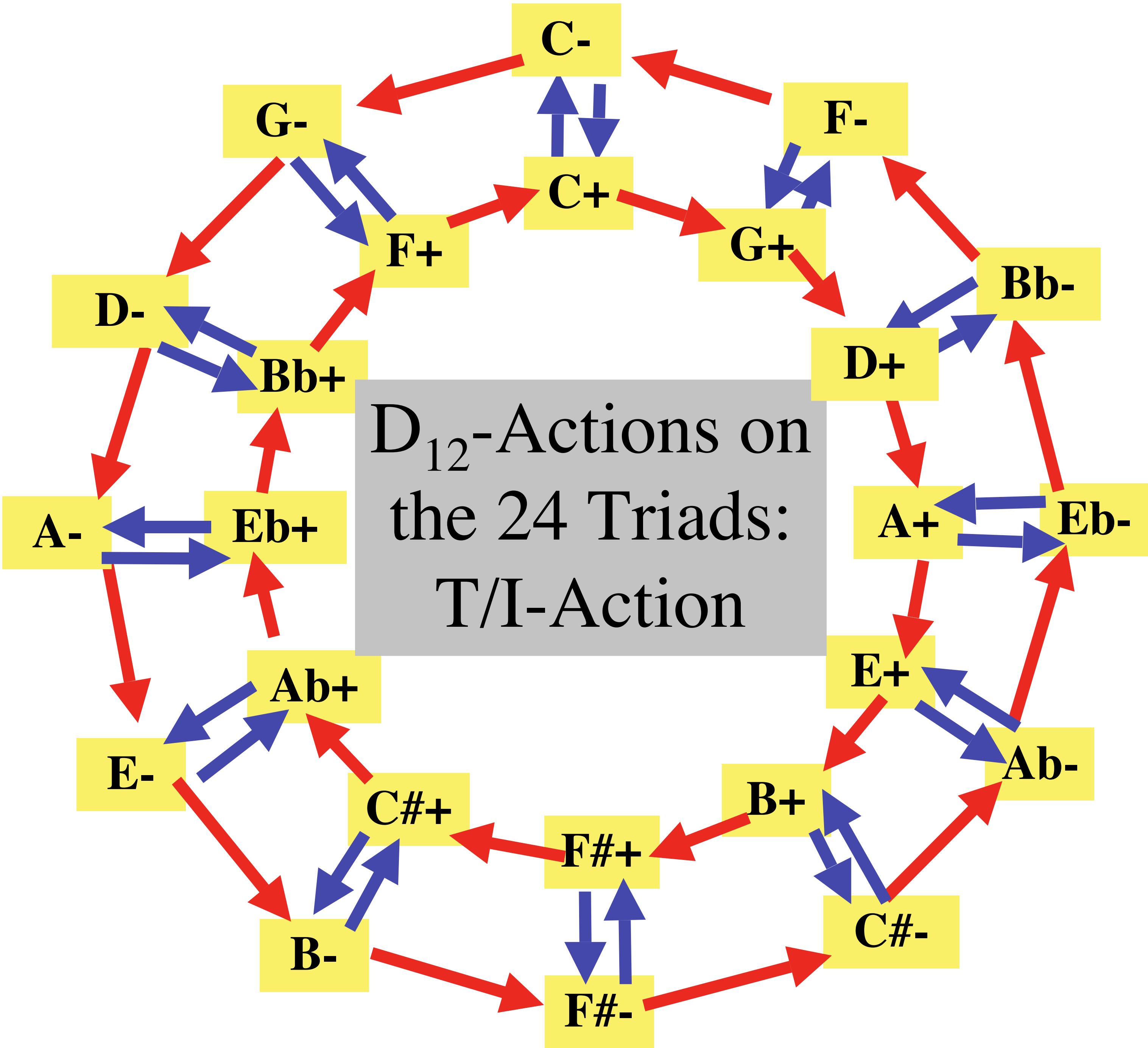
6 common tones

Minimal voiceleading

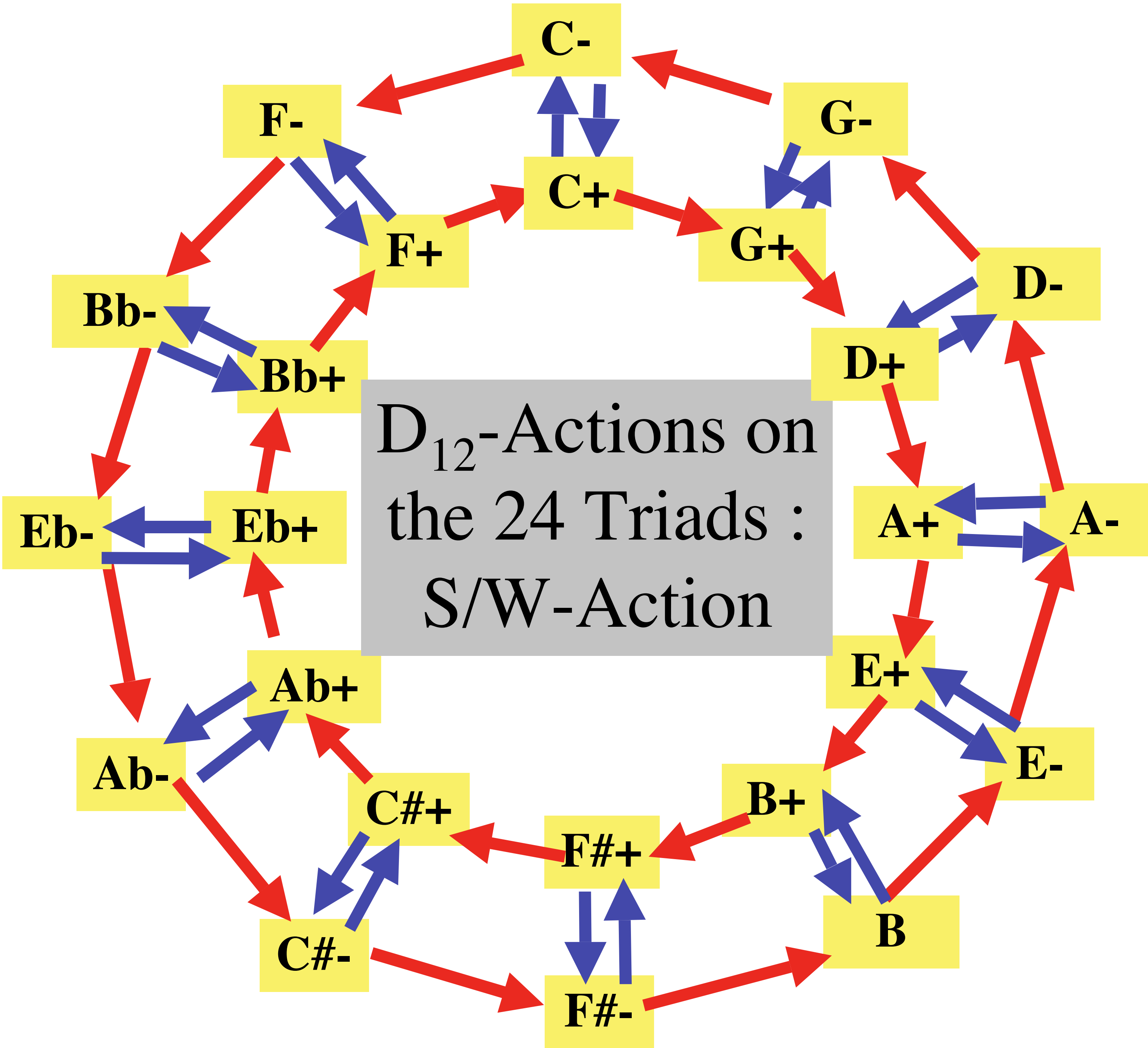




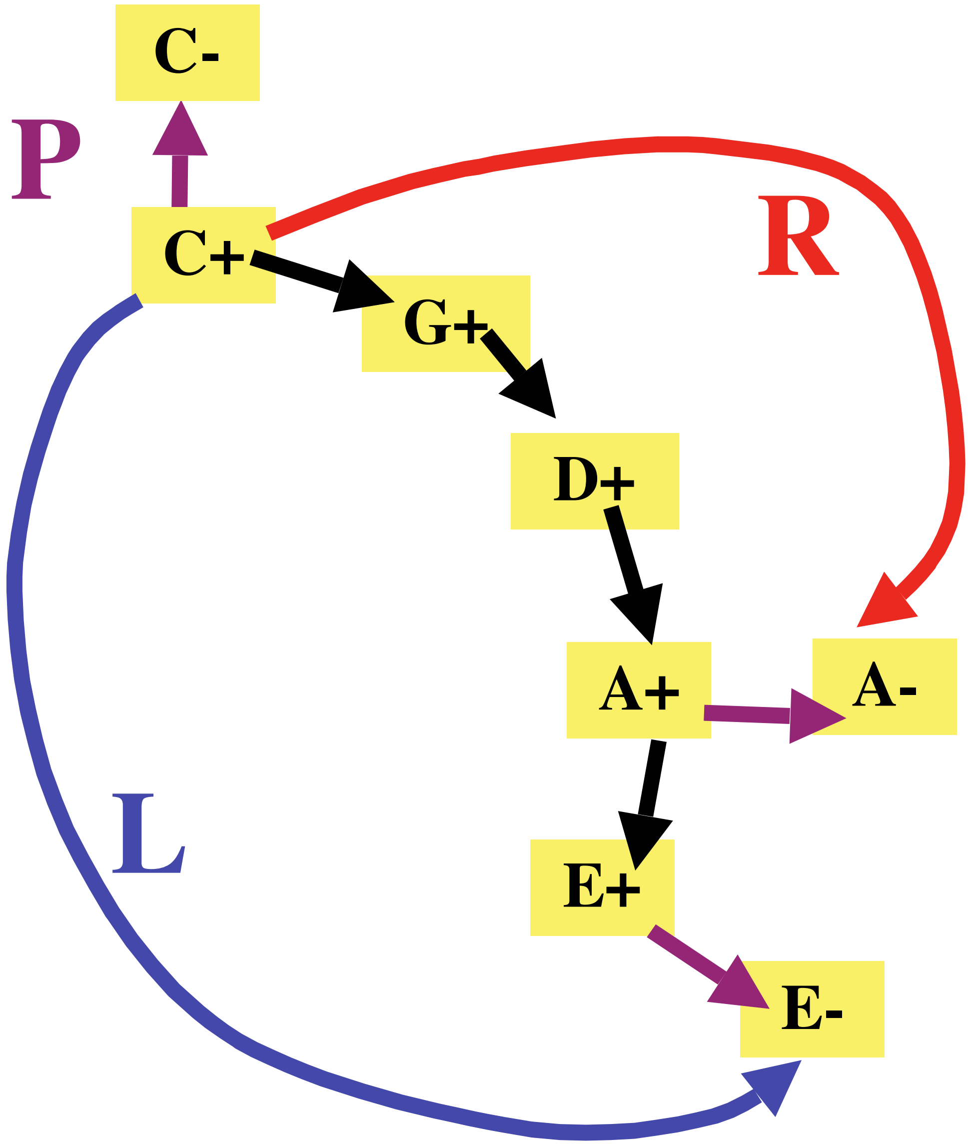
Caley-Graph for D_{12}
$T_0 I$
$T_1 I$
$T_{11} I$
T_0
T_{11}
T_1
$T_2 I$
$T_{10} I$
T_{10}
T_2
$T_3 I$
T_9
T_3
$T_9 I$
T_8
T_4
$T_4 I$
$T_8 I$
T_7
T_5
T_6
$T_5 I$
$T_7 I$
$T_6 I$



D_{12}-Actions on the 24 Triads: T/I-Action
C-
F-
Bb-
Eb-
Ab-
C#-
F#-
B-
E-
A-
D-
G-
C+
G+
D+
A+
E+
B+
F#+
C#+
Ab+
Eb+
Bb+
F+


D12-Actions on
the 24 Triads :
S/W-Action
C-
G-
D-
A-
E-
B
F#-
C#-
Ab-
Eb-
Bb-
F-
C+
G+
D+
A+
E+
B+
F#+
C#+
Ab+
Eb+
Bb+
F+

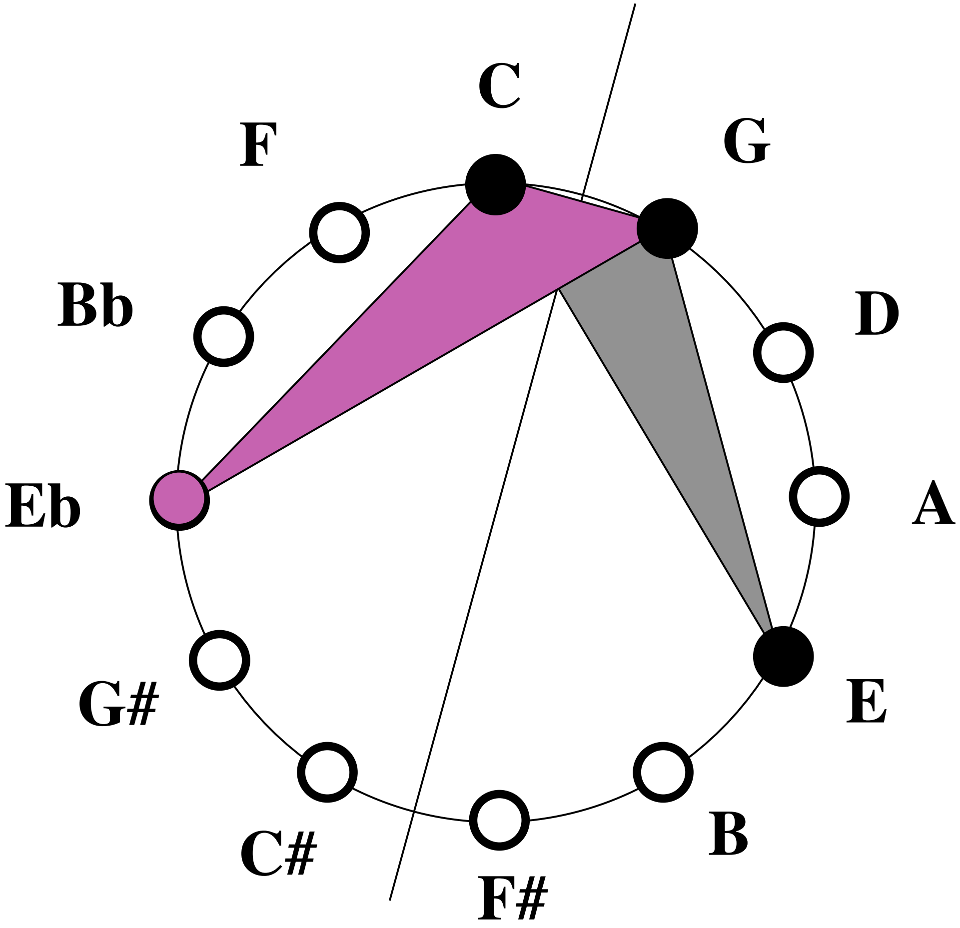
Neo-Riemannian Transformations As Alternative Generators

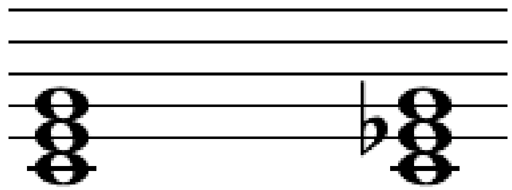


Parallel
Leading-Tone
Relative

Hugo Riemann

(Neo-)Riemannian Operation **P = „Parallel“**

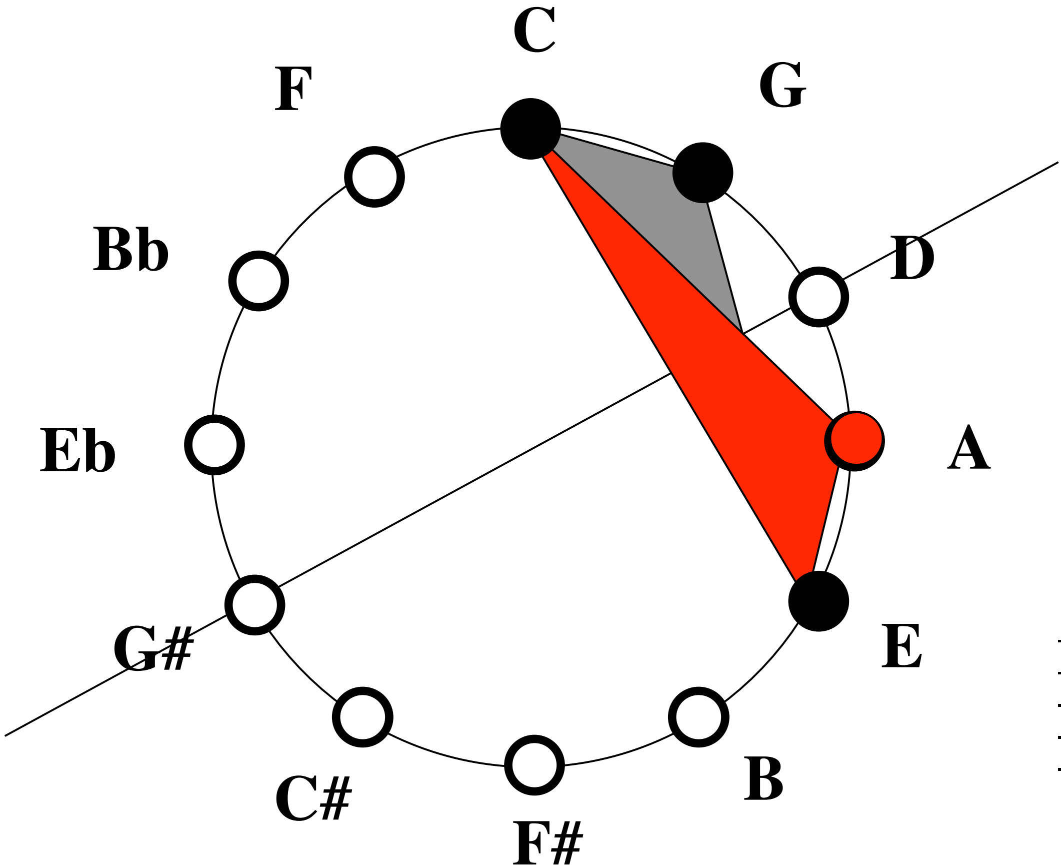


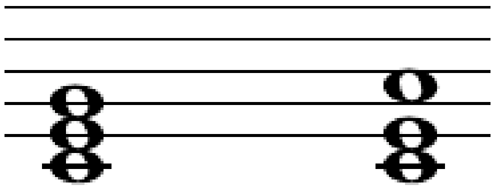


R. Cohn: „The overdetermined triad“: Voice leading parsimony

Hugo Riemann

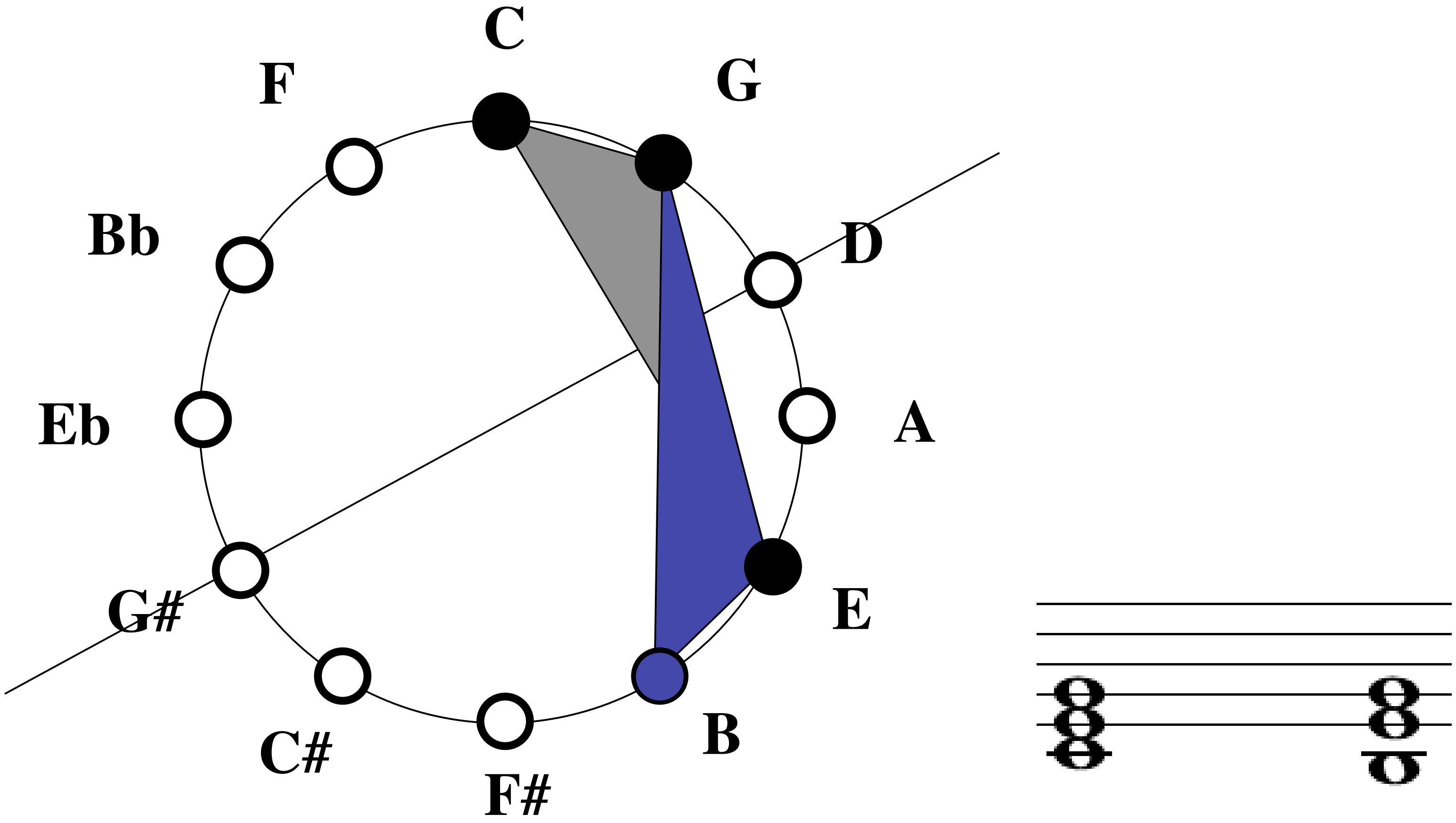
(Neo-)Riemannian Operation **R = „Relative“**



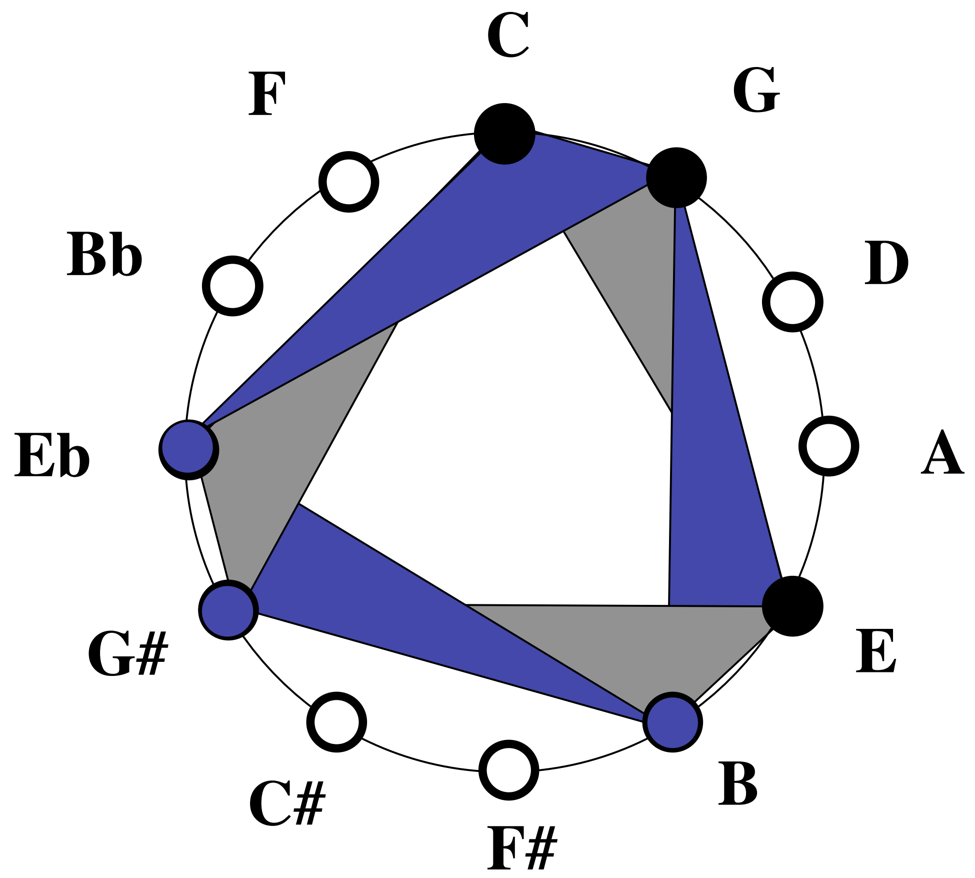


Hugo Riemann

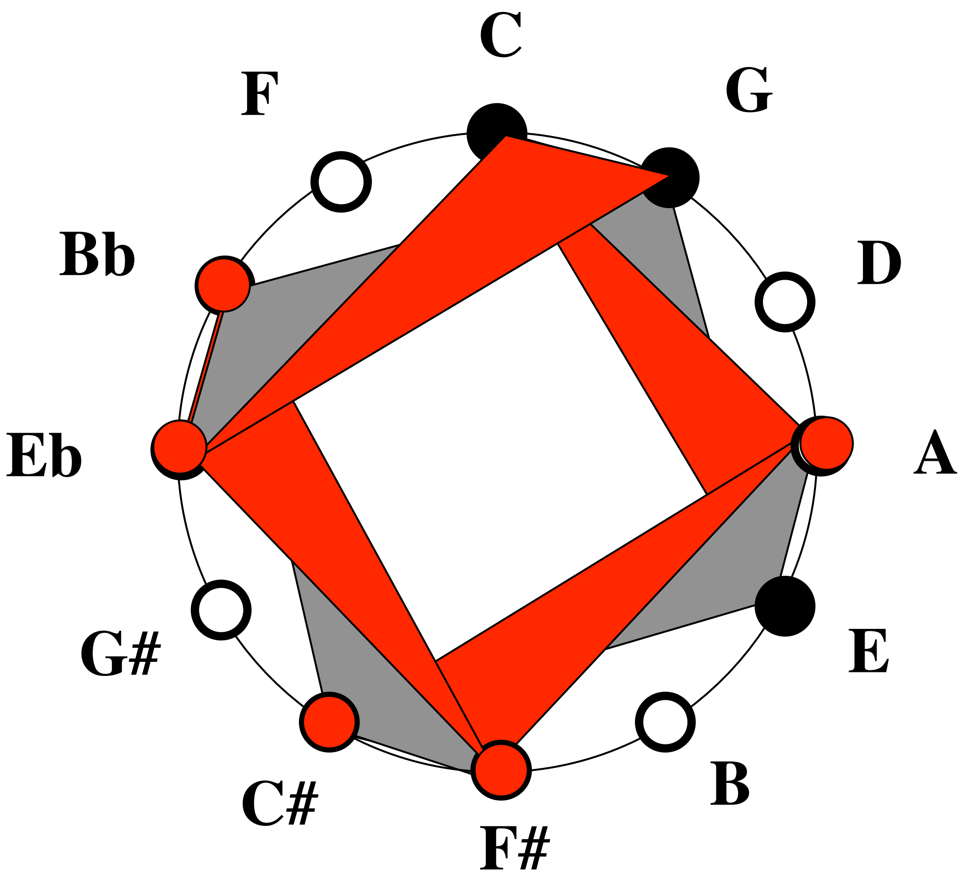
(Neo-)Riemannian Operation **L = „Leading-Tone“**

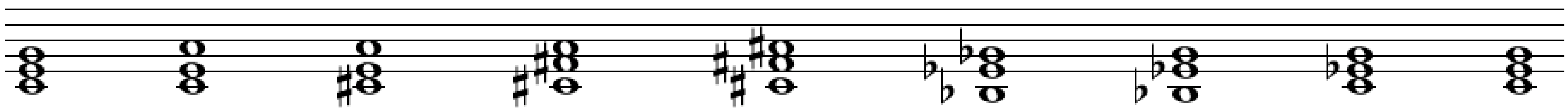


Hexatonic System **<L, P>**



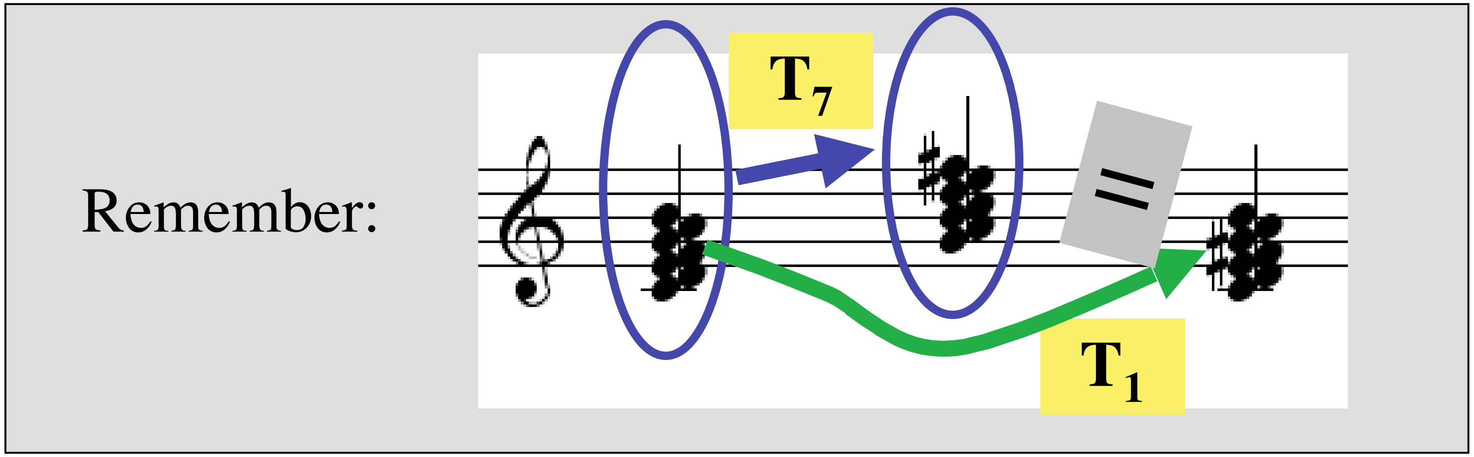
Octatonic System **<R, P>**

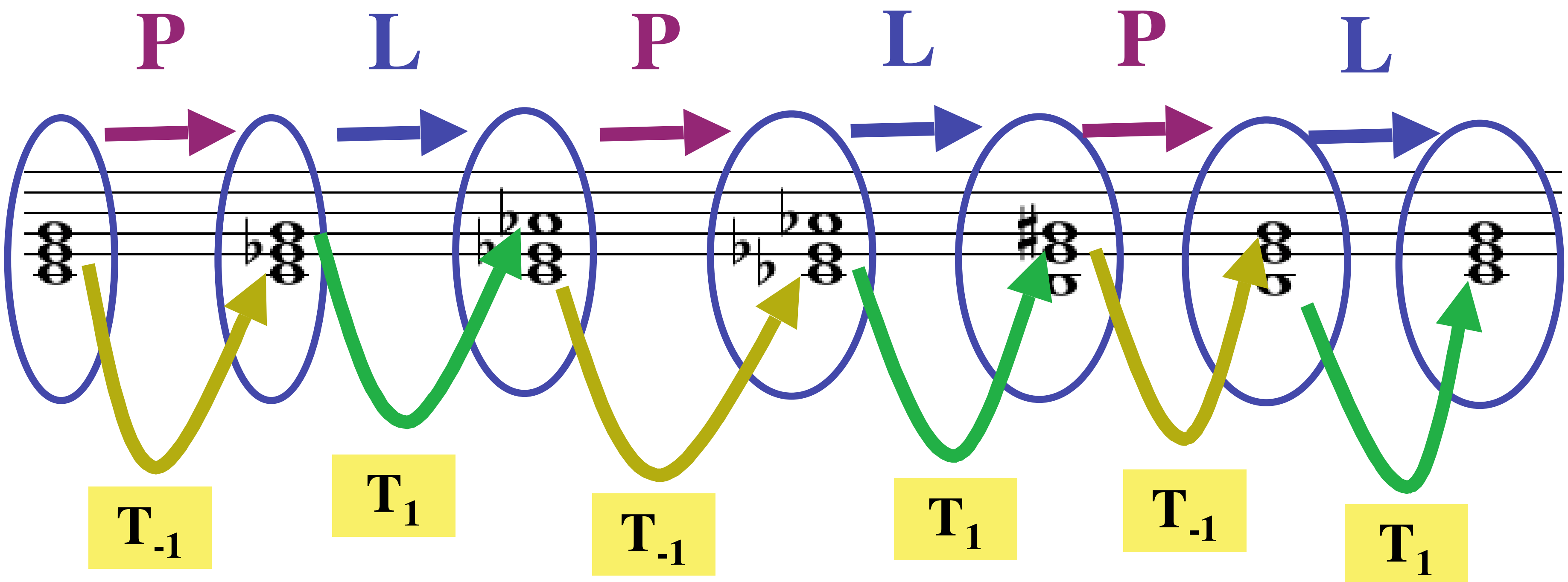




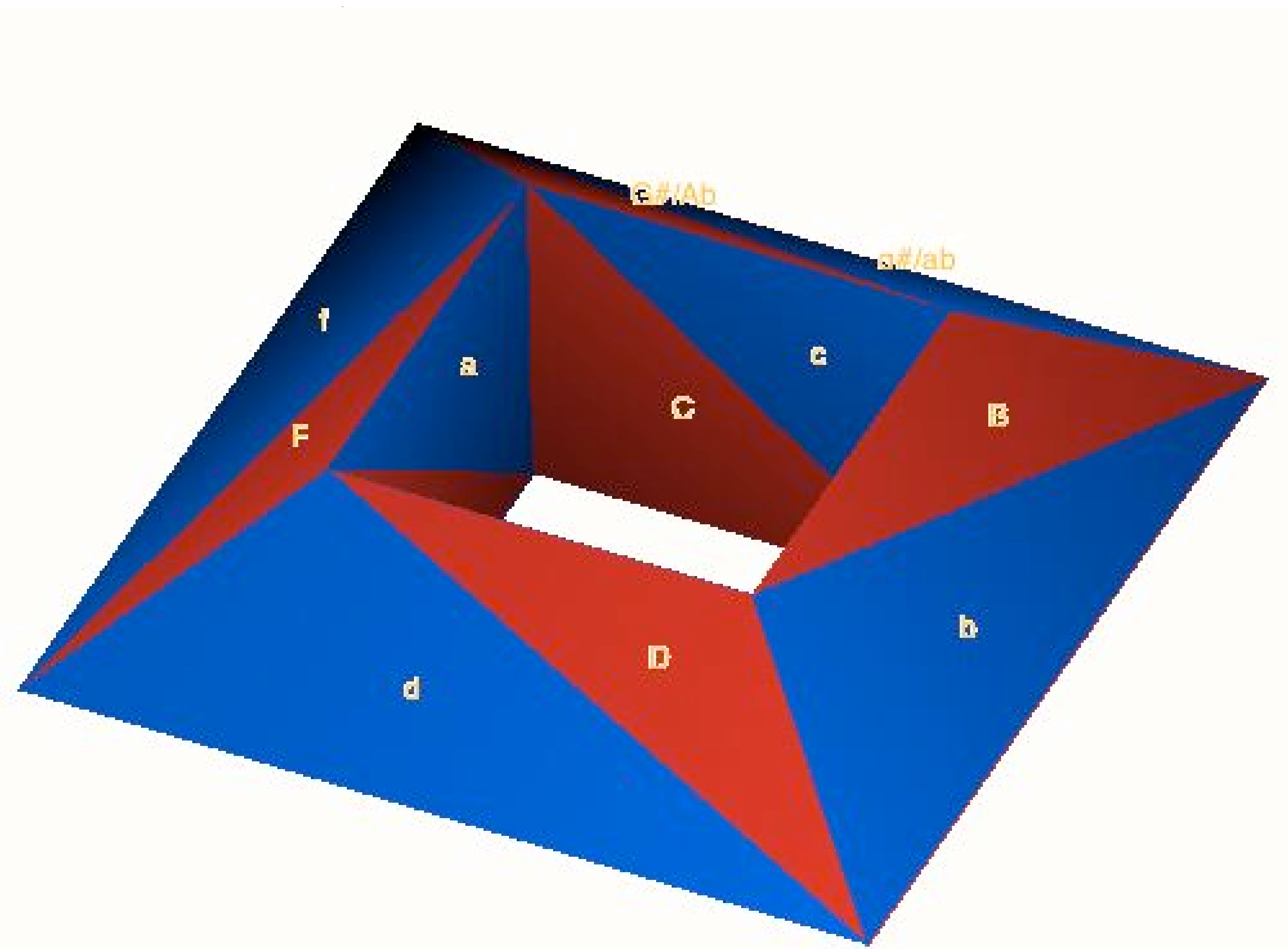
Hexatonic System (Cohn 1996):

Solidarity between Transformation and Voiceleading Parsimony

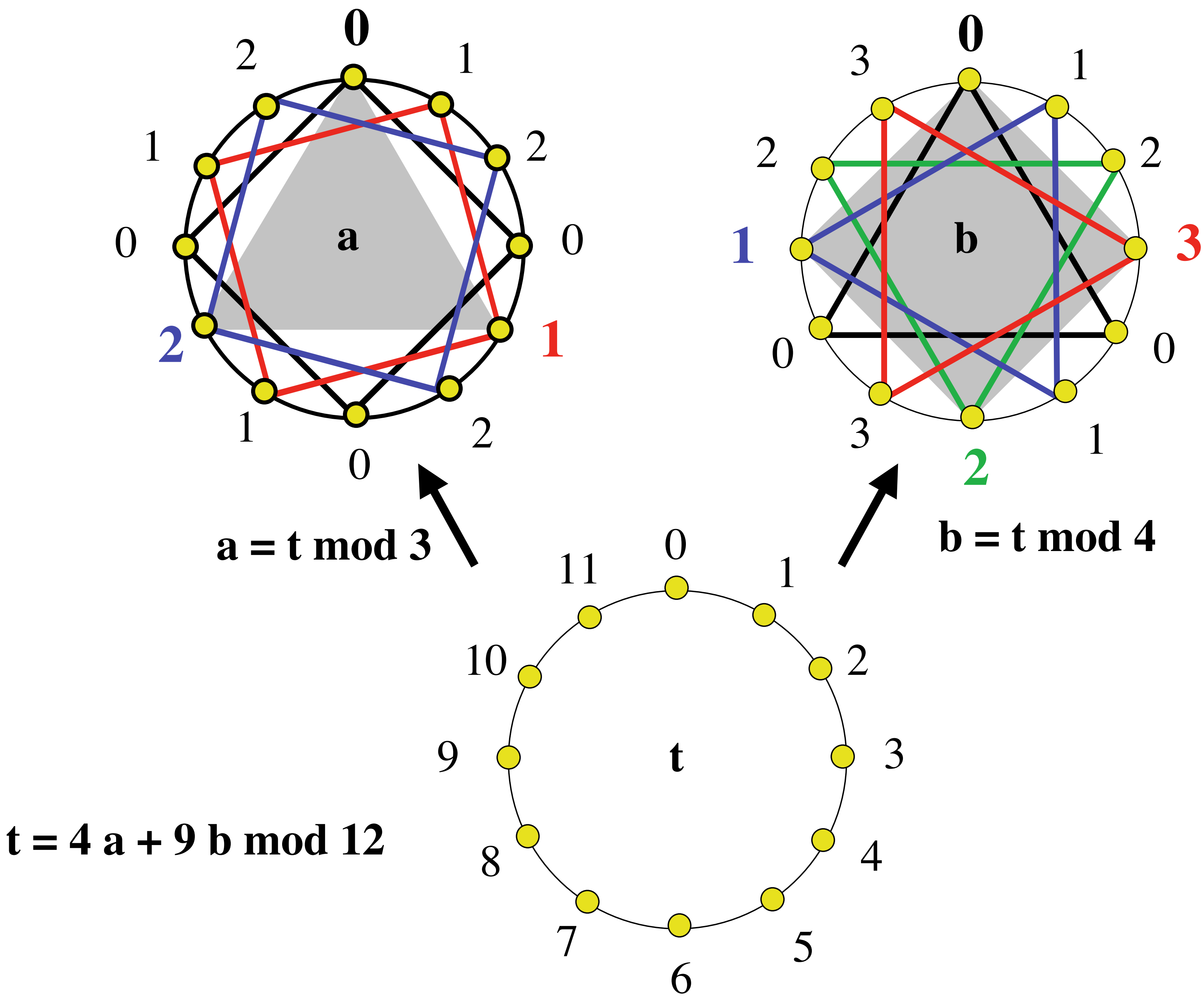




Decomposition, Stabilizers, Conjugation etc.



G#/Ab
g#/ab
f
a
c
C
B
F
b
D
d



a
b
t
a = t mod 3
b = t mod 4
t = 4 a + 9 b mod 12
0
1
2
3
4
5
6
7
8
9
10
11

Full Symmetry Group of affine Transformations of Z_{12}

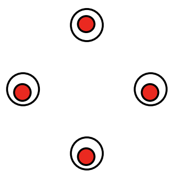
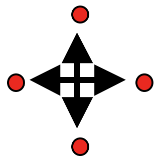
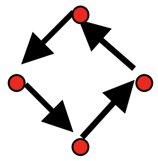
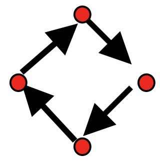
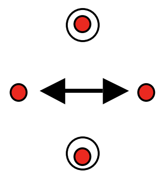
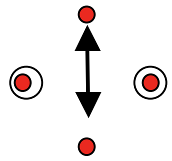
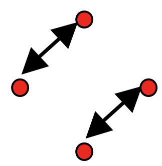
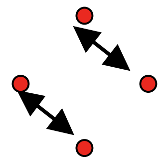
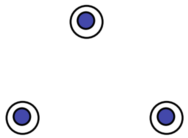
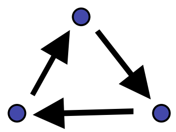
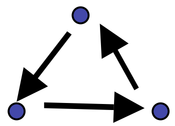
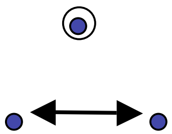
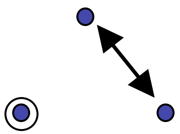
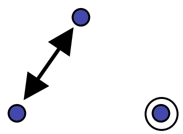
$$Sym(Z_{12}) = Z_{12} \rtimes Z_{12}^{*}$$

Translations

Kleinian Four-Group

1	5	7	11
5	1	11	7
7	11	1	5
11	7	5	1

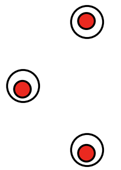
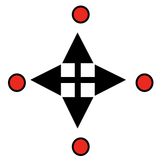
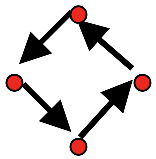
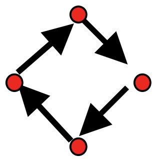
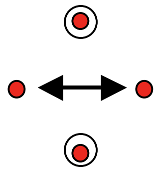
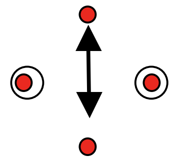
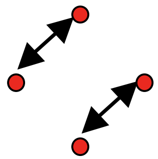
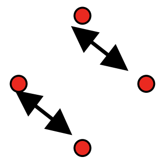
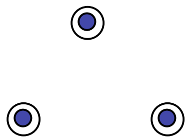
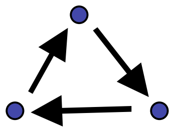
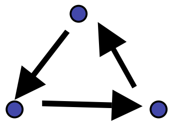
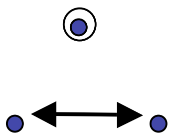
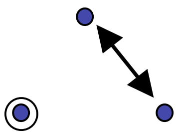
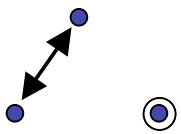
(t, u) (s, v) = (t + u s, u v)

								
	1,0	1,6	1,9	1,3	7,0	7,6	7,9	7,3
	1,4	1,10	1,1	1,7	7,4	7,10	7,1	7,7
	1,8	1,2	1,5	1,11	7,8	7,2	7,5	7,11
	5,0	5,6	5,9	5,3	11,0	11,6	11,9	11,3
	5,4	5,10	5,1	5,7	11,4	11,10	11,1	11,7
	5,8	5,2	5,5	5,11	11,8	11,2	11,5	11,11

$$\gamma: G \times X \longrightarrow X$$

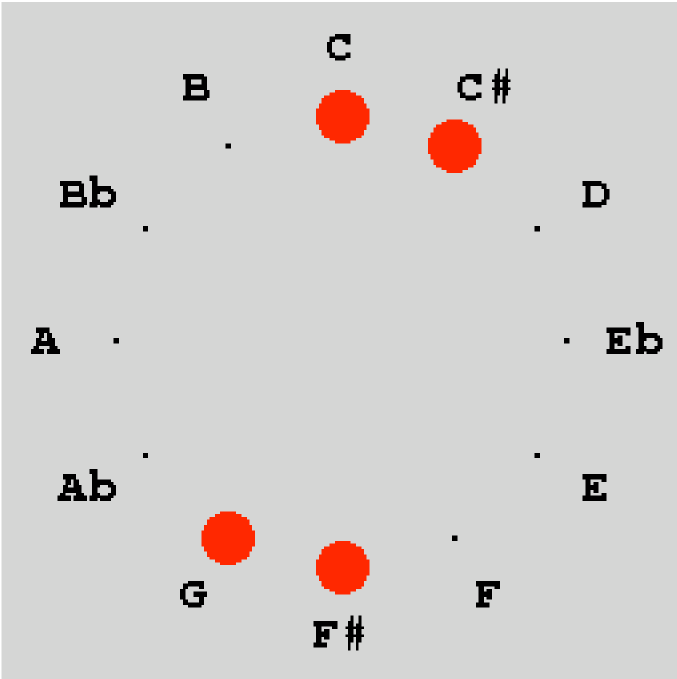
Stabilizer of an Element x of X:

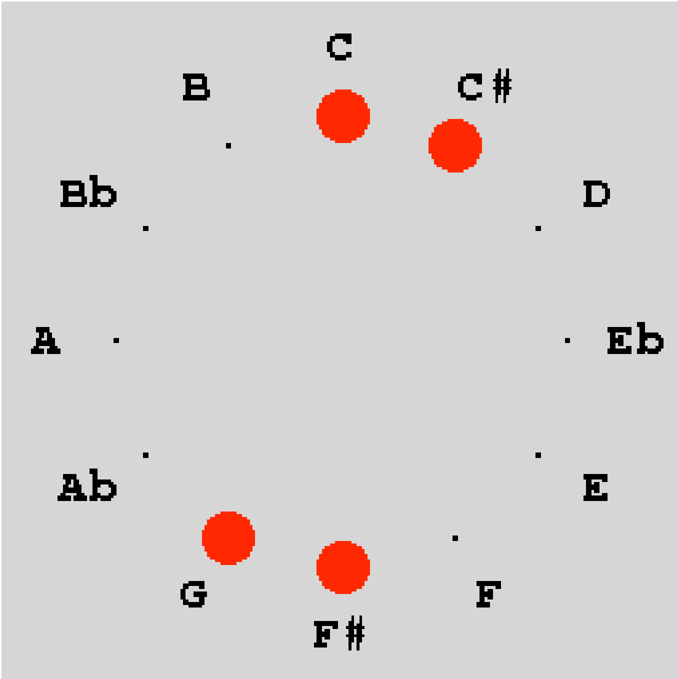
Subgroup H of all g with $\gamma_g(x) = x$

								
	1,0	1,6	1,9	1,3	7,0	7,6	7,9	7,3
	1,4	1,10	1,1	1,7	7,4	7,10	7,1	7,7
	1,8	1,2	1,5	1,11	7,8	7,2	7,5	7,11
	5,0	5,6	5,9	5,3	11,0	11,6	11,9	11,3
	5,4	5,10	5,1	5,7	11,4	11,10	11,1	11,7
	5,8	5,2	5,5	5,11	11,8	11,2	11,5	11,11

Internal Symmetries of Chords and Scales

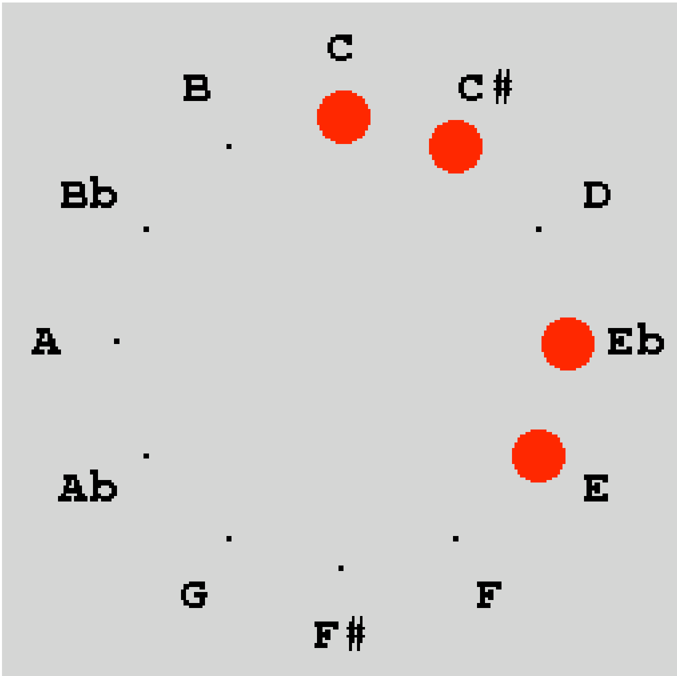
Example: Octatonic Strategies in Bartok (c.f. Cohn 1991)
Mikrokosmos Book IV: *From the Island of Bali*

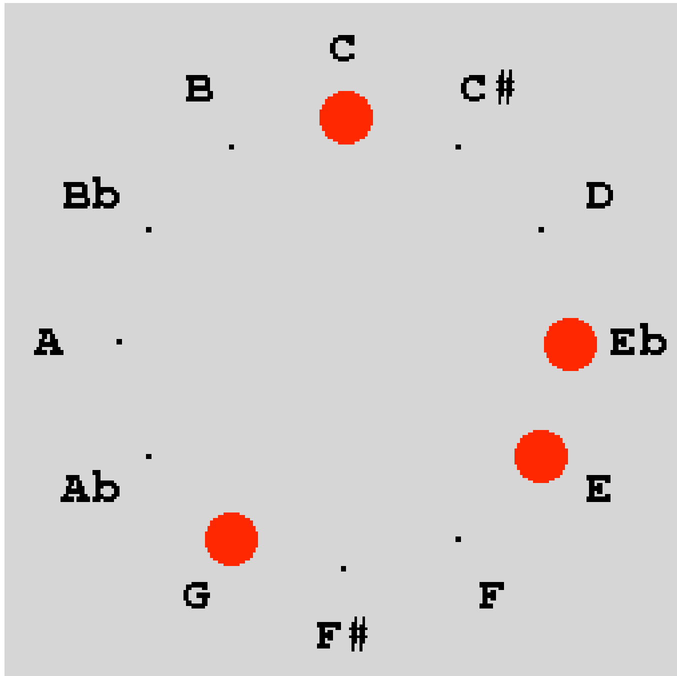




Internal Symmetries of Chords and Scales

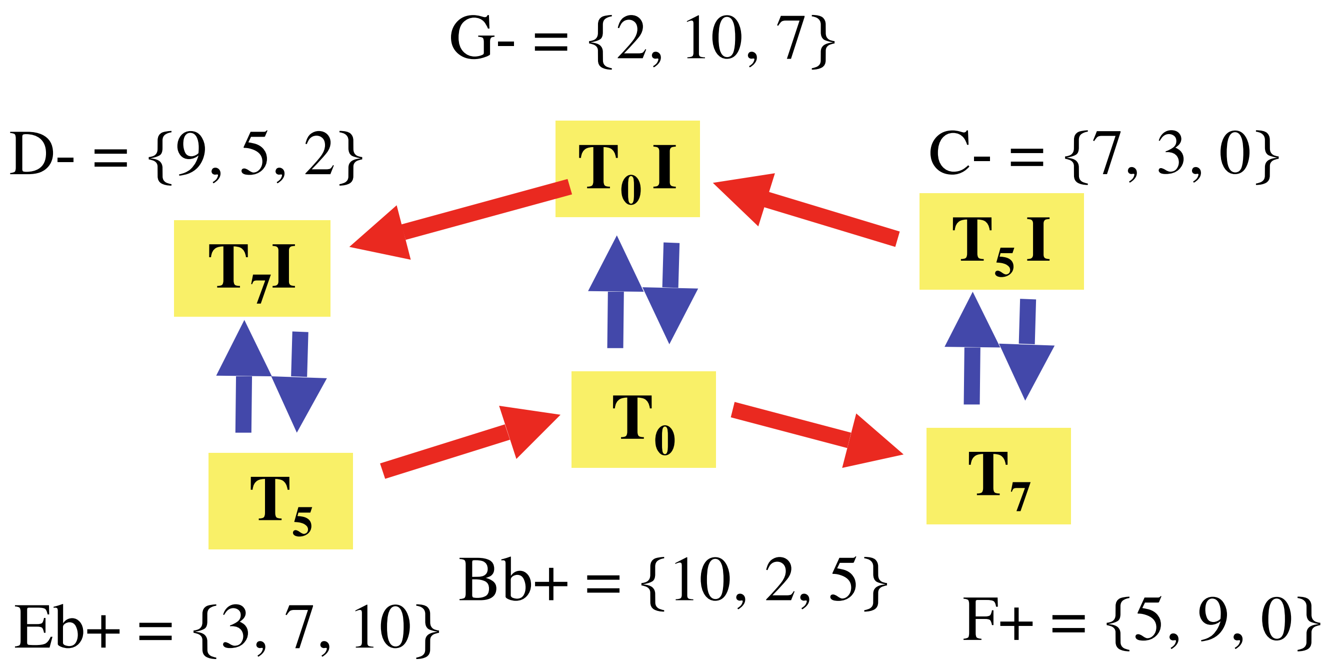
Example: Octatonic Strategies in Bartok (c.f. Cohn 1991)
Sonata for two Pianos and Percussion



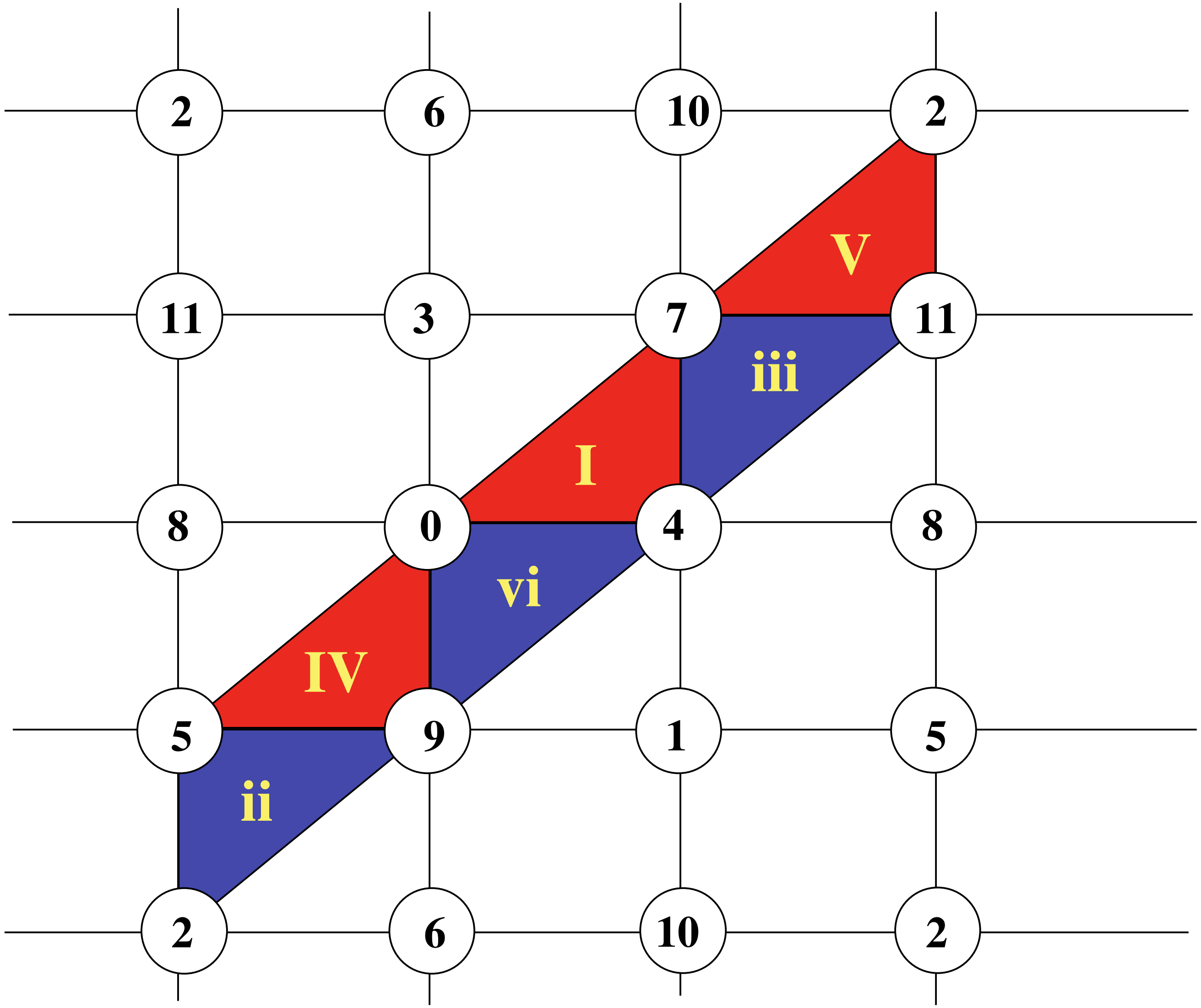


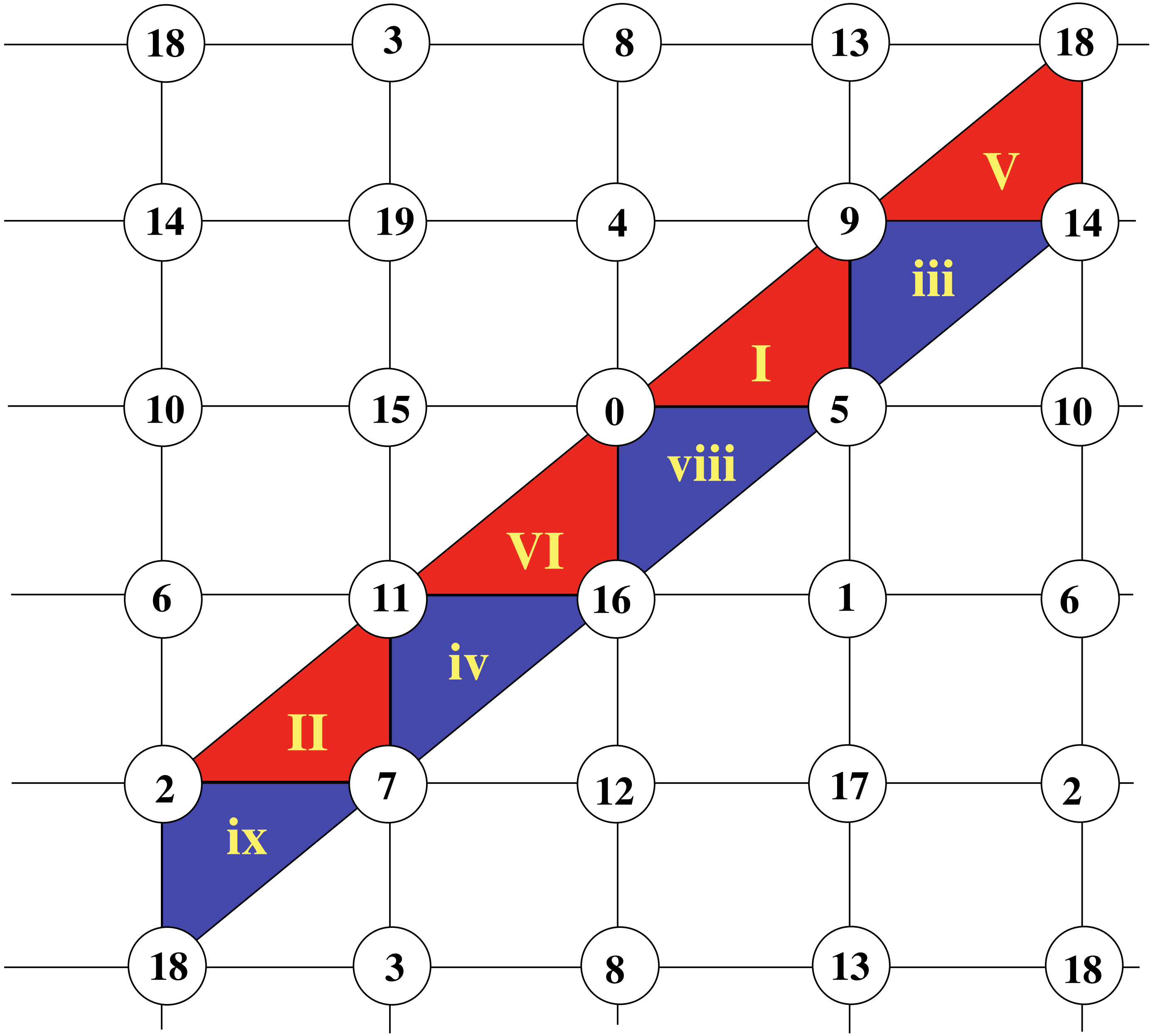
Internal Symmetries of the Diatonic

Riemann 1878, Balzano 1980, Mazzola 1985, Cohn 1997

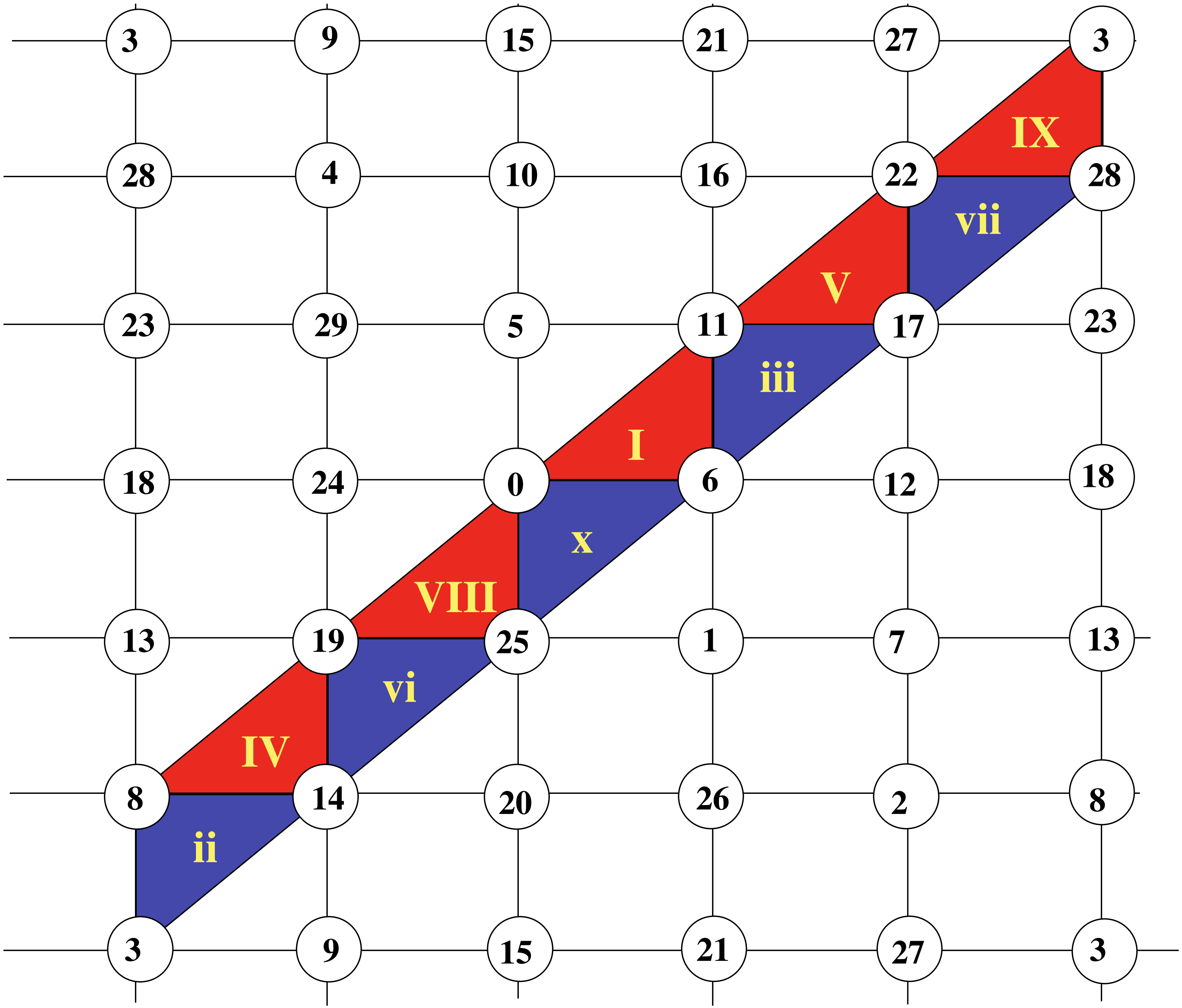


The Balzano - Family $Z_{k(k+1)}$



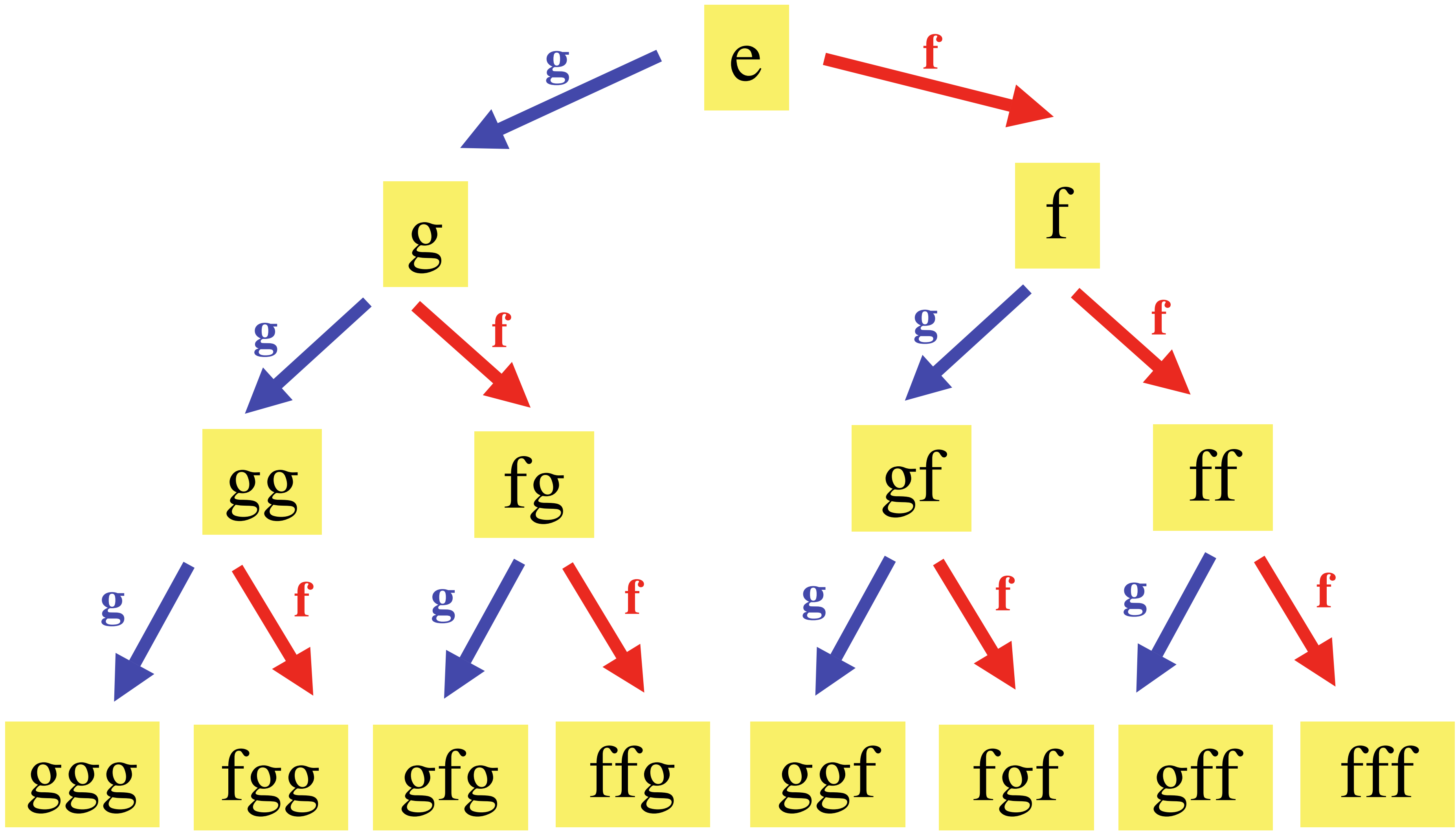


18
3
8
13
18
V
14
19
4
9
14
iii
I
10
15
0
5
10
viii
VI
6
11
16
1
6
iv
II
2
7
12
17
2
ix
18
3
8
13
18


3 9 15 21 27 3
28 4 10 16 22 28
23 29 5 11 17 23
18 24 0 6 12 18
13 19 25 1 7 13
8 14 20 26 2 8
3 9 15 21 27 3
IX
vii
V
iii
I
x
VIII
vi
IV
ii

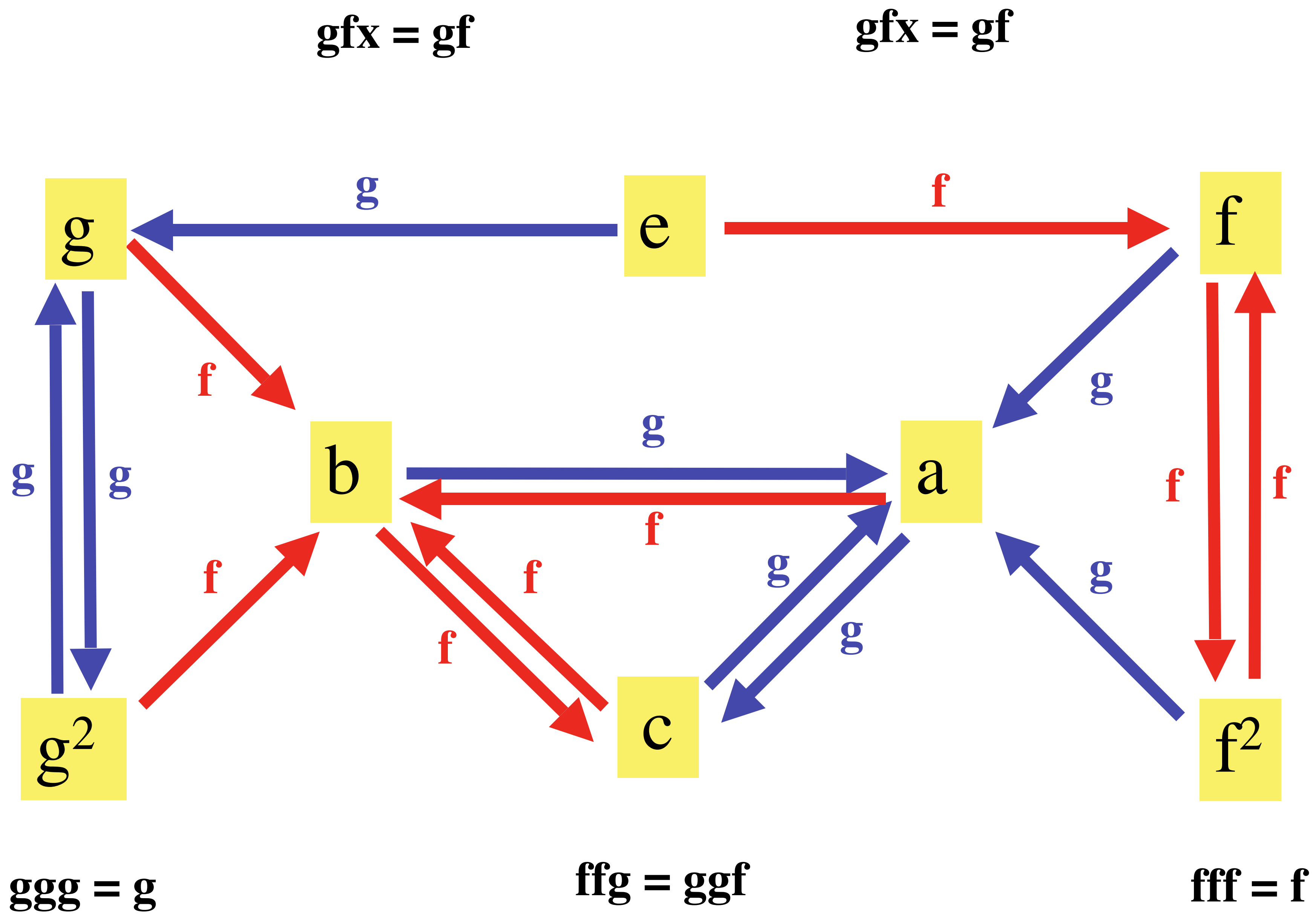
Monoids
and Monoid Actions

Caley-Graph for a Free Monoid <f, g>

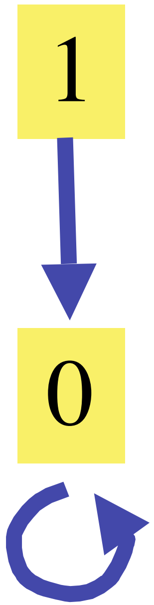


Example: Rhythms with two durational Values: 2, 3
c.f. Chemillier & Truchet (2004)

Caley-Graph for the Triadic Monoid

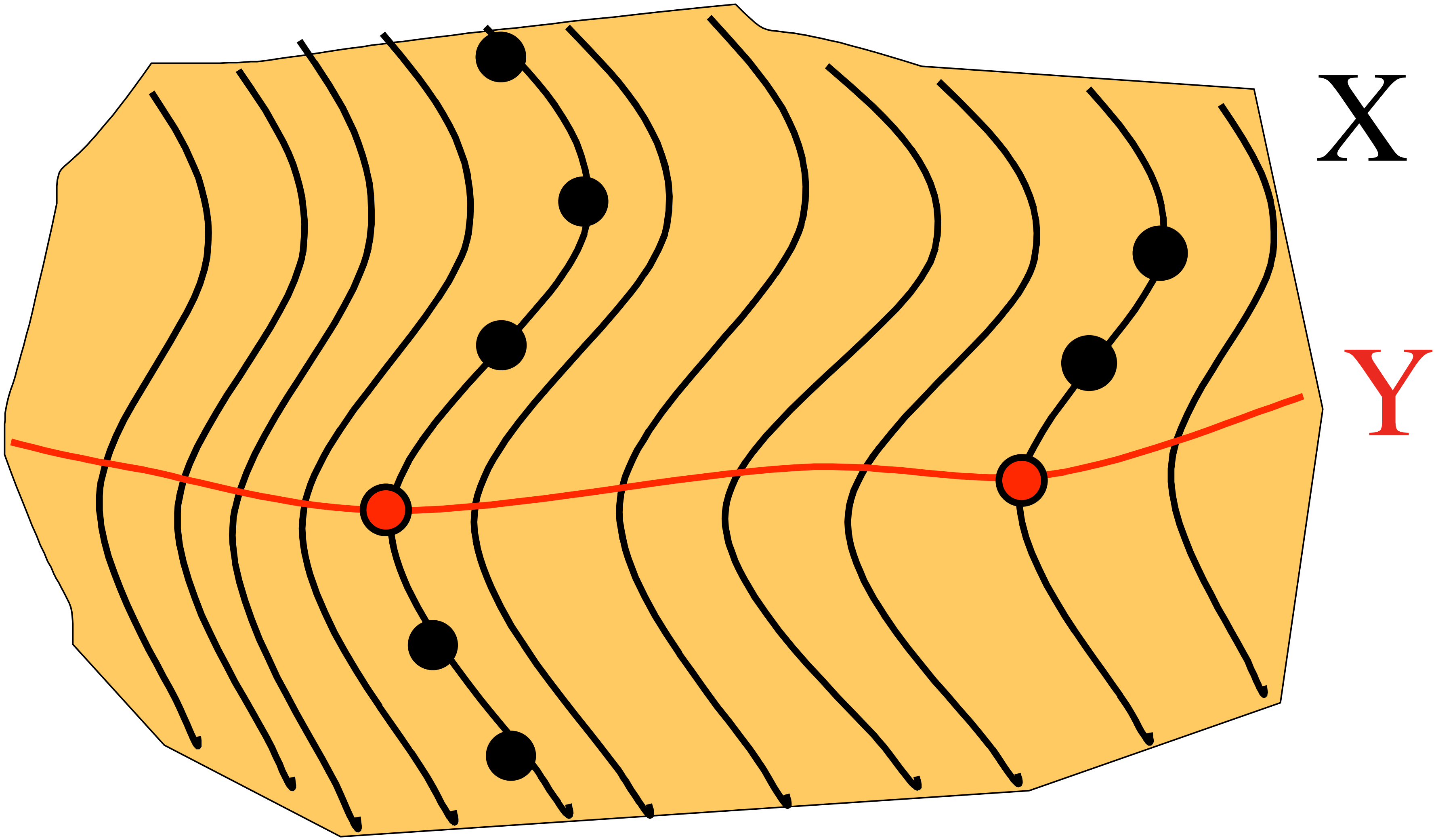


Caley-Graph for Z_2^*



*	0	1
0	0	0
1	0	1

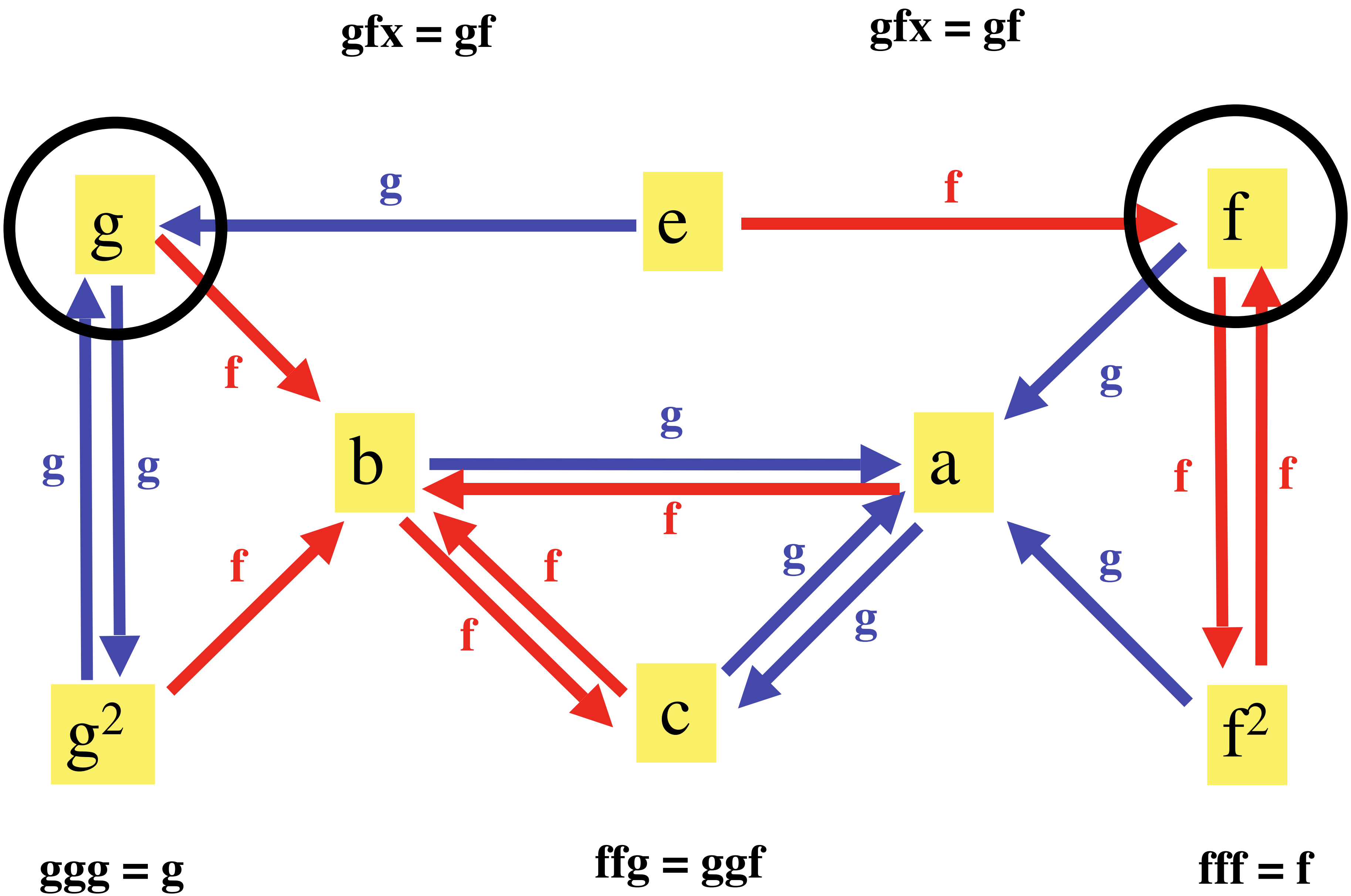
What is a Z_2^* - Action $\boldsymbol{\mu: Z_2^* \times X \longrightarrow X}$?



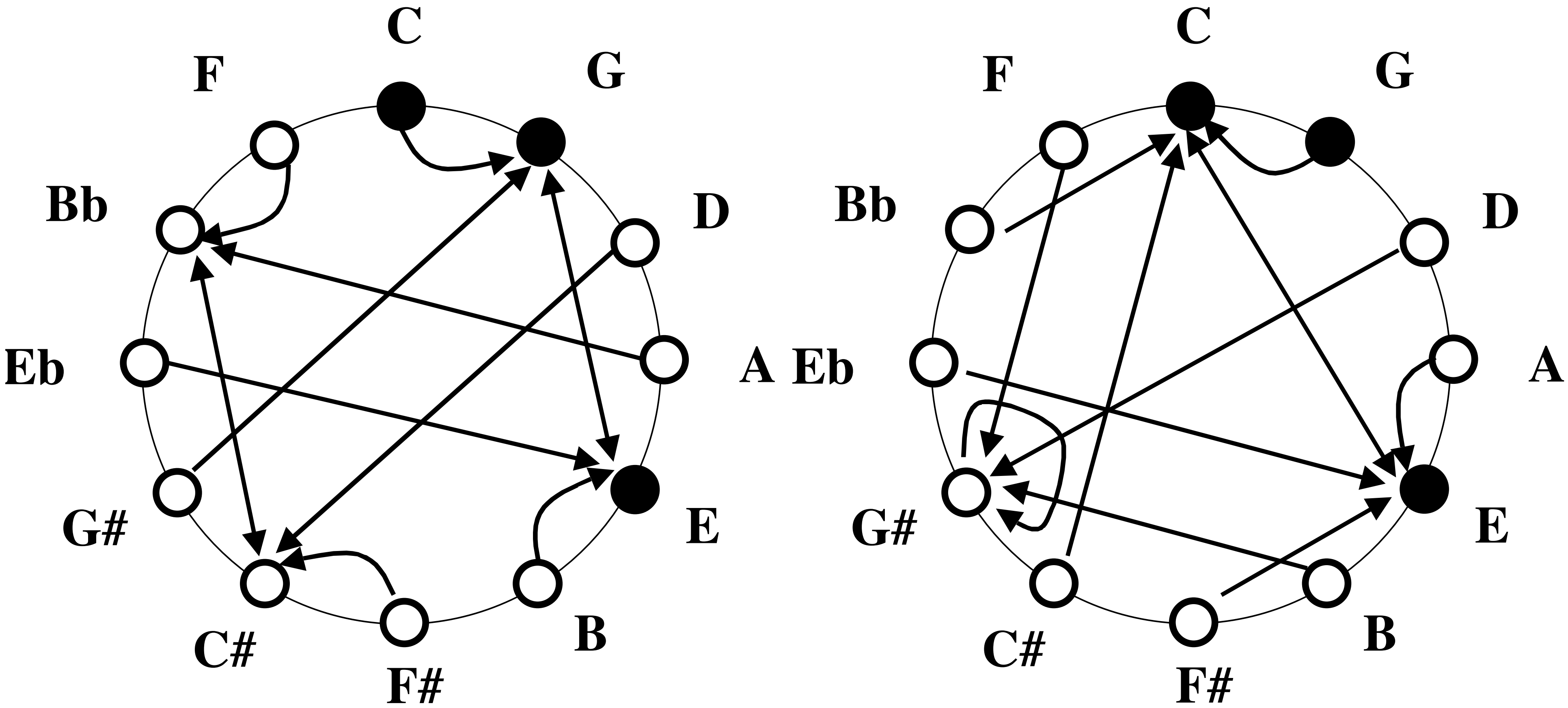
Answer: (1) $\mathbf{\mu_1}$ is the identical map id_X of **X**.

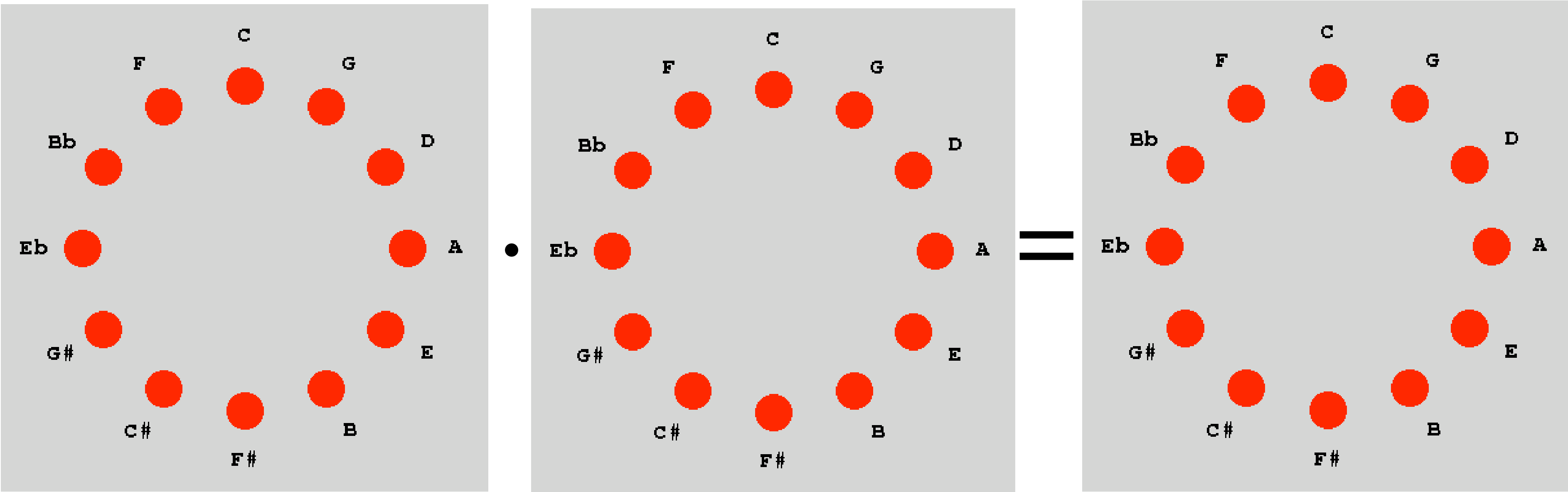
(2) $\mathbf{\mu_0}$ is a projection to a subset **Y** of **X**.

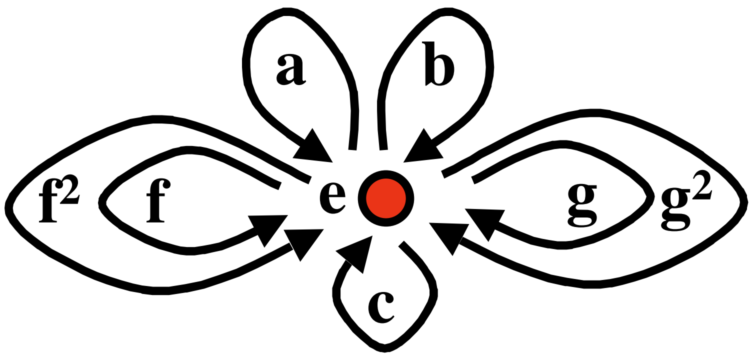
Caley-Graph for the Triadic Monoid



Generators of a Triadic Action

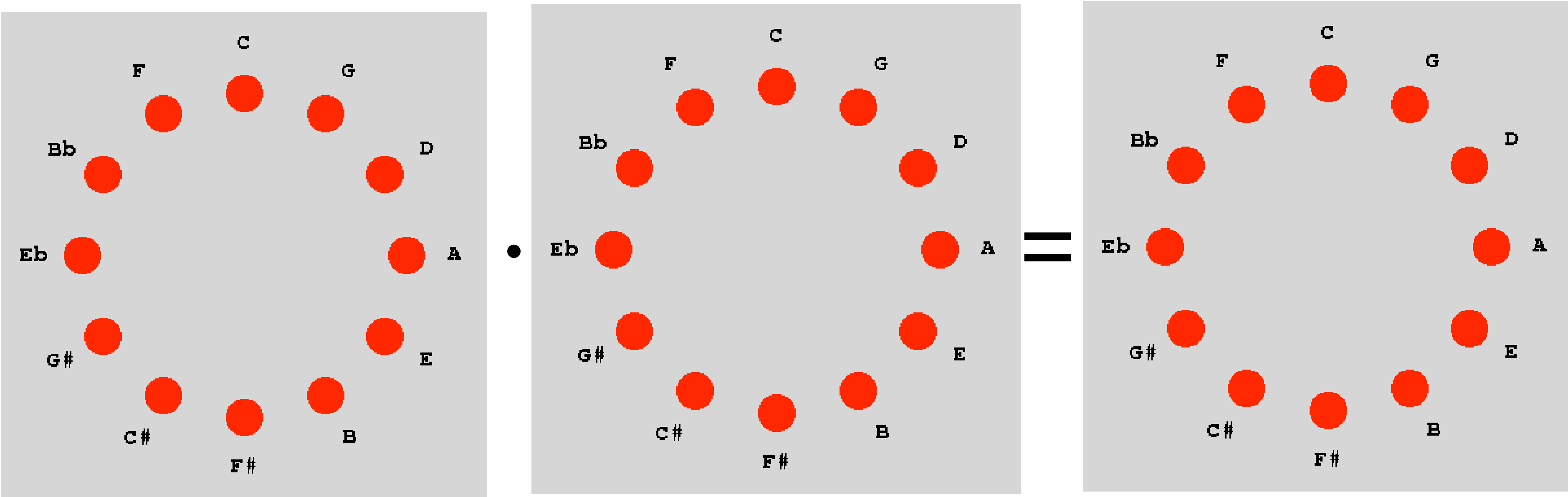


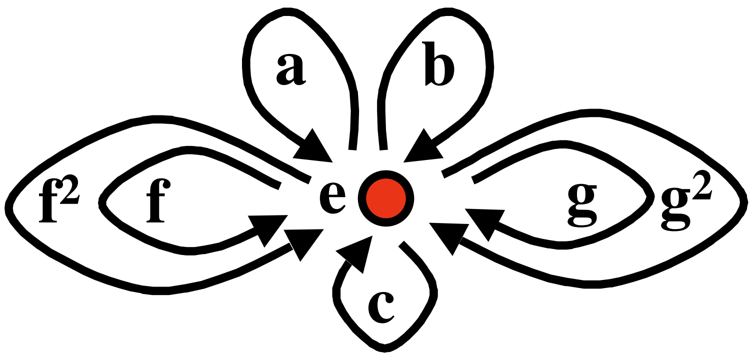




Concatenation Table

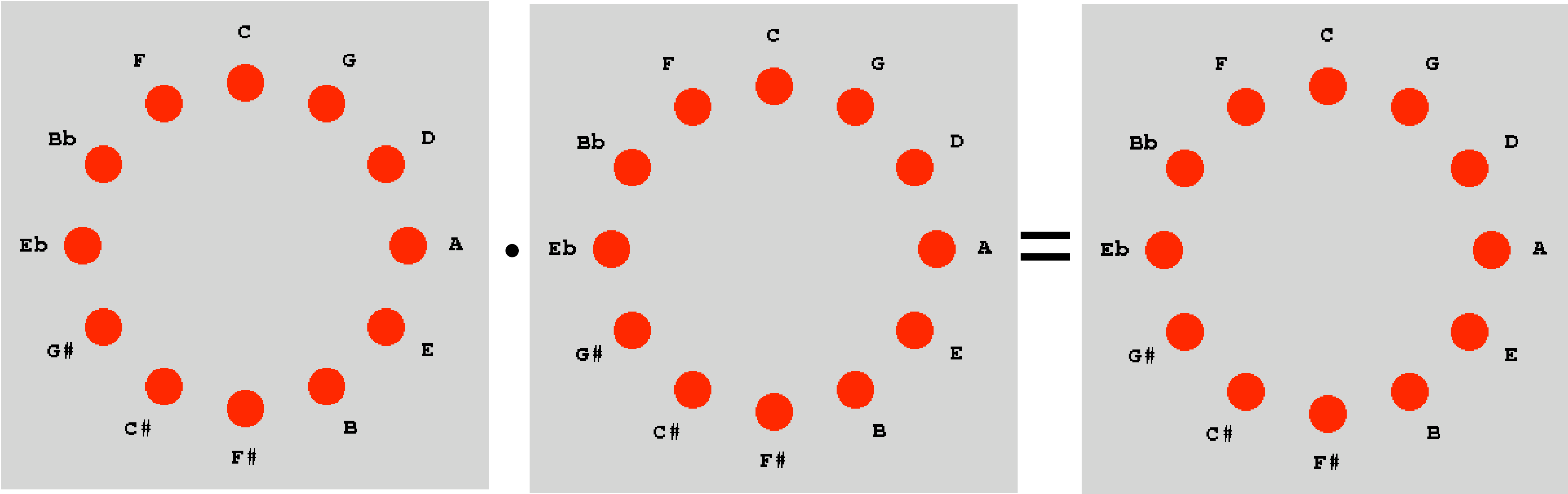
$\cdot$	a	b	c	e	f	f^2	g	g^2
a	a	a	a	a	a	a	a	a
b	b	b	b	b	b	b	b	b
c	c	c	c	c	c	c	c	c
e	a	b	c	e	f	f^2	g	g^2
f	b	c	b	f	f^2	f	b	b
f^2	c	b	c	f^2	f	f^2	c	c
g	c	a	a	g	a	a	g^2	g
g^2	a	c	c	g^2	c	c	g	g^2



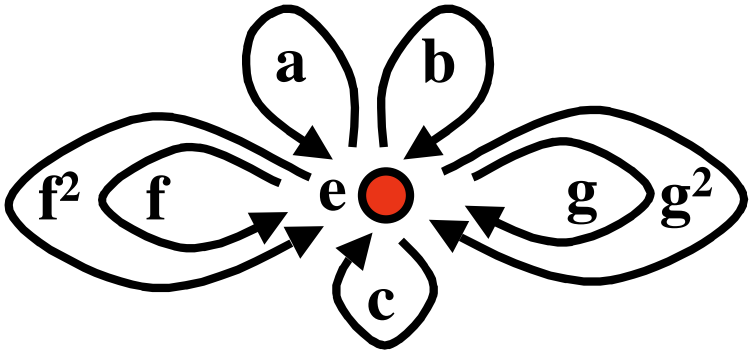


Concatenation Table

$\cdot$	a	b	c	e	f	f^2	g	g^2
a	a	a	a	a	a	a	a	a
b	b	b	b	b	b	b	b	b
c	c	c	c	c	c	c	c	c
e	a	b	c	e	f	f^2	g	g^2
f	b	c	b	f	f^2	f	b	b
f^2	c	b	c	f^2	f	f^2	c	c
g	c	a	a	g	a	a	g^2	g
g^2	a	c	c	g^2	c	c	g	g^2



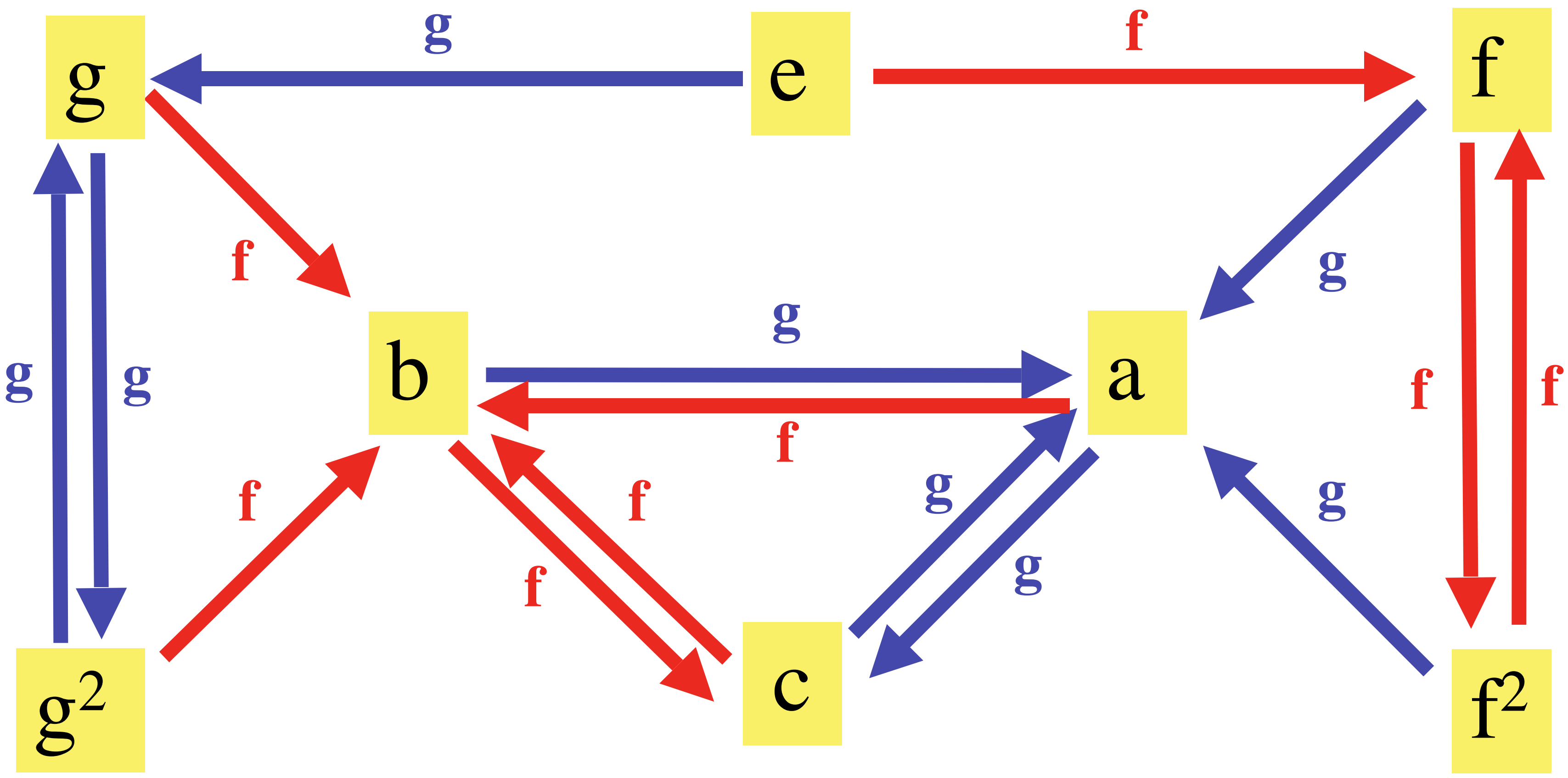
Concatenation Table



$\cdot$	a	b	c	e	f	f^2	g	g^2
a	a	a	a	a	a	a	a	a
b	b	b	b	b	b	b	b	b
c	c	c	c	c	c	c	c	c
e	a	b	c	e	f	f^2	g	g^2
f	b	c	b	f	f^2	f	b	b
f^2	c	b	c	f^2	f	f^2	c	c
g	c	a	a	g	a	a	g^2	g
g^2	a	c	c	g^2	c	c	g	g^2

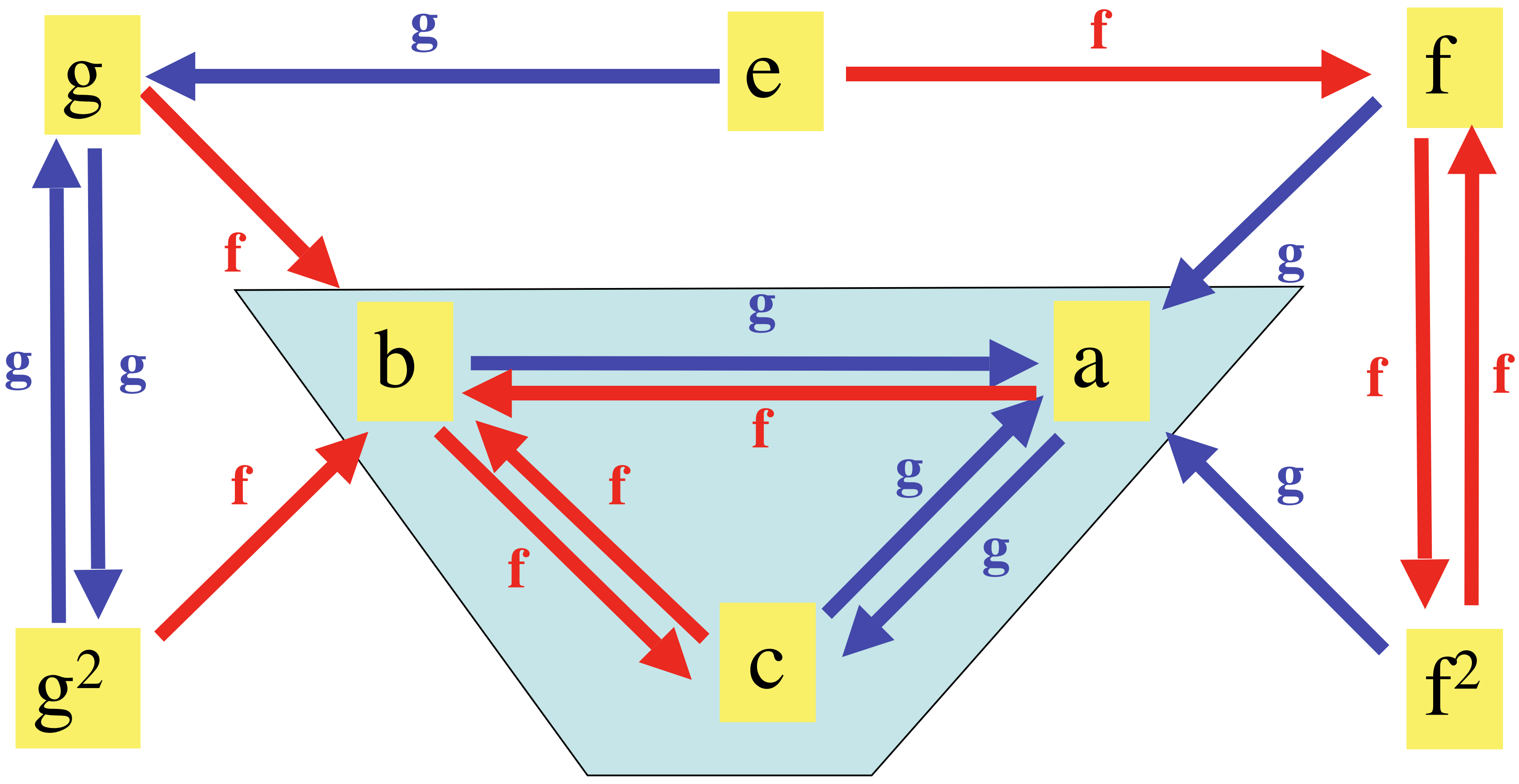
Cosieves B:

mB ⊂ B for all Elements m of M



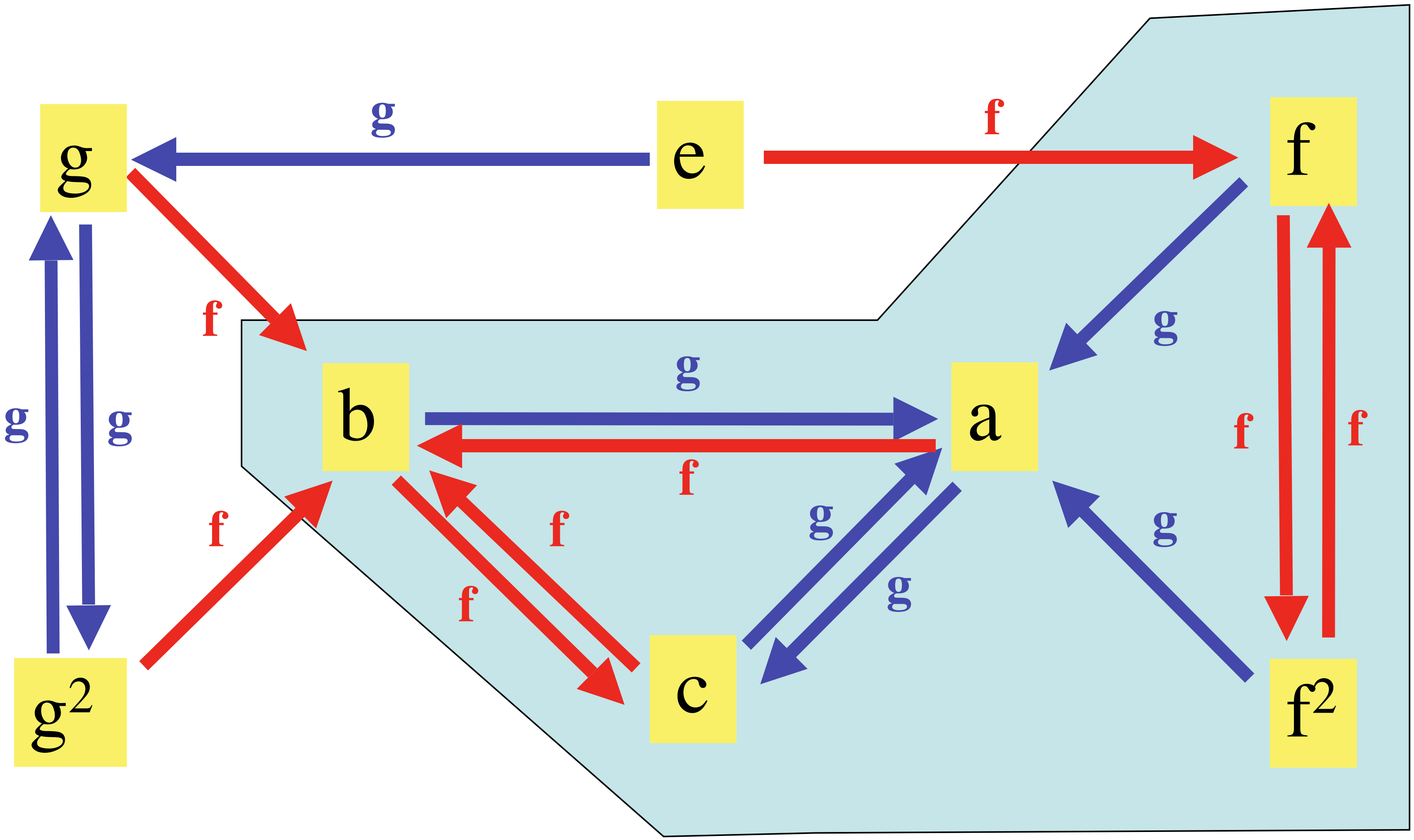
Cosieves B:

mB ⊂ B for all Elements m of M



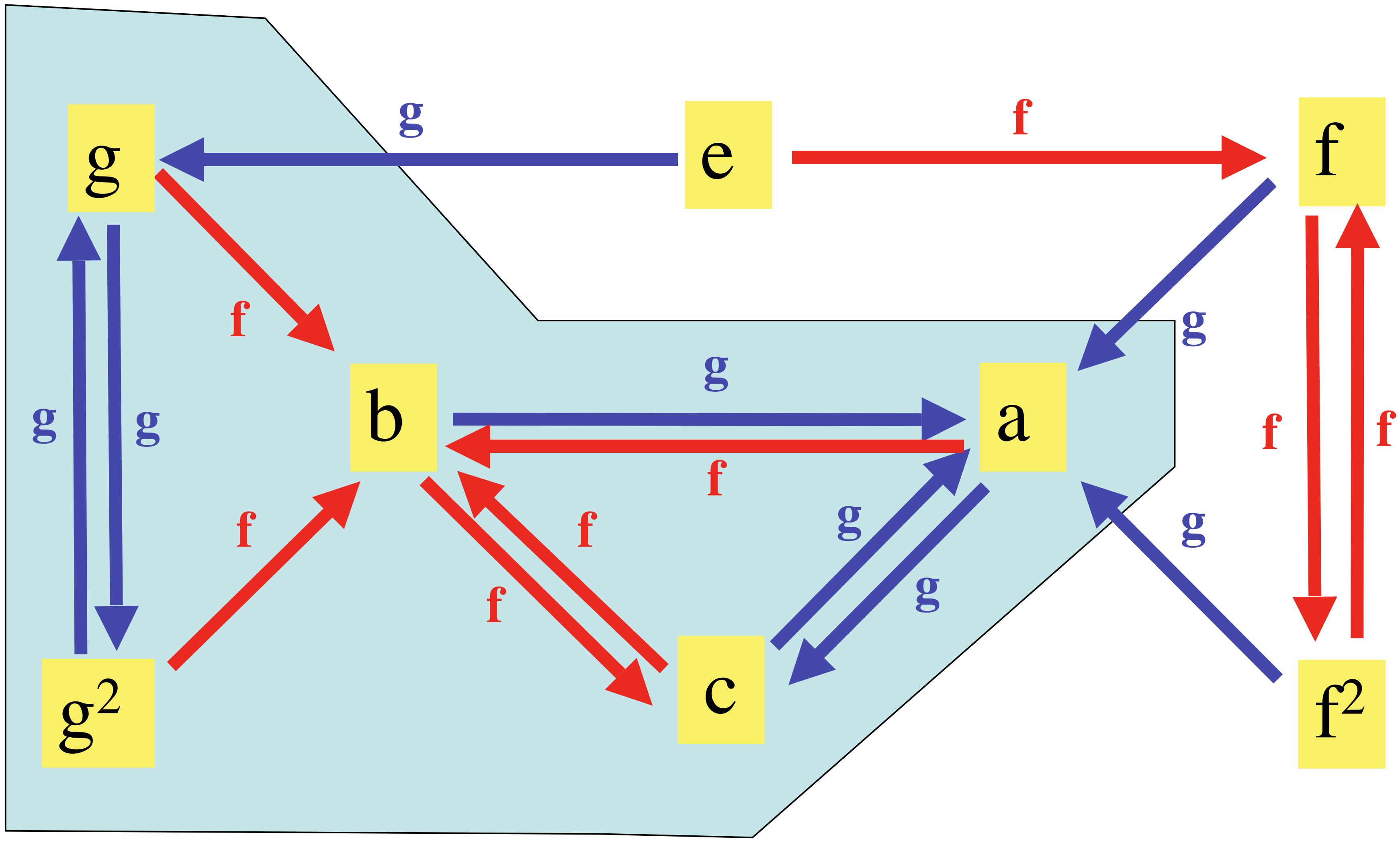
Cosieves B:

mB ⊂ B for all Elements m of M



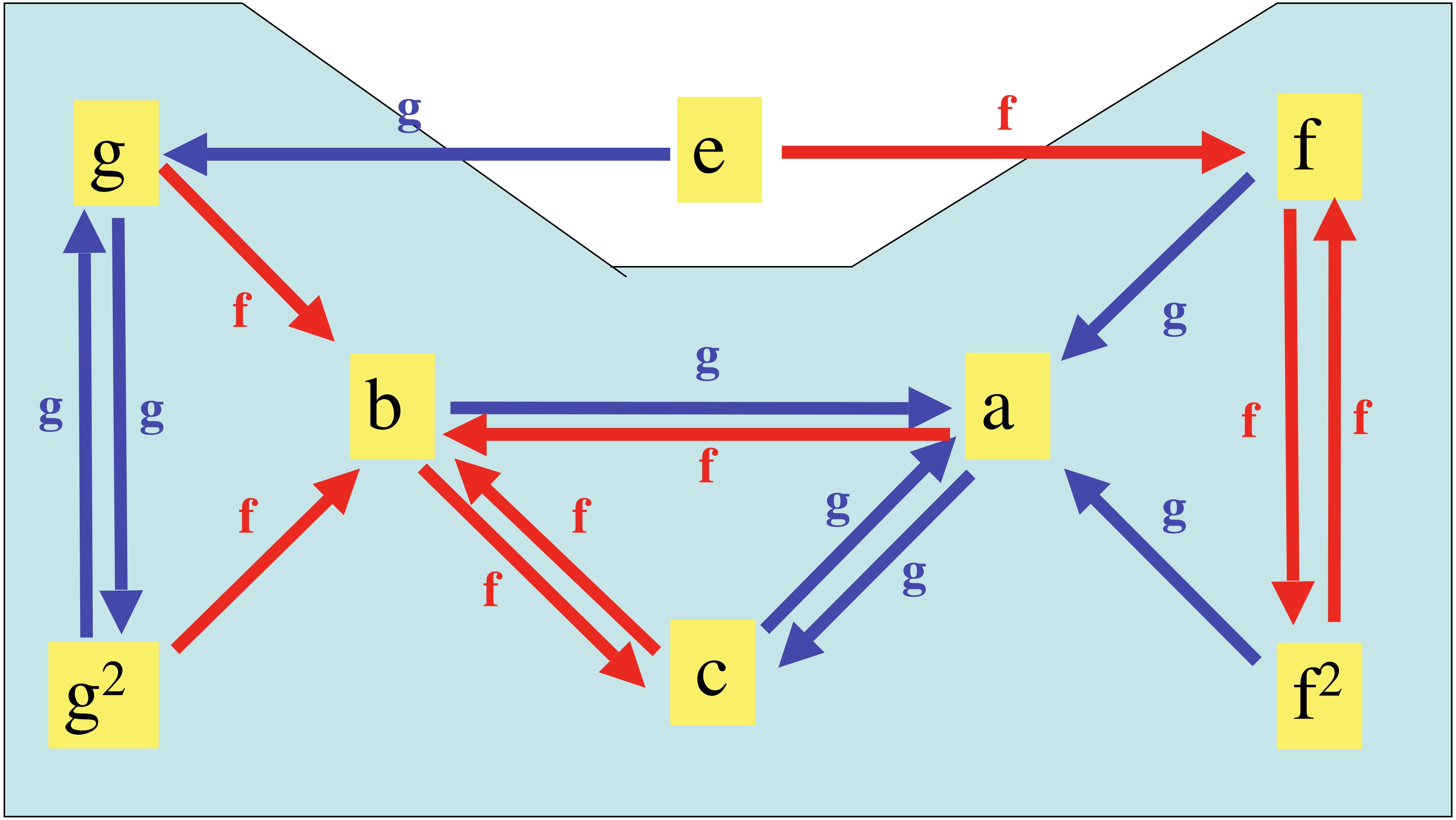
Cosieves B:

mB ⊂ B for all Elements m of M



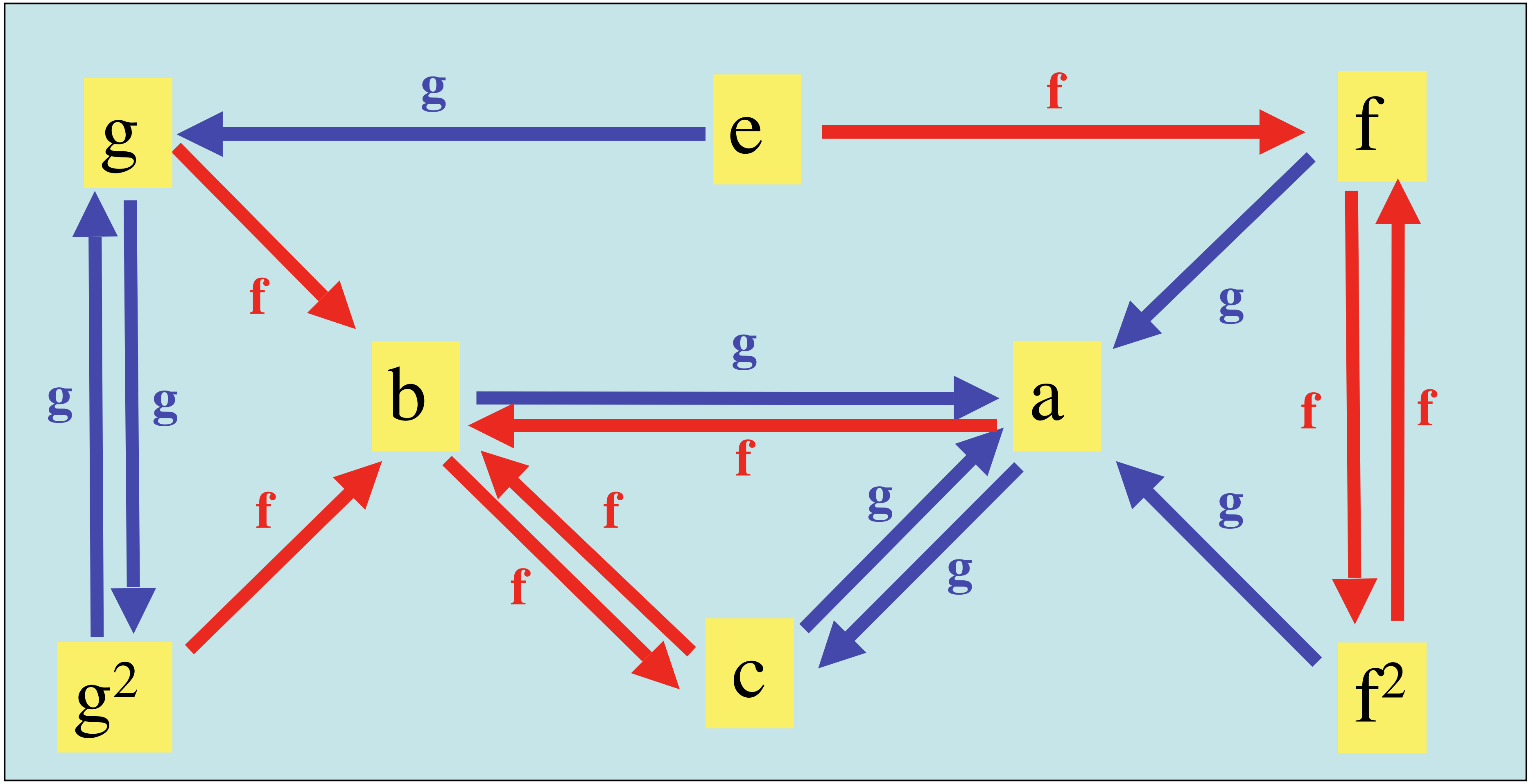
Cosieves B:

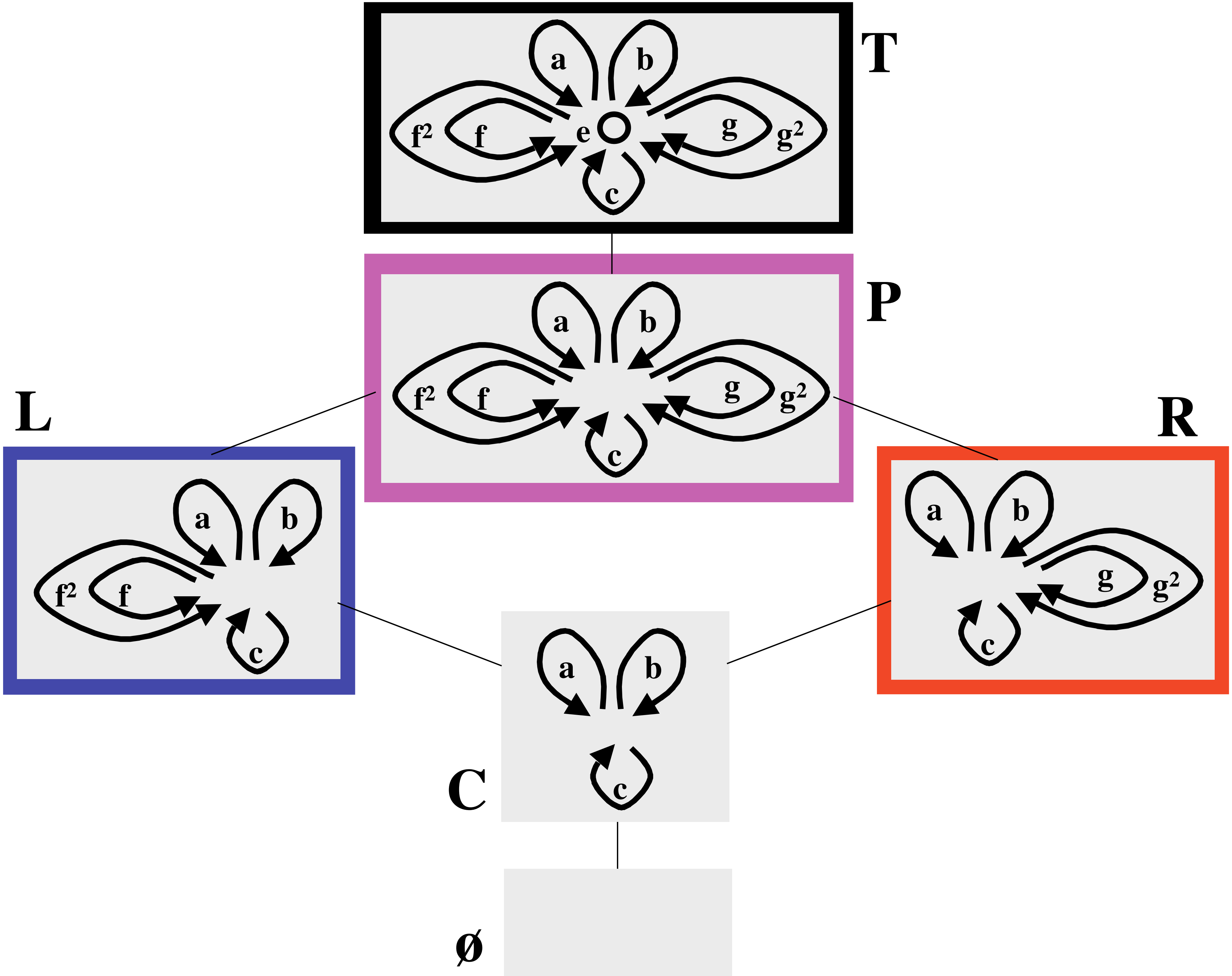
mB ⊂ B for all Elements m of M



Cosieves B:

mB ⊂ B for all Elements m of M




T
a
b
f²
f
e
g
g²
c
P
a
b
f²
f
g
g²
c
L
a
b
f²
f
c
R
a
b
g
g²
c
C
a
b
c
ø

Action of M on Sieves

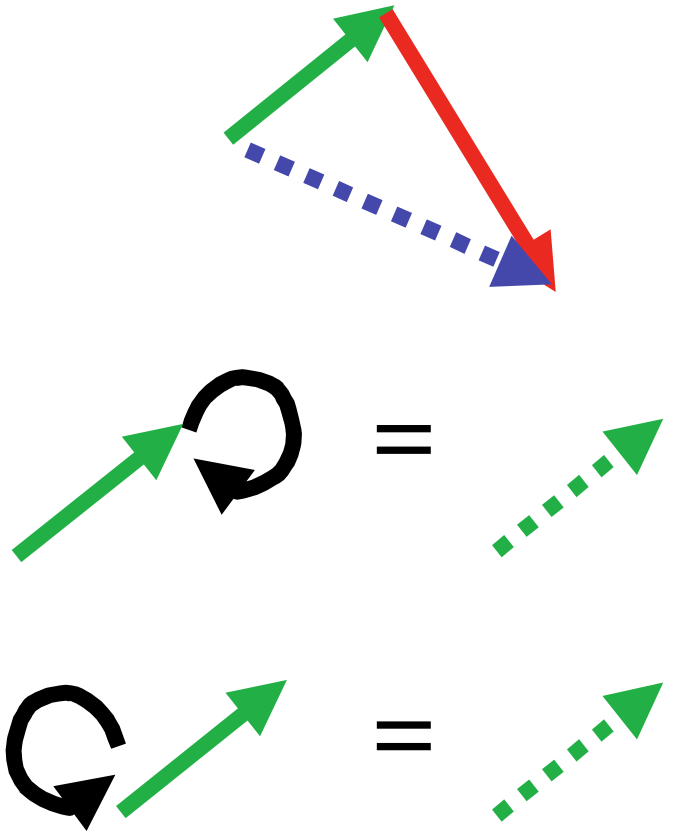
Go to the blackgoard :-)

From Monoids to Categories

Recall:

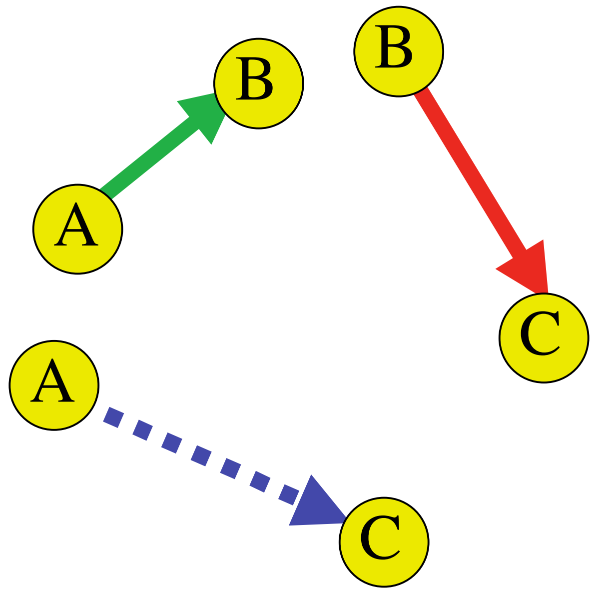
A **Monoid** is Semigroup

with a neutral Element.



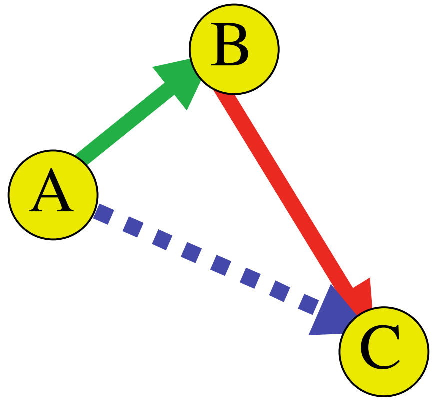
From Monoids to Categories

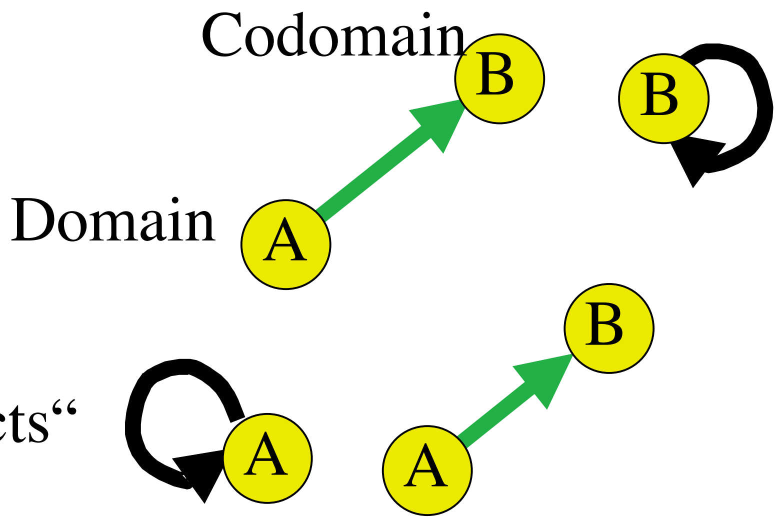
In a **Category** arrows are sorted:



From Monoids to Categories

In a **Category** arrows are sorted:

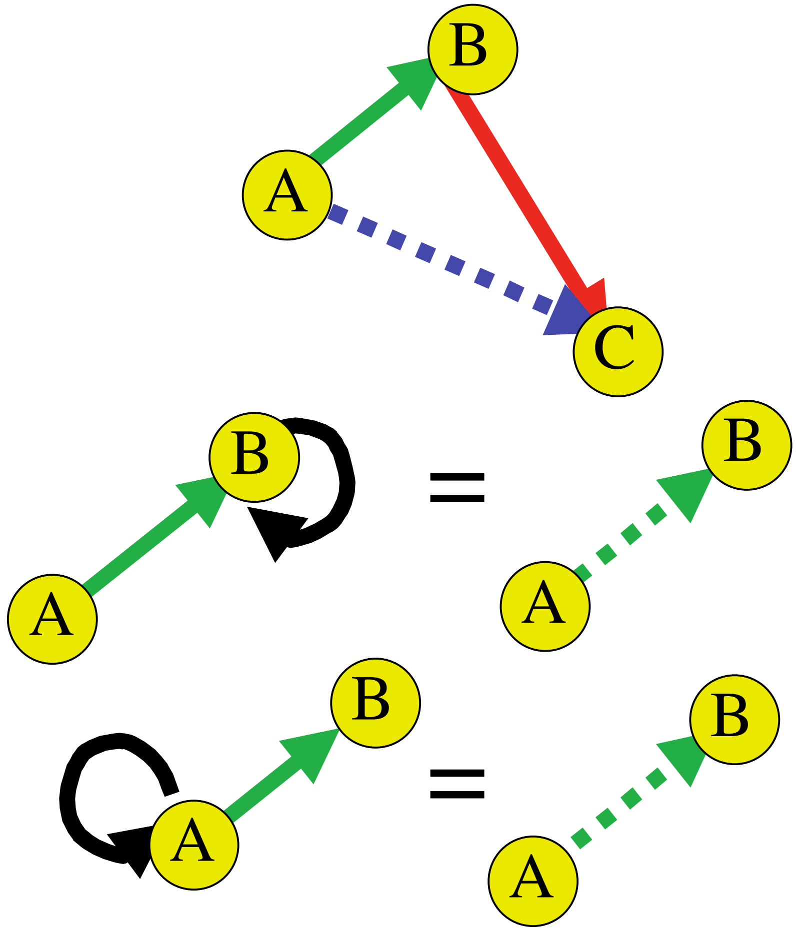




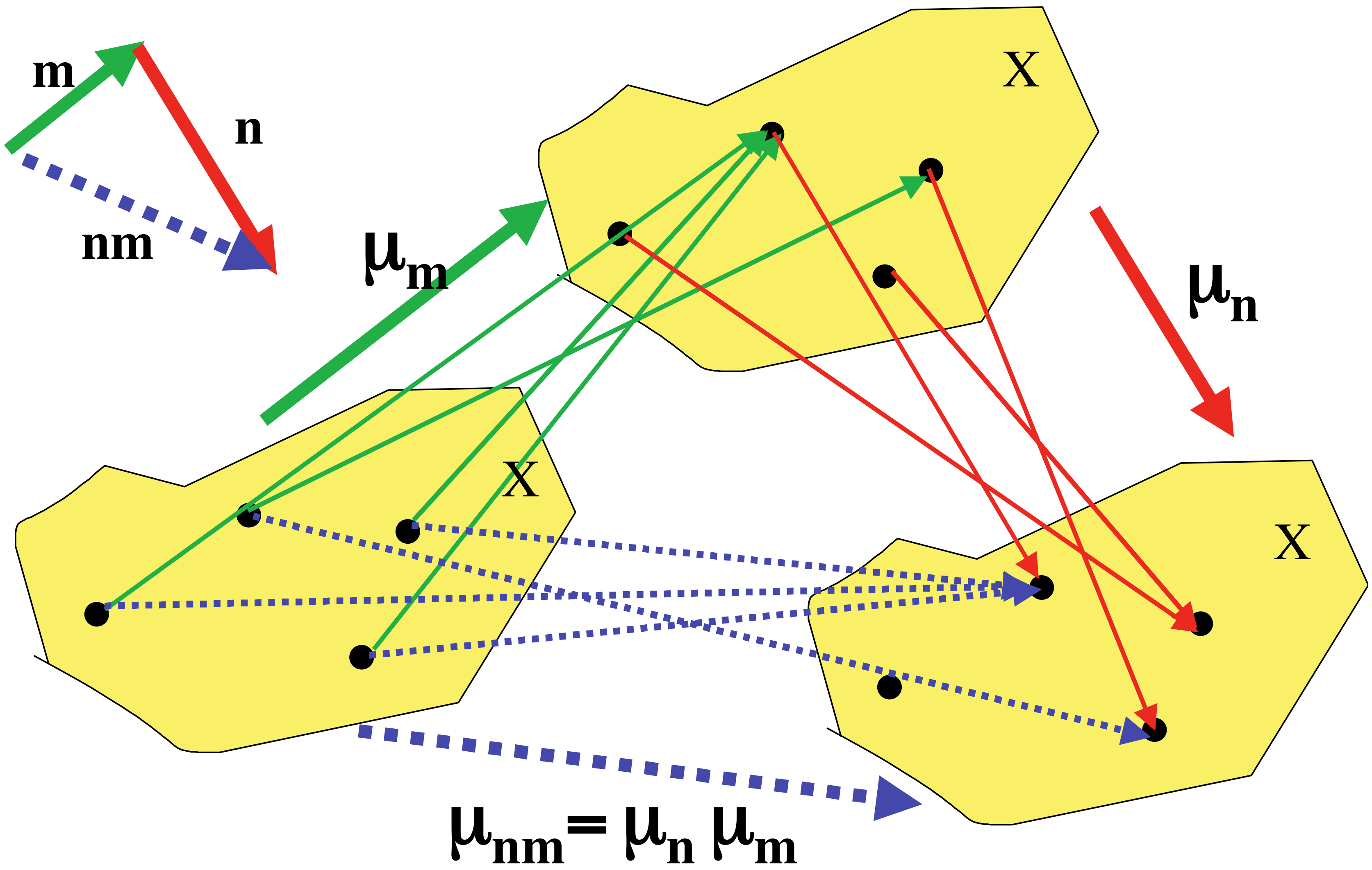
Identities = „Objects“

From Monoids to Categories

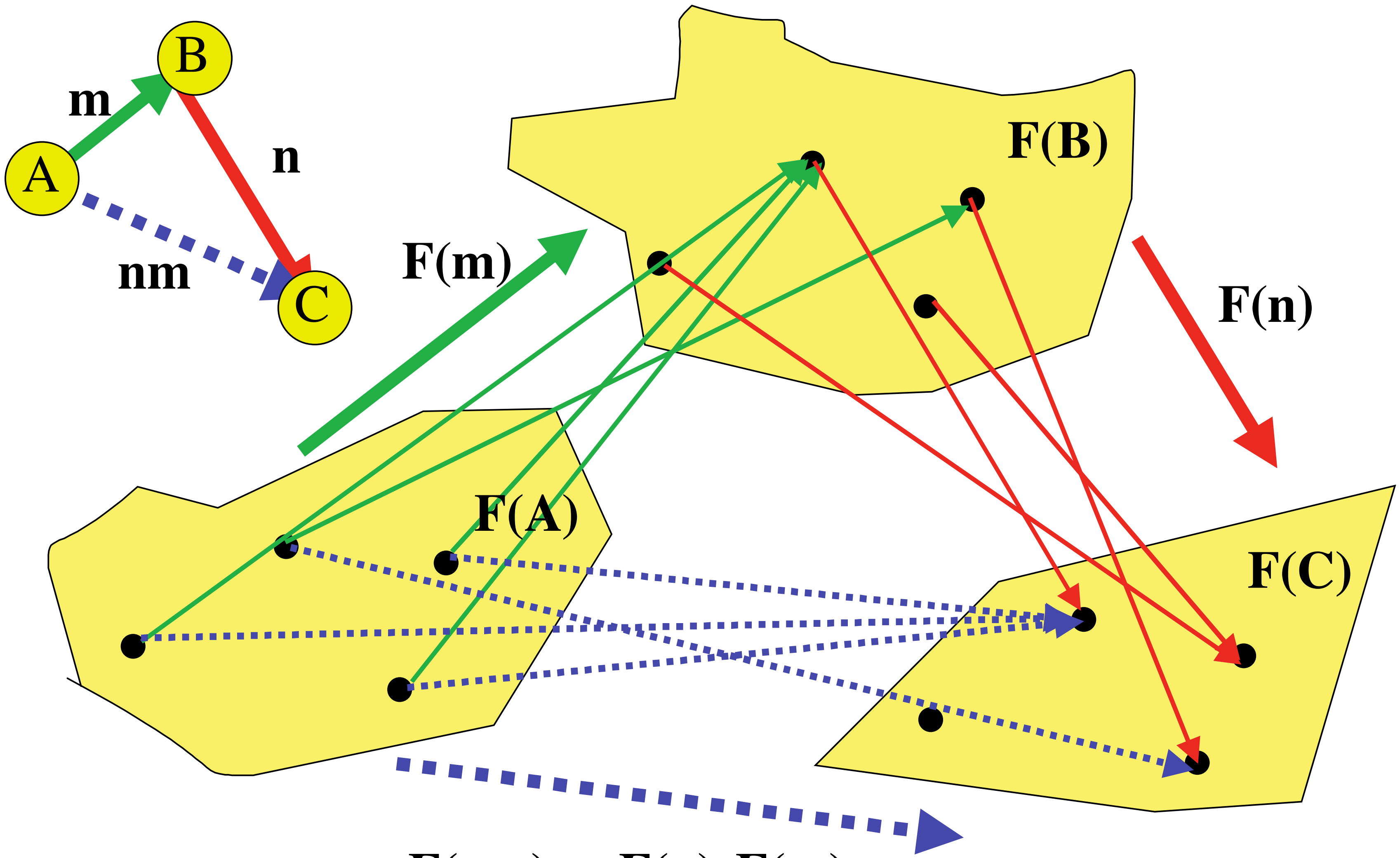
In a **Category** arrows are sorted:



Monoid Action $\mu: \mathbf{M} \times \mathbf{X} \longrightarrow \mathbf{X}$ on a Set X

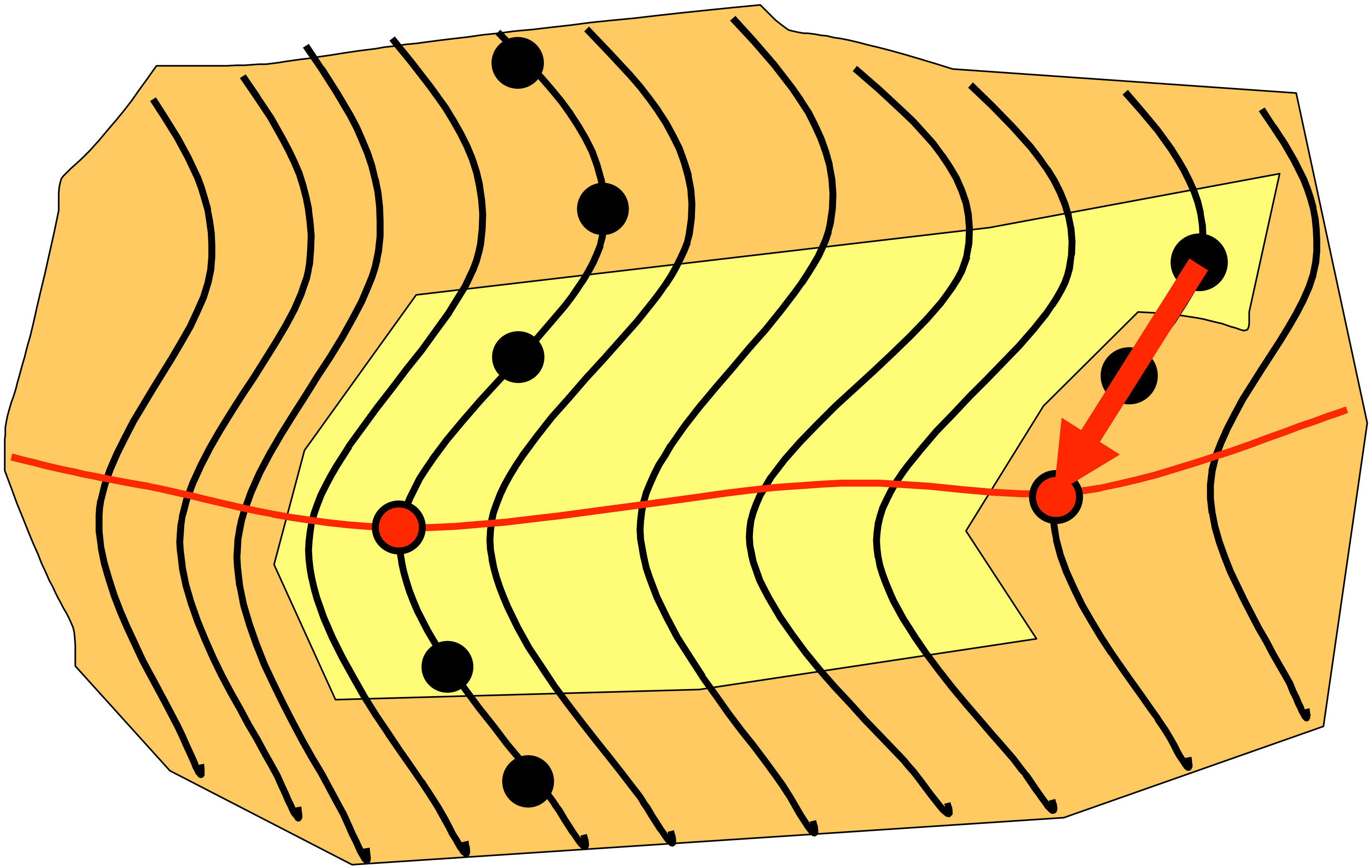


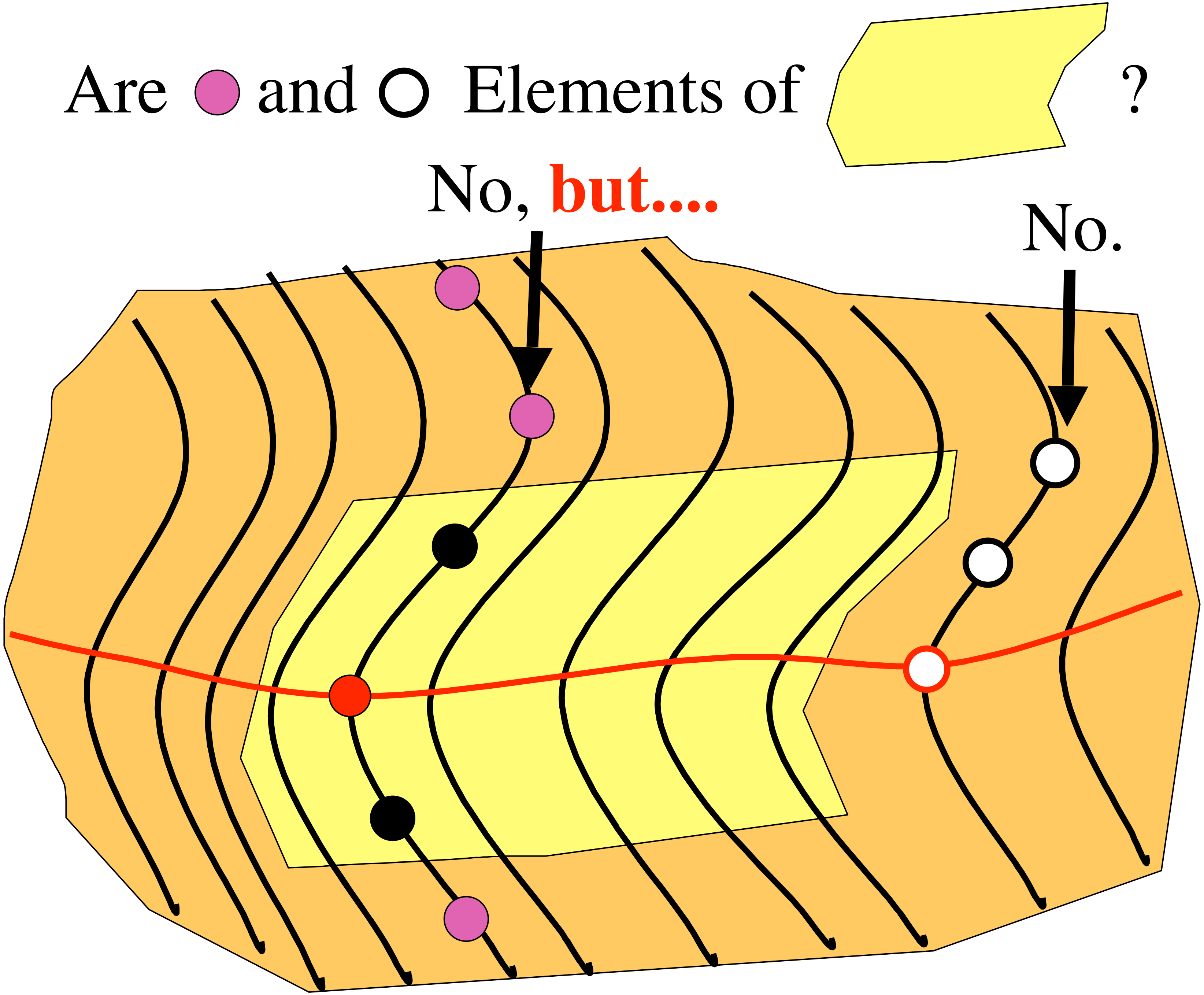
Functor F: C ⟶ Sets



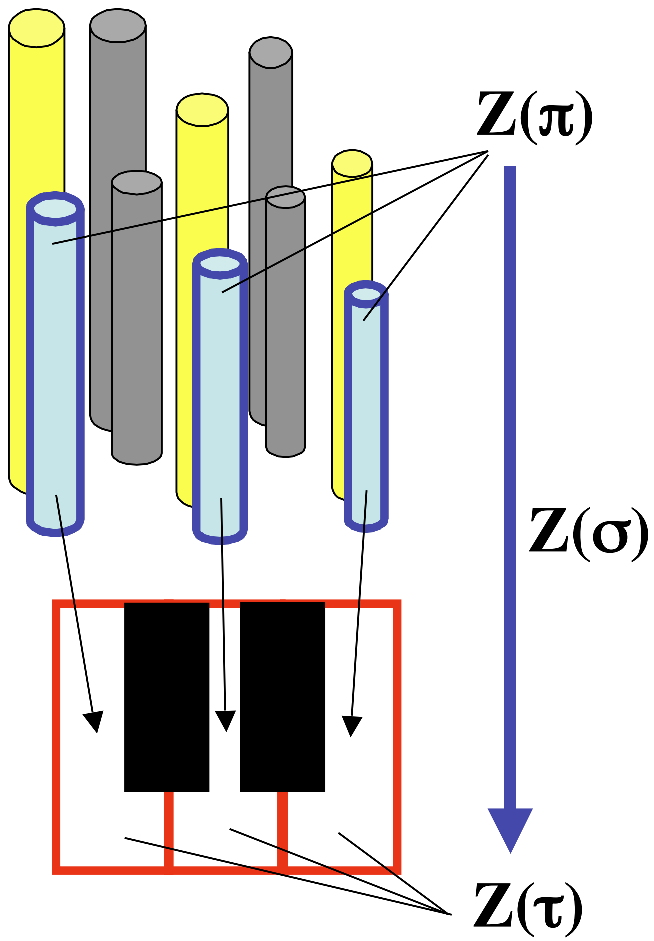
F(nm) = F(n) F(m)

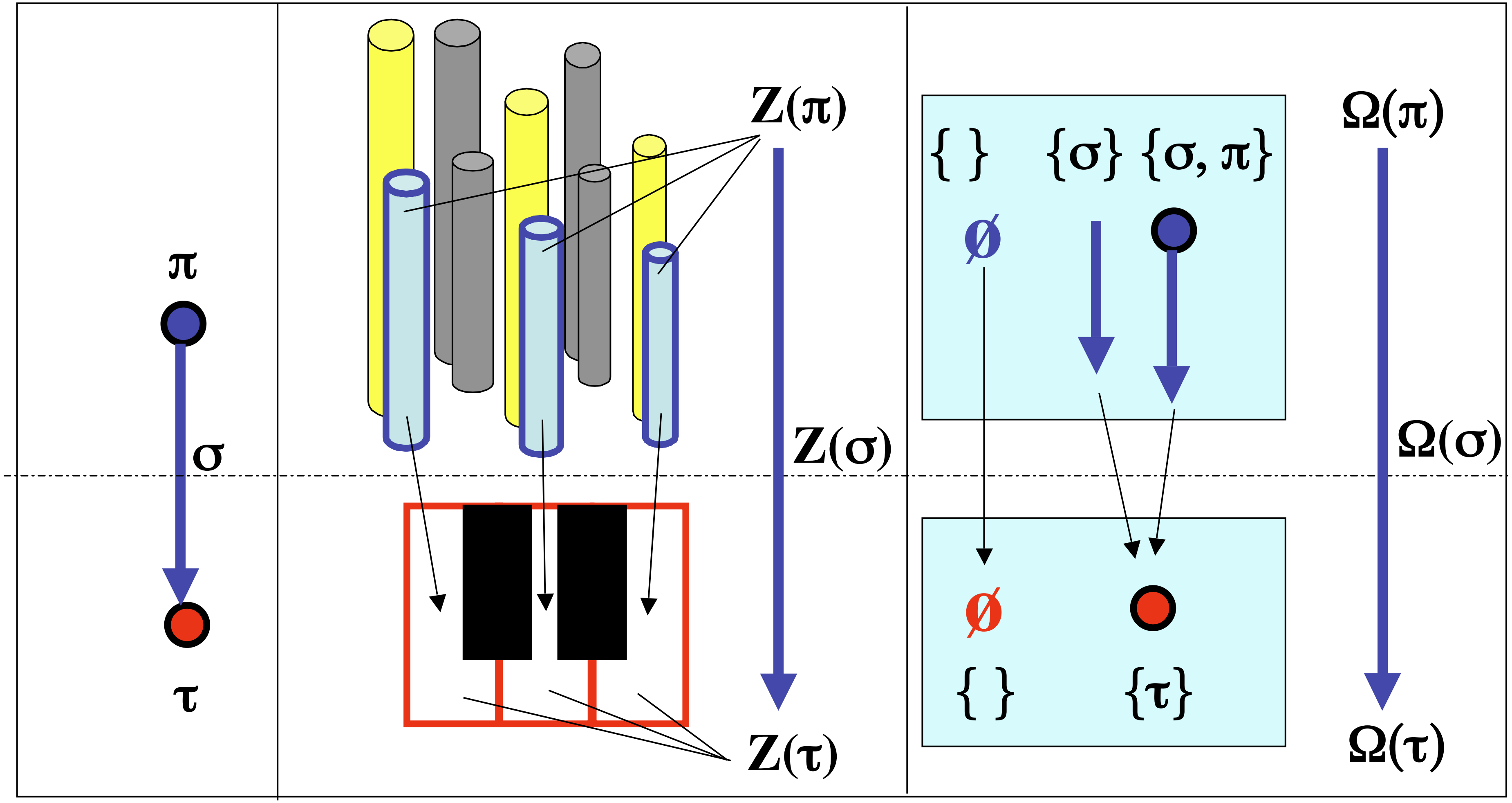
Is not a sub-object of this Projection





Are
and
Elements of
?
No, **but....**
No.


Z(π)
Z(σ)
Z(τ)

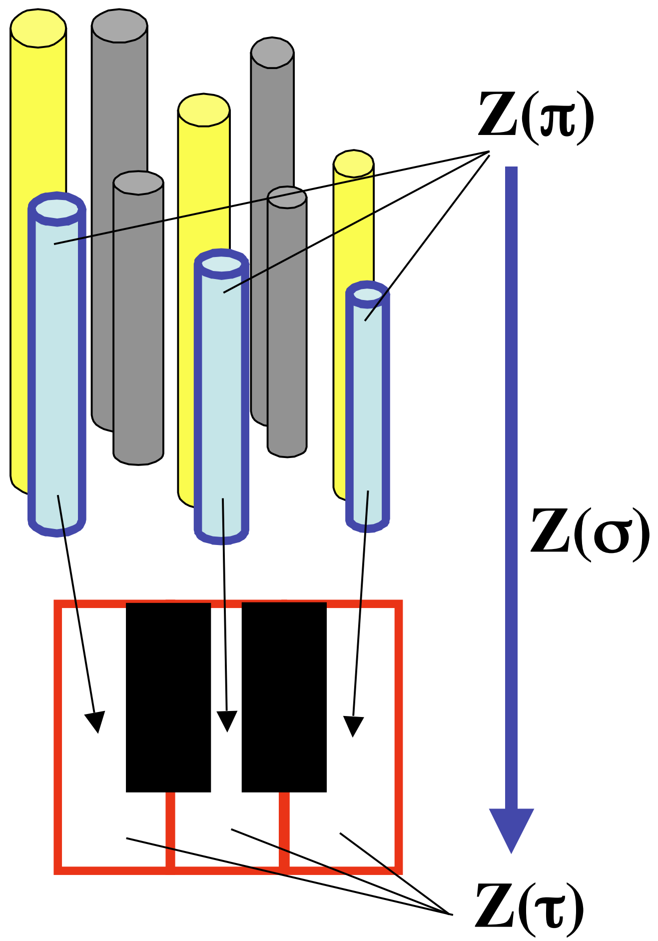


Abstract Diagram

Play-Action

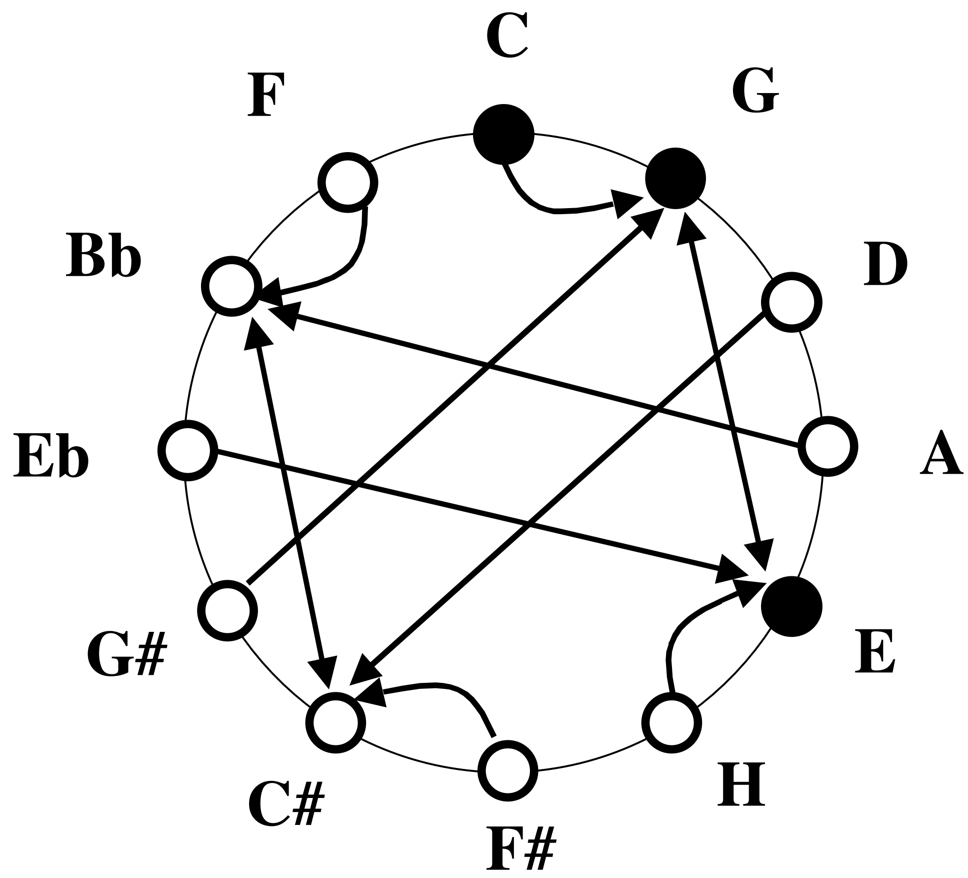
Evaluation-Action

Play Action

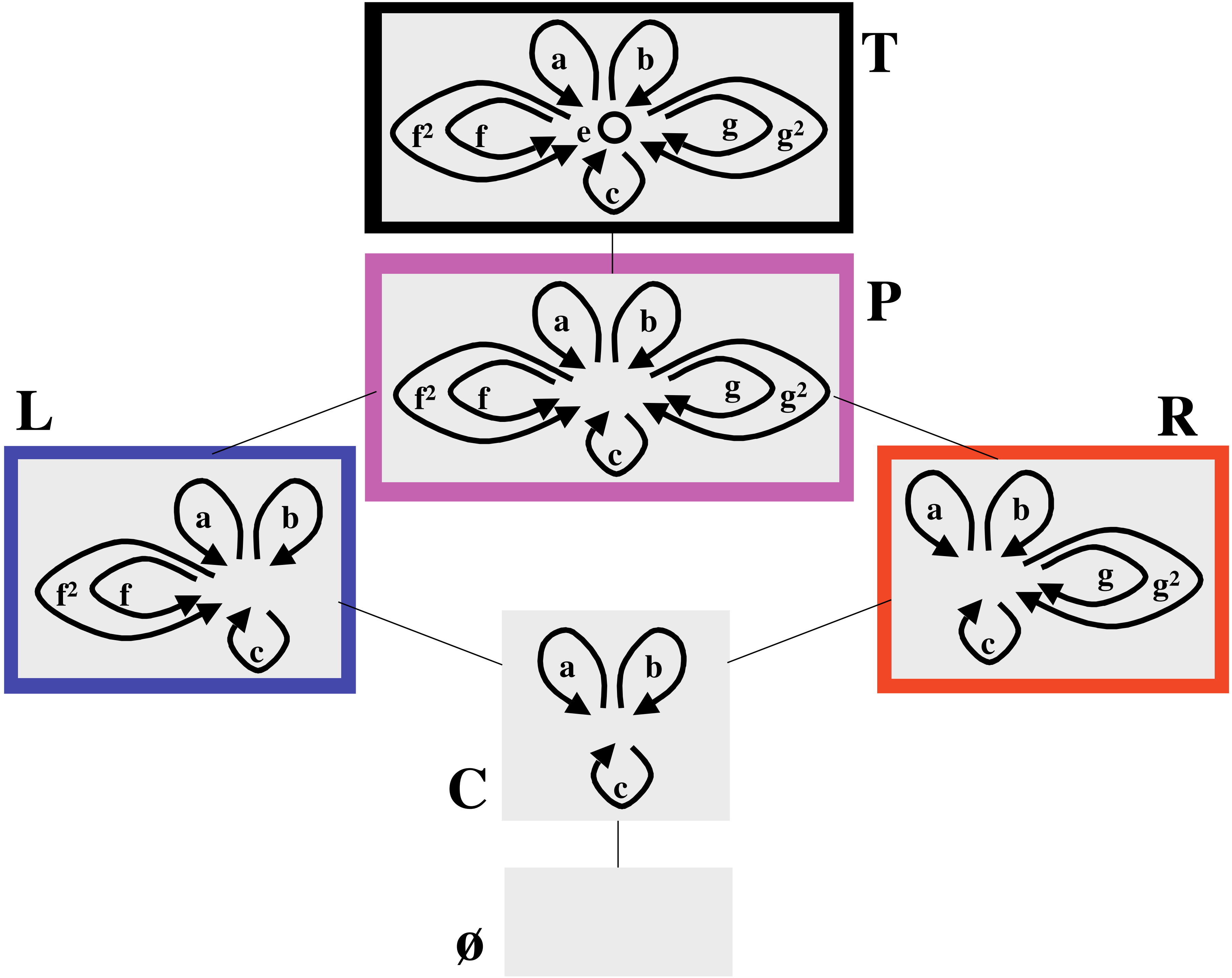


Pipes are mapped to keys

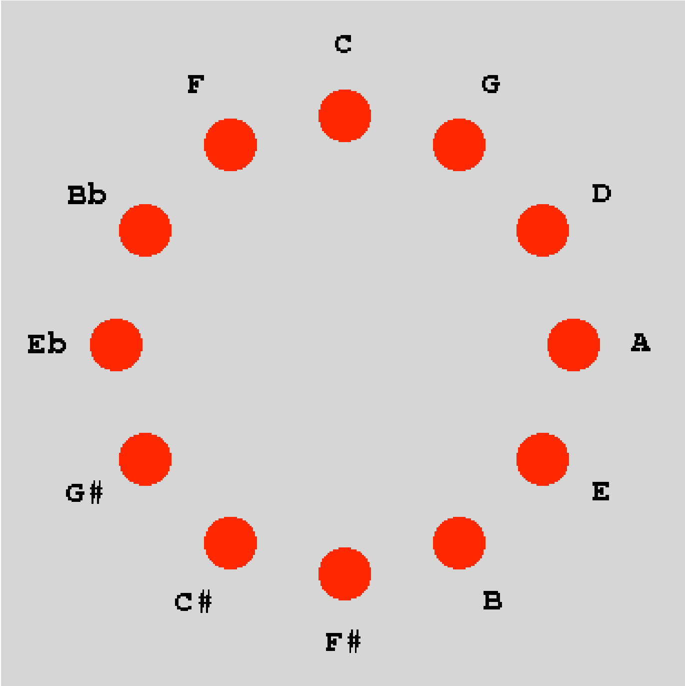
Perspectival Action

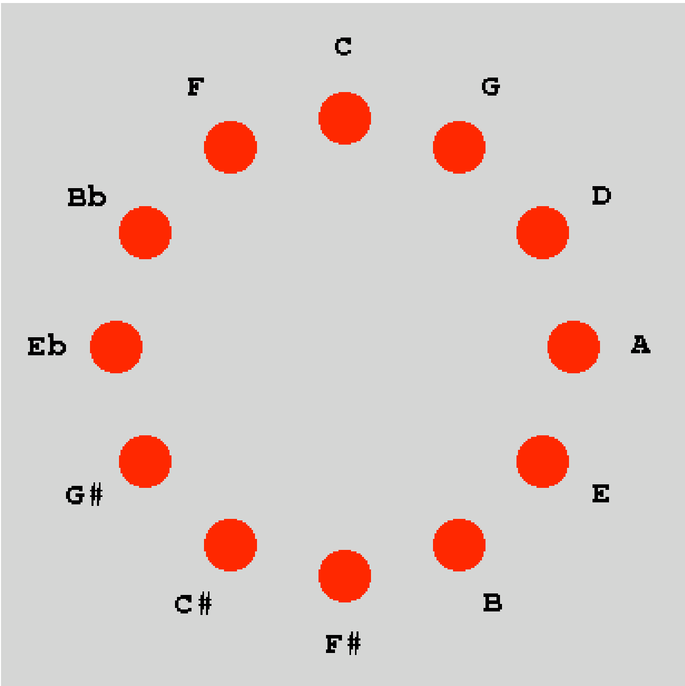


Tones are mapped to tones

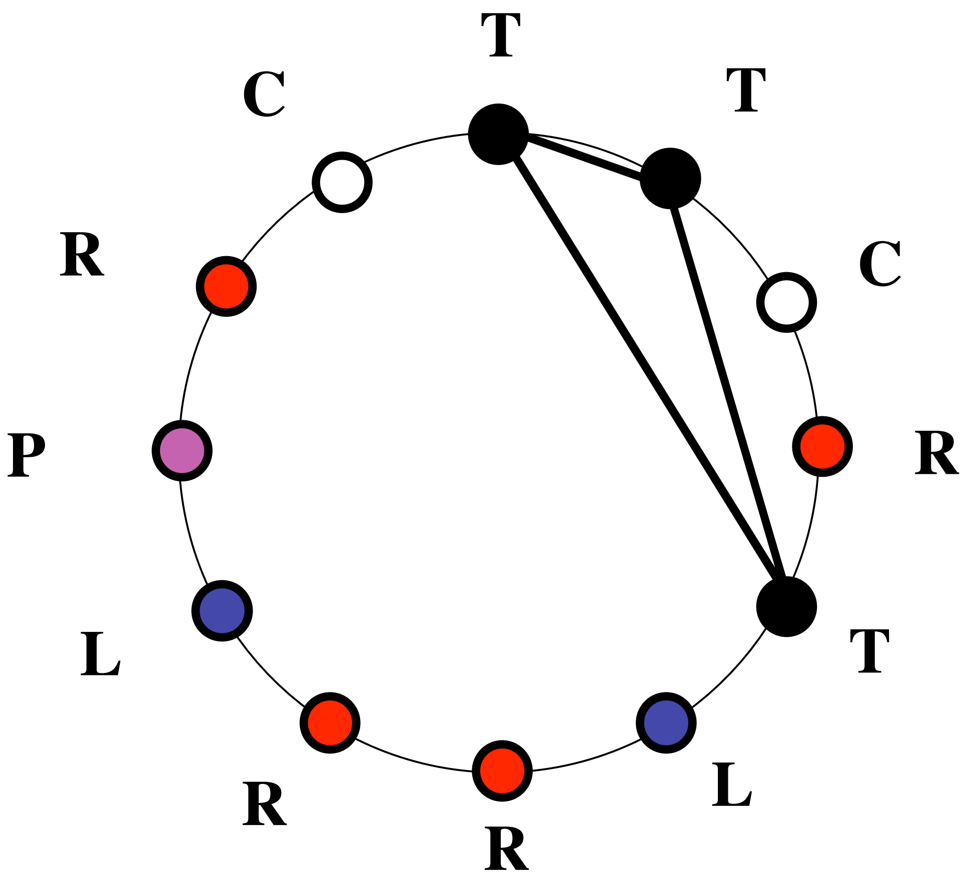


T
a
b
f²
f
e
g
g²
c
P
a
b
f²
f
g
g²
c
L
a
b
f²
f
c
R
a
b
g
g²
c
a
b
C
c
ø



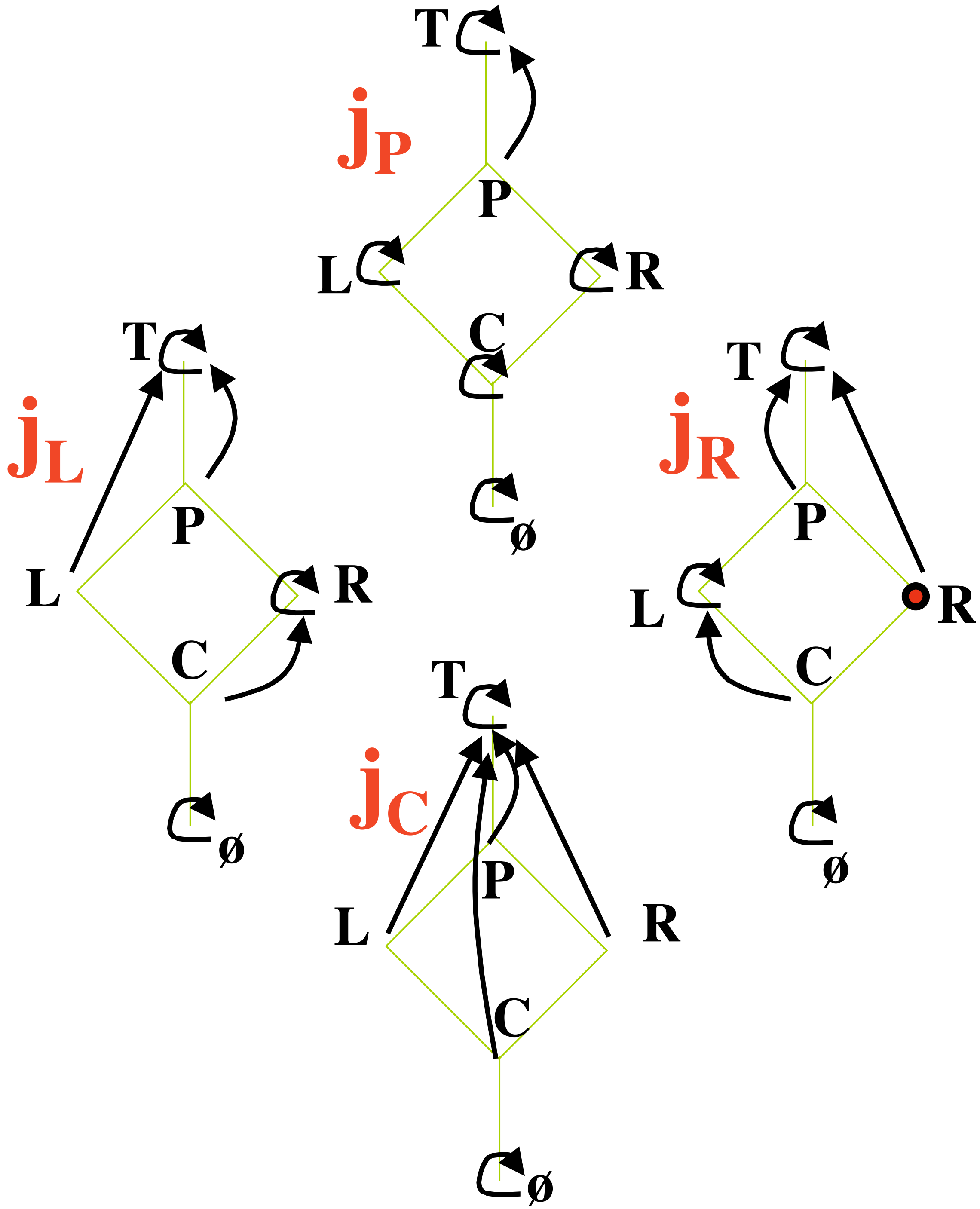


tic Function of {0, 4, 1}

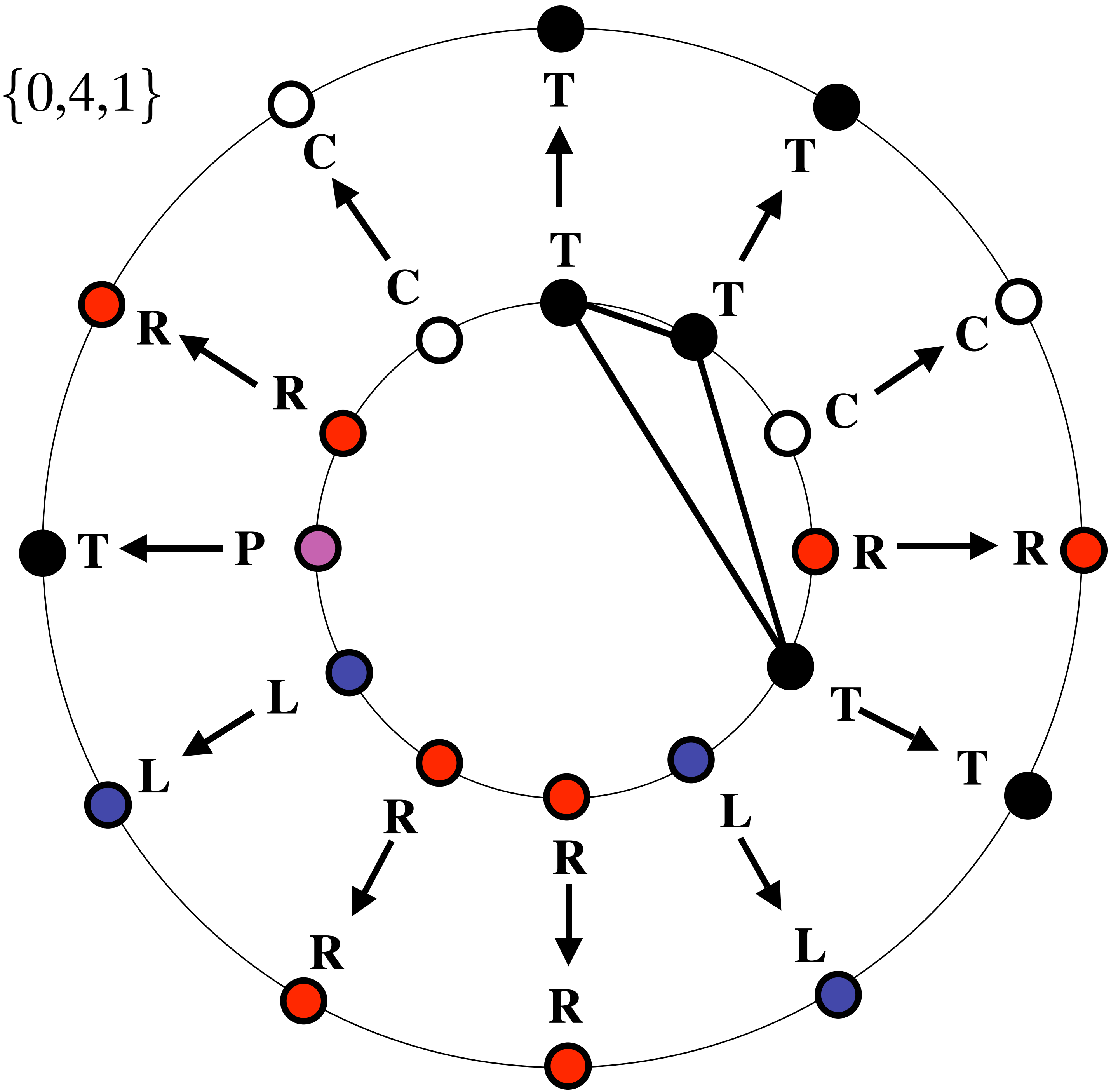


Lawvere-Tierney Topologies

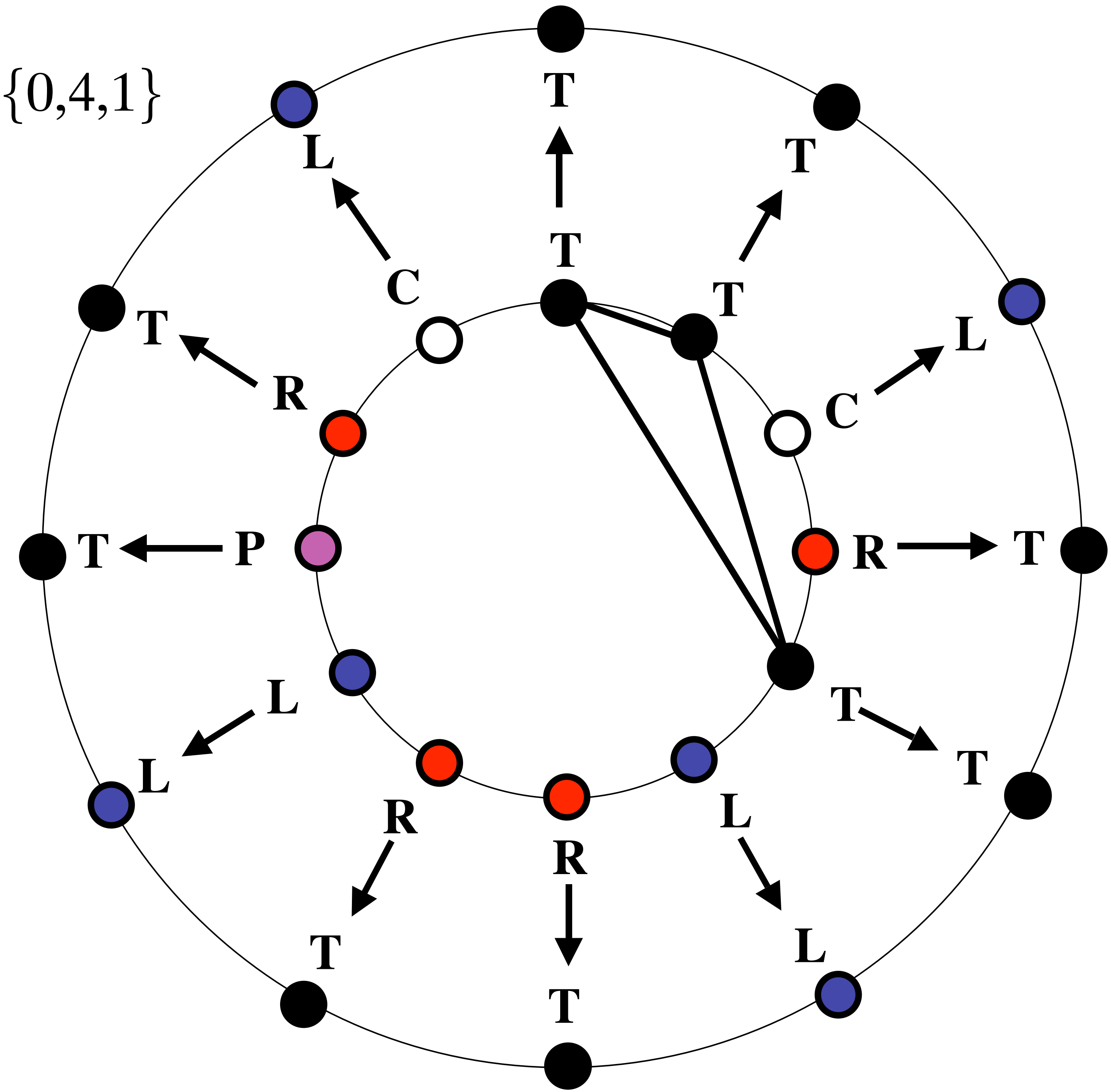
as Upgrade Operations



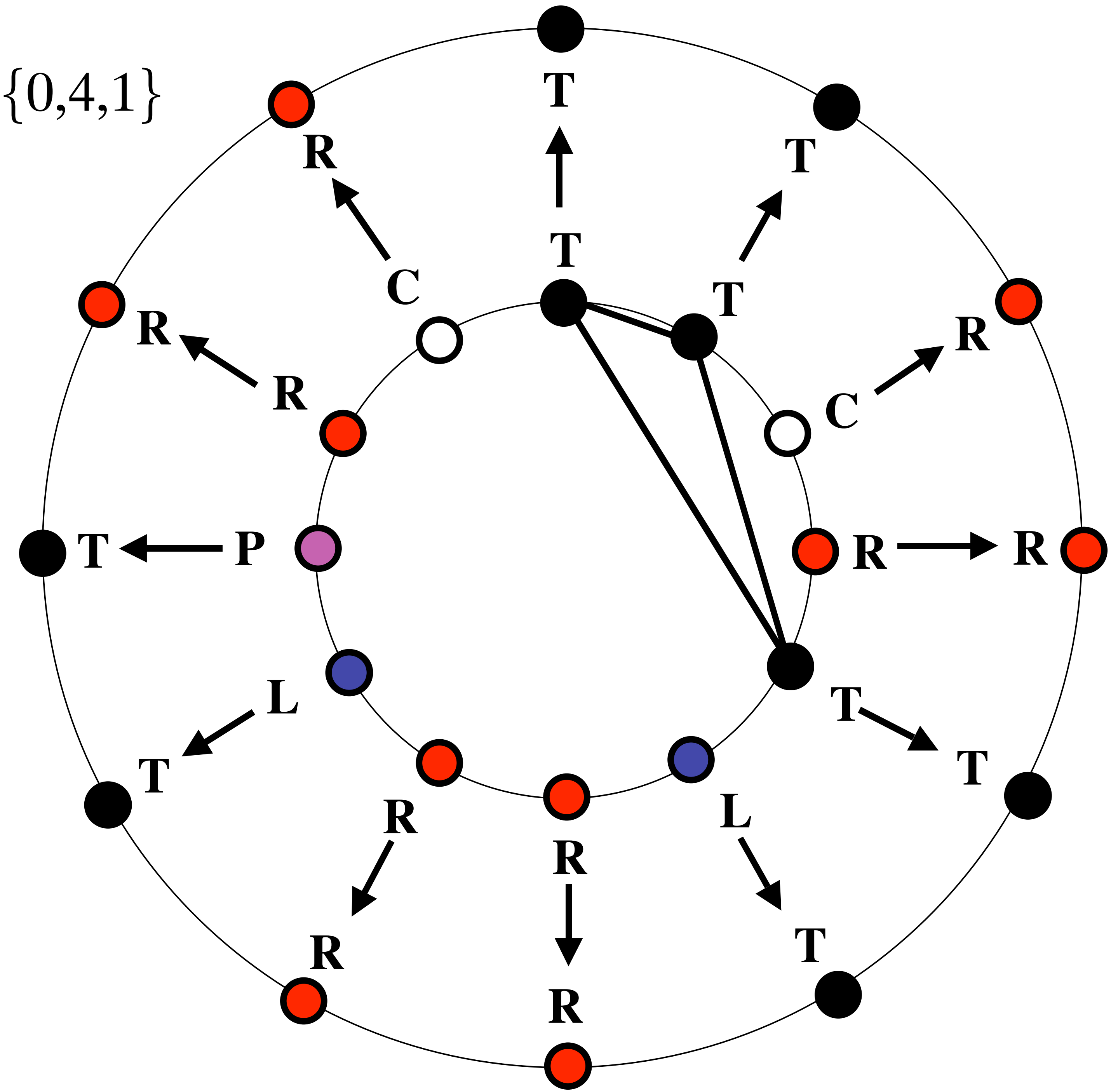
$j_P \circ \chi_{\{0,4,1\}}$



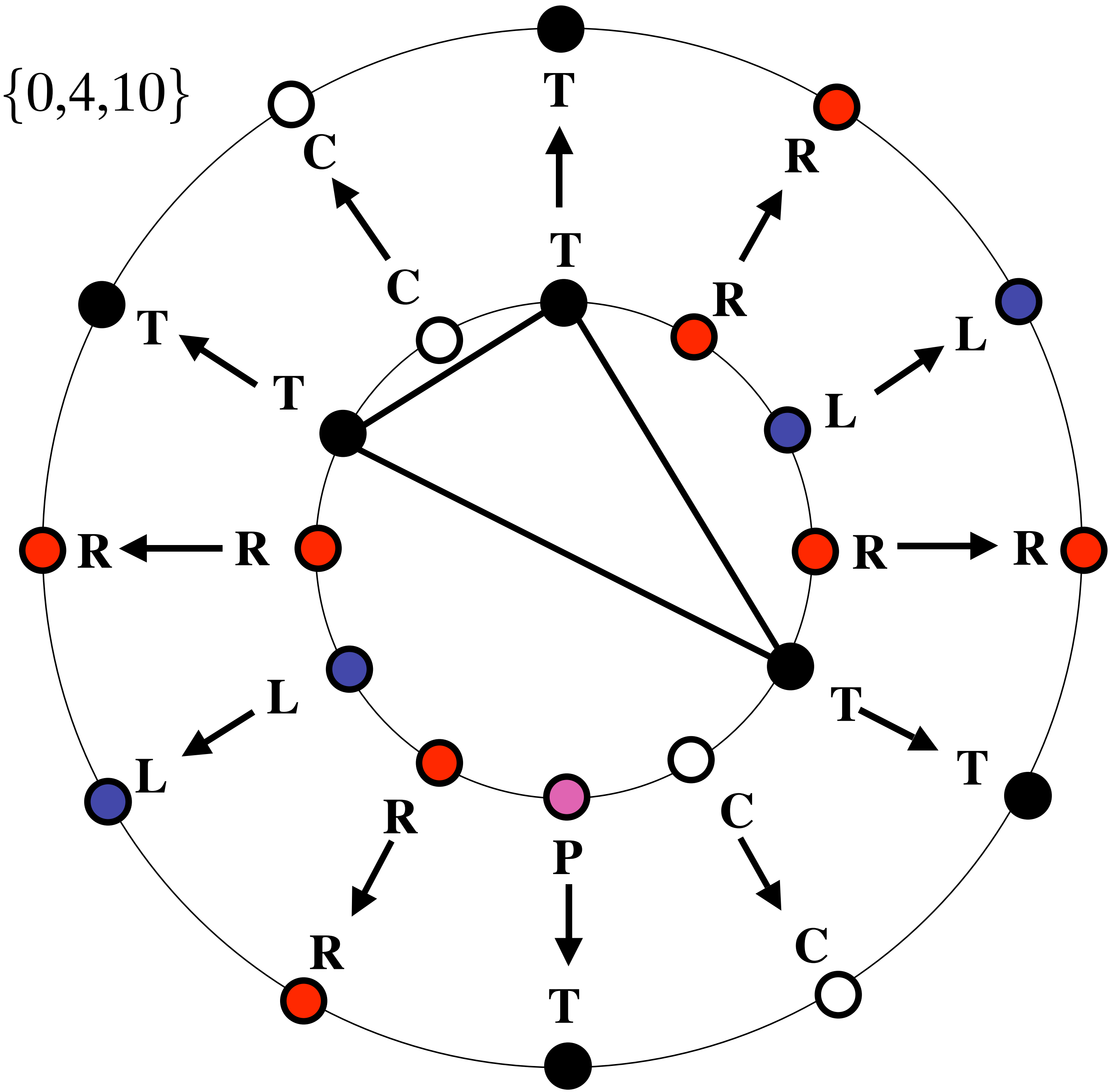
$j_R \circ \chi_{\{0,4,1\}}$



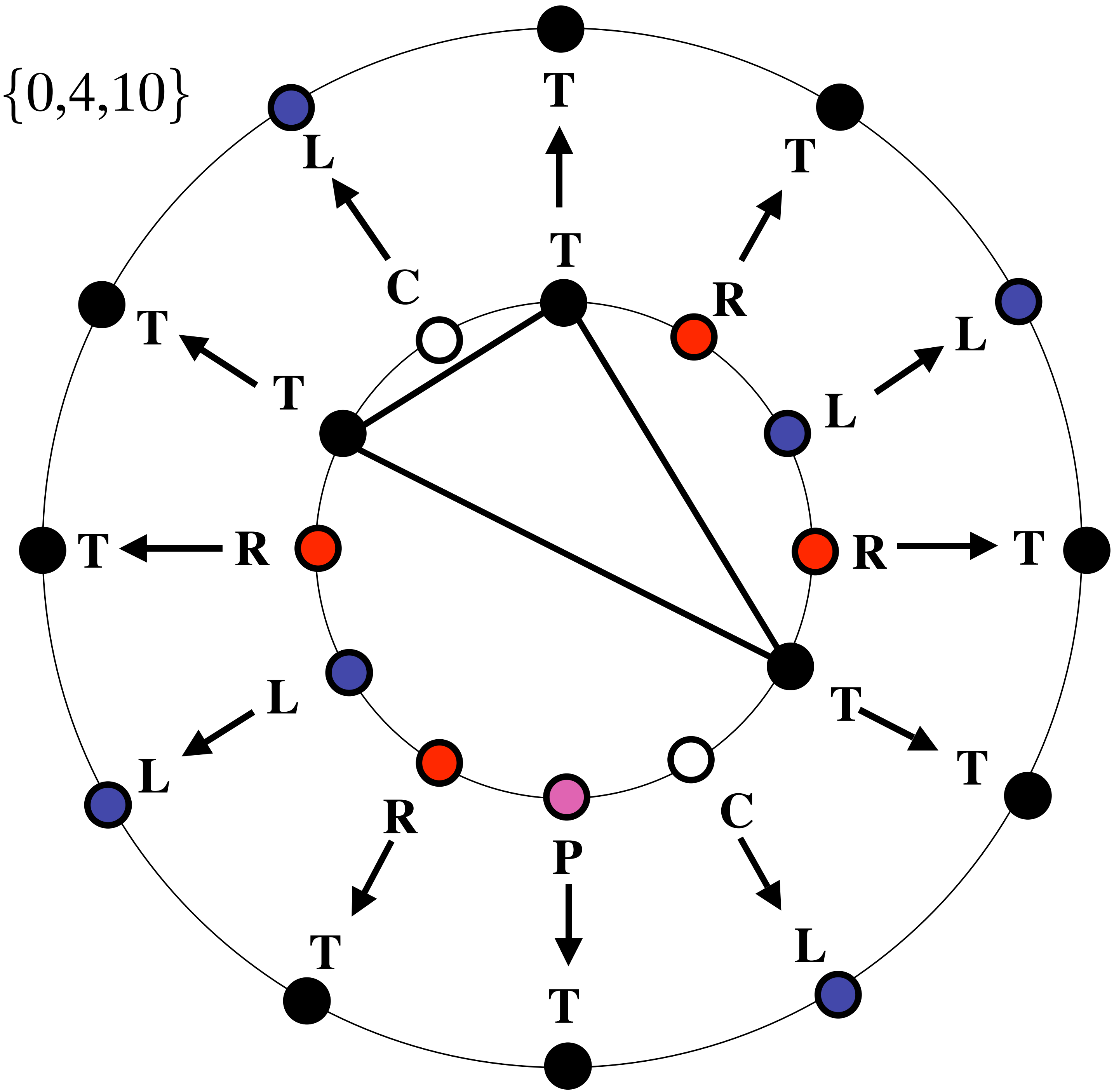
$j_L \circ \chi_{\{0,4,1\}}$



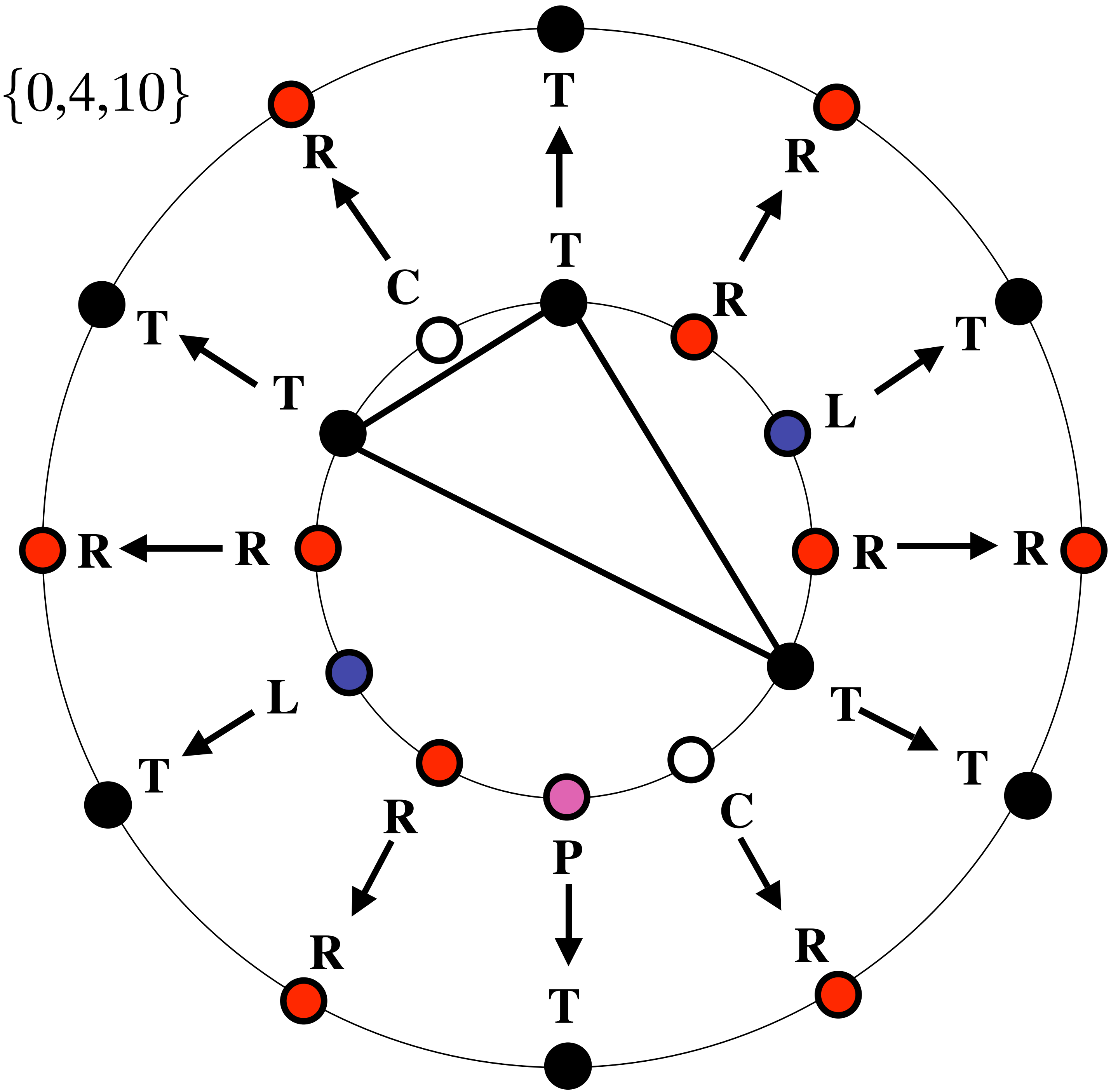
$j_P \circ \chi_{\{0,4,10\}}$



$j_R \circ \chi_{\{0,4,10\}}$

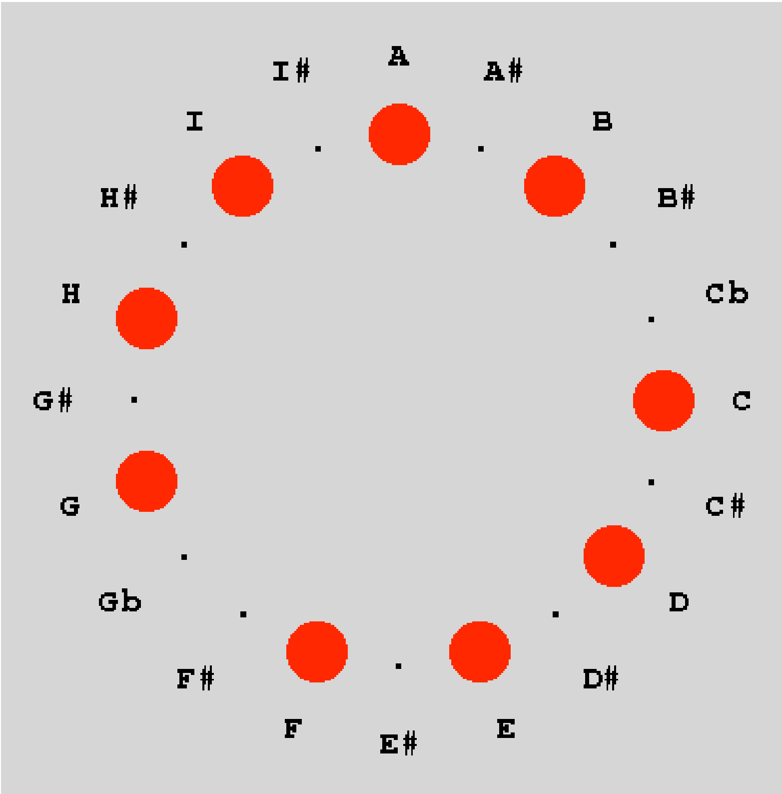


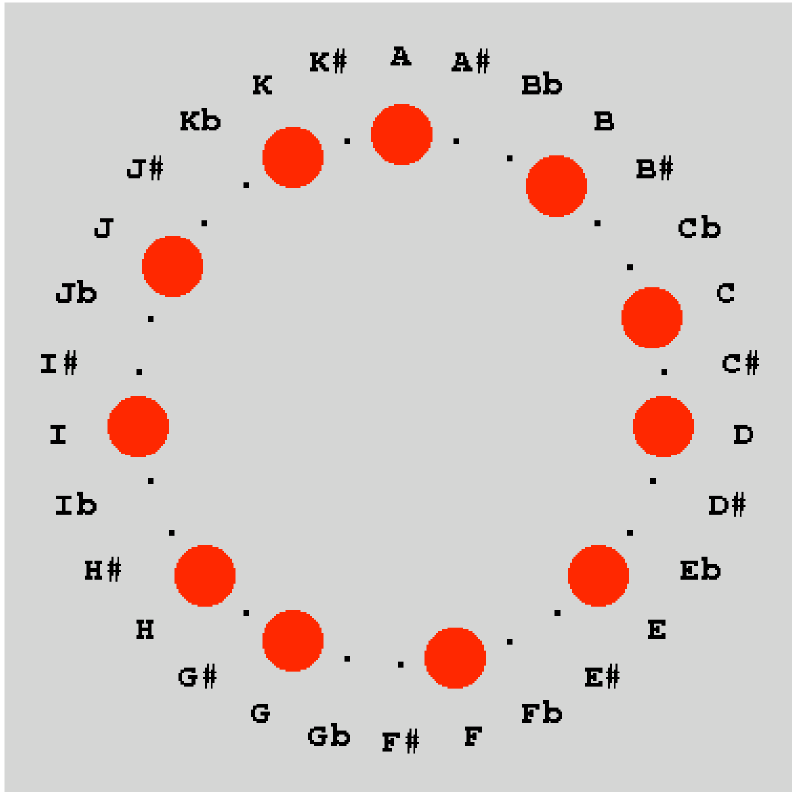
$j_L \circ \chi_{\{0,4,10\}}$

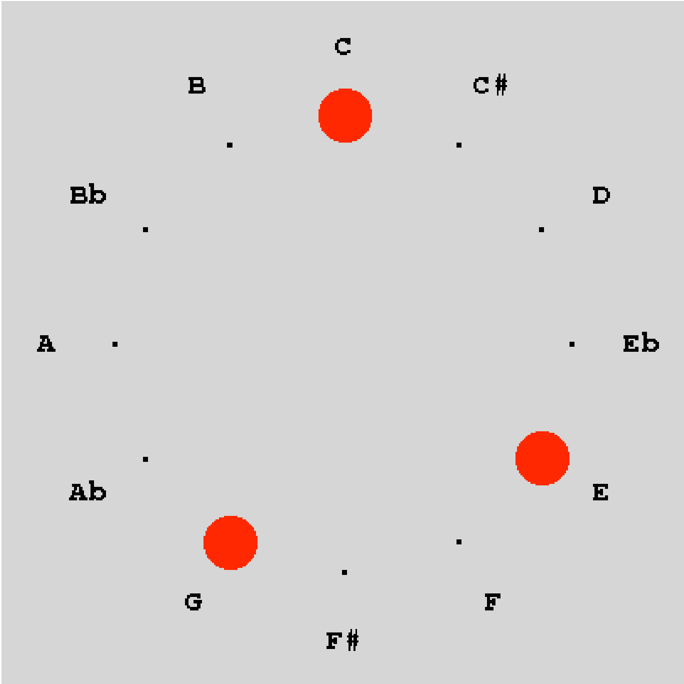


Fifth - Transposition through Chromatic Alteration

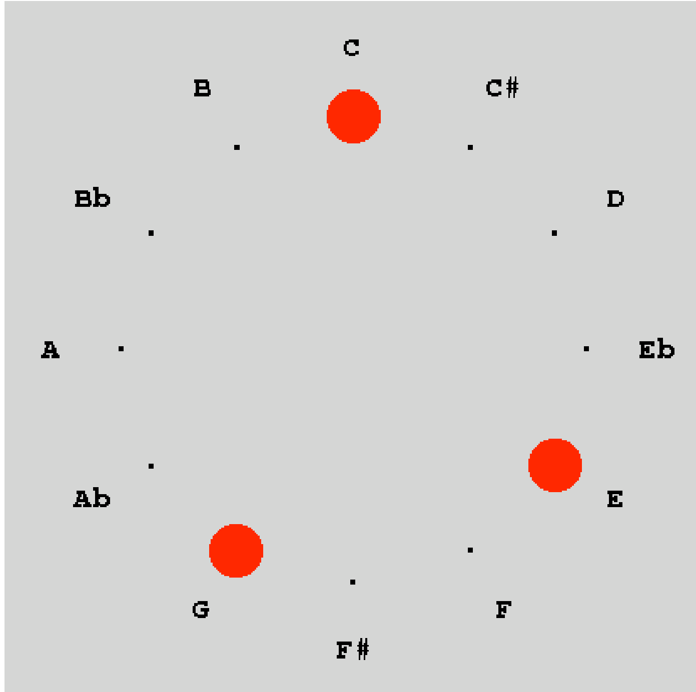
$$Z_{20} = Z_{4(4+1)} \qquad Z_{30} = Z_{5(5+1)}$$

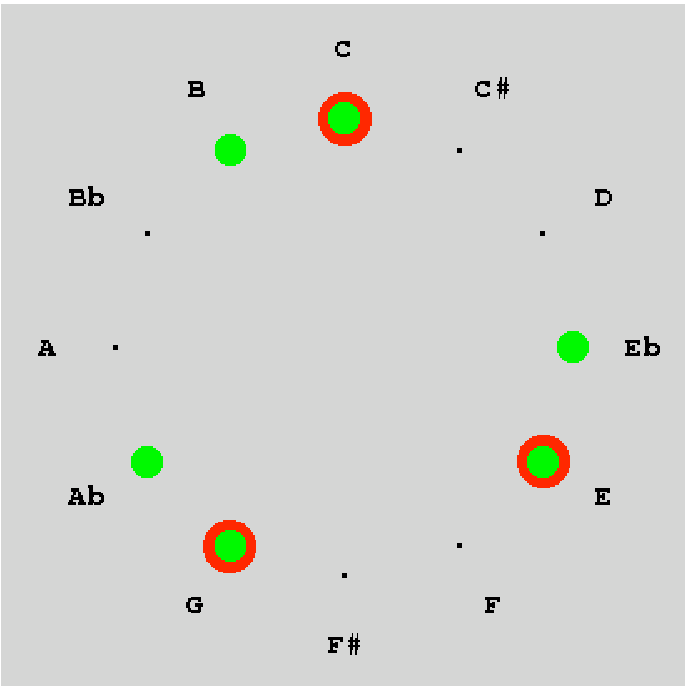




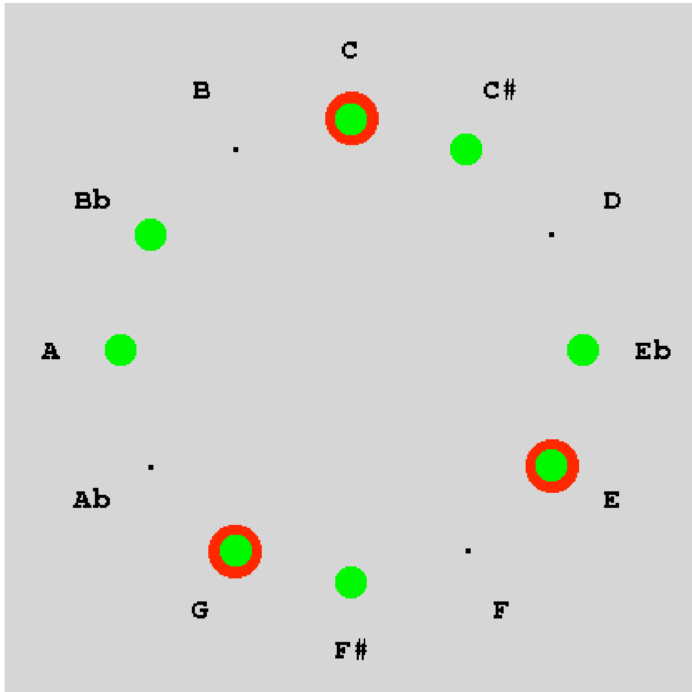


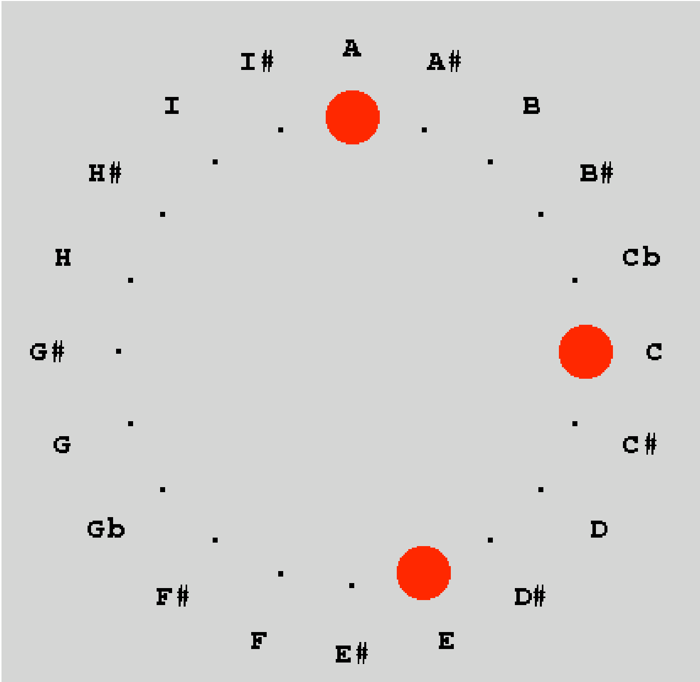
Triadic Action on Z_{12}



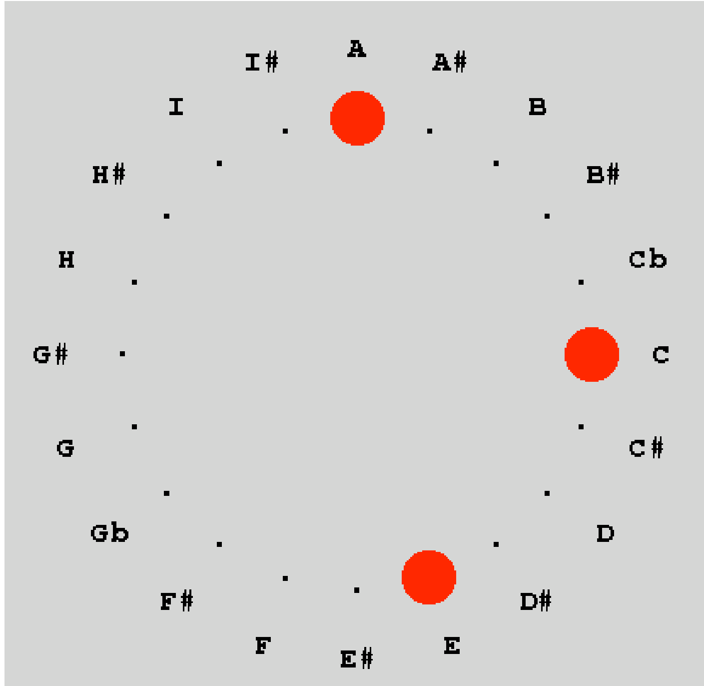


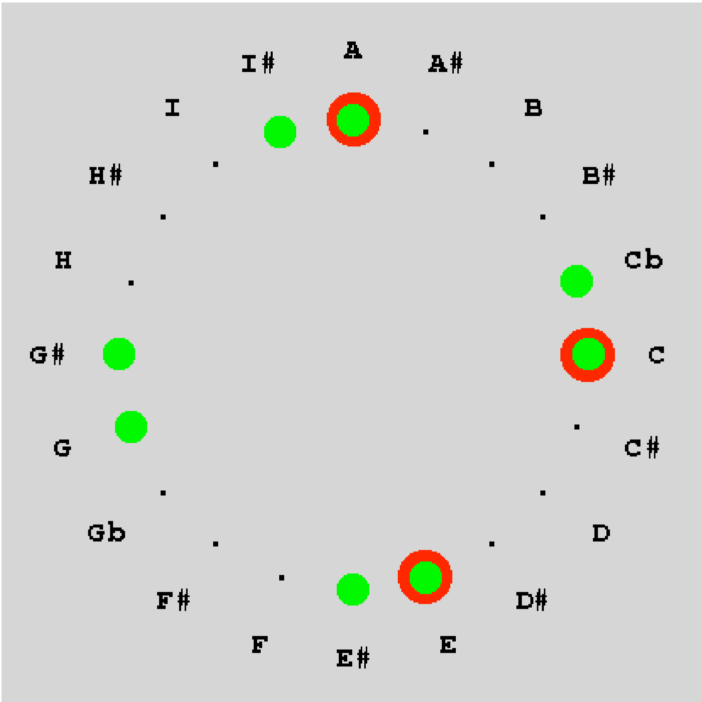
Upgrades



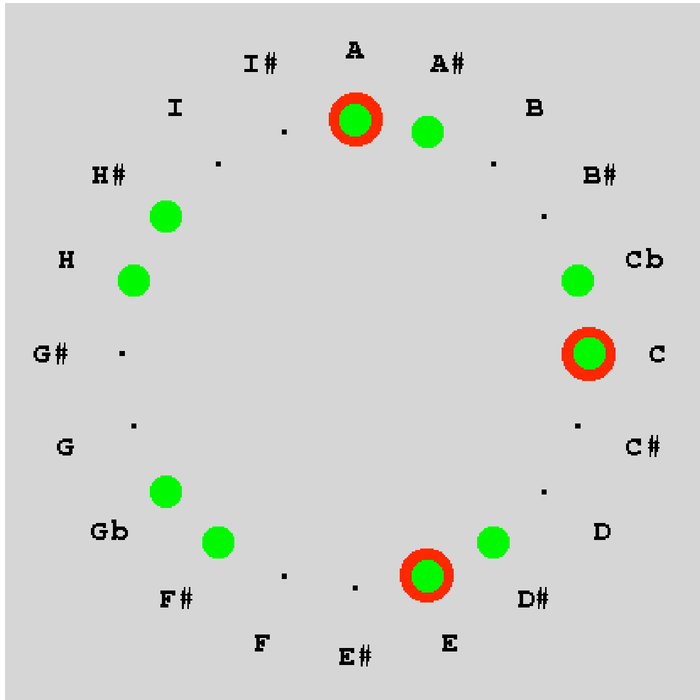


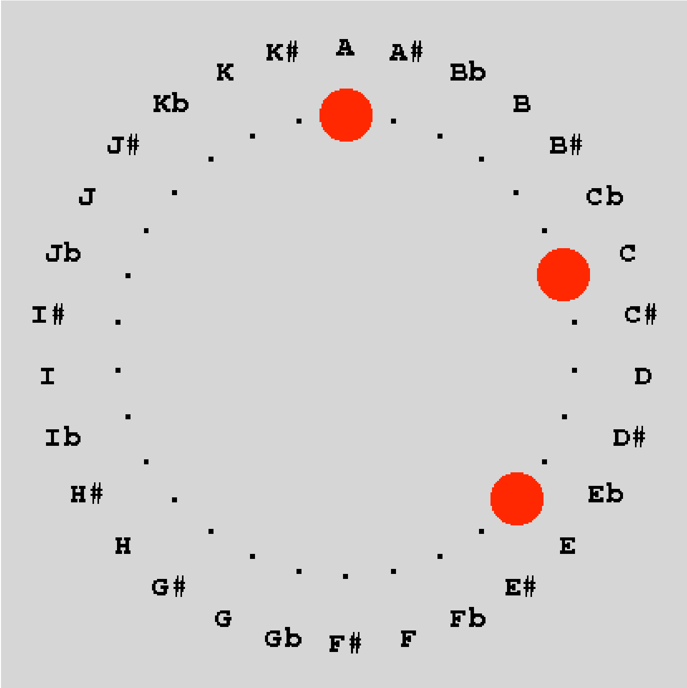
Triadic Action on Z_{20}



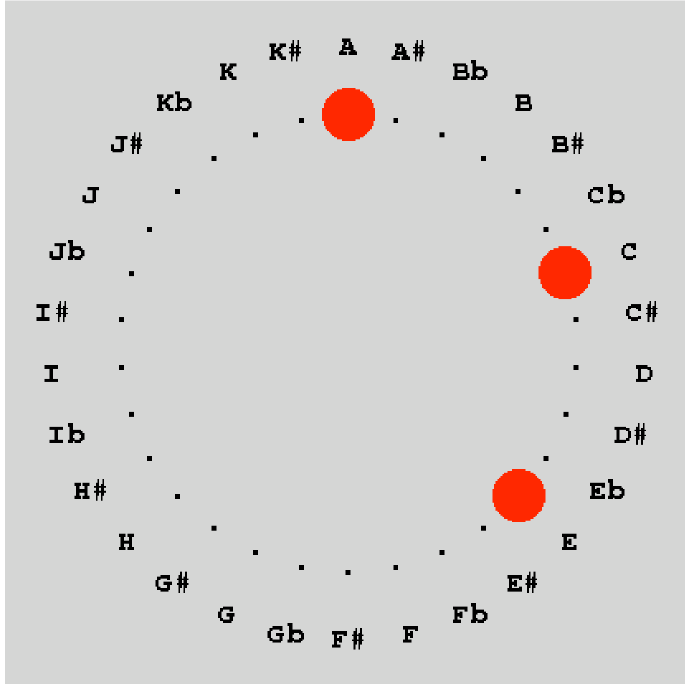


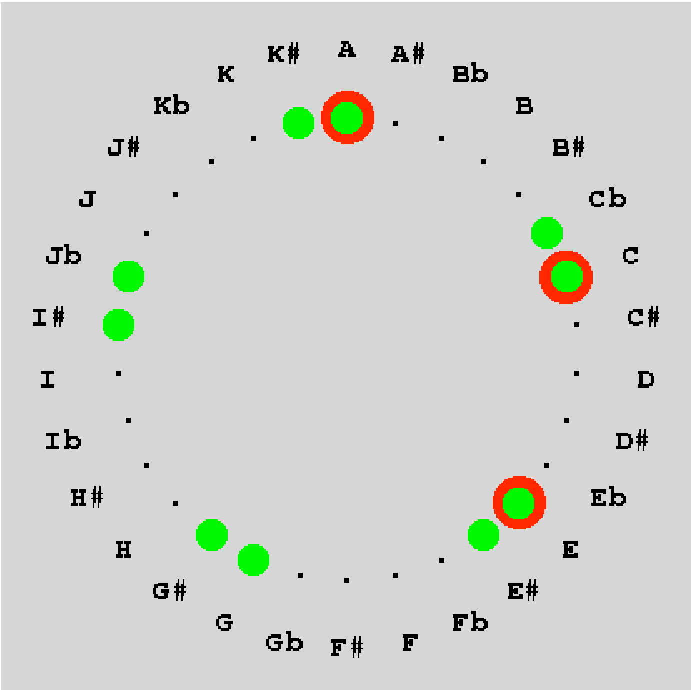
Upgrades





Triadic Action on Z_{30}





Upgrades