Dodecaphonic Knots and Topology of Words

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Olivier Messiaen Modes de valeurs et d’intensité for piano (1949). Extension of the twelve-tone technique to all parameters (not only pitch classes but rhythms, articulation, dynamics and so on).

Jean Barraqué, Le temps restitué (1957). Multiple rows (*Proliferation of series*)

Serialism with Microtones (Alain Bancquart, Jean-Etienne Marie, Ben Johnston). Extension to N-tone rows in equal or non-equal temperaments.
Voici le mode:

I

(1a Division I est utilisée dans la portée supérieure du Piano)

II

(1a Division II est utilisée dans la portée médiane du Piano)

III

(1a Division III est utilisée dans la portée inférieure du Piano)

O. Messiaen, Modes de valeurs et d’intensité
Twelve-tone Technique

Identify the 12 notes with $\mathbb{Z}_n$ i.e. 0=do, 1= do #, etc.

- Choose a 12-tone row.
- Example: Jean Barraqué ...au-delà du hasard
- $P = (0, 8, 7, 1, 4, 2, 10, 3, 11, 5, 6, 9)$
- Apply the 4 basic following transformations
  - Transpose up or down: $T_n(x) = x + n \mod 12$
  - Retrograde = reverse in time $R(P) = 9, 6, 5, 11$, etc.
  - Inversion = reverse in pitch $I_n(x) = -x + n \mod 12$
  - Retrograde Inverse = composition $RI$

$(P, I, R, RI) = \text{Klein group} = \text{dihedral group } D_2$ with presentation

$$D_2 = \langle a, b \mid a^2 = b^2 = (ab)^2 = 1 \rangle$$

with $1 = P$, $a = I$, $b = R$

- Material for composition = at most $4 \times 12 = 48$ rows, called derivative forms
Enumeration of Tone Rows

Action of the dihedral group $D_n$ on the permutation group $S_n$

Theorem

In the $n$-tone equal temperament ($n \geq 3$), under the equivalence of the derived forms (i.e., under the action of the dihedral group), there are

$$\begin{cases} \frac{1}{4}(n-1)! + 2^{(n-4)/2} \frac{1}{n} \left( \frac{n}{2} + 1 \right)! & \text{if } n \text{ is even} \\ \frac{1}{4}(n-1)! + 2^{(n-5)/2} \left( \frac{n-1}{2} \right)! & \text{if } n \text{ is odd} \end{cases}$$

tone rows of $n$ pitch classes.

In particular, for $n = 12$, $|S| = 9,985,920$ rows.

The problem of counting the tone rows of $S$ is equivalent to count the number of orbits of the group $G = D_{12} \times \mathbb{Z}_2$ generated by the transformations $T$, $I$ and $R$ on the series $P$. $D_{12}$ denotes the dihedral group (of order 24). Burnside’s lemma says that the number of orbits is the average number of fixed points.
Tone Rows with Limited Derivative Forms

**Remark**

12!/48 = 479,001, 600/48 = 9,979,200 is not the number of 12-tone rows because some tone rows have less than 48 derivative series.

**Example 1**

A = (0, 4, 8, 11, 3, 7, 1, 9, 5, 2, 10, 6) verifies

\[ T_6(A) = R(A) = (6, 10, 2, 5, 9, 1, 7, 3, 11, 8, 4, 0) \]

Thus \[ T_n(A) = R T_{n+6}(A) \] and \[ I_n(A) = R I_{n+6}(A) \].

The series A has only 24 derivative forms.

**Example 2**

A = (0, 3, 11, 2, 10, 1, 6, 9, 5, 8, 4, 7) is equal to \[ R I_7(A) \].
How to build a Chord diagram?

1. Take a row of twelve pitch classes: 0, 8, 6, 7, 2, 4, 9, 11, 1, 5, 3, 10
2. Place the numbers on a circle
3. Join the tritone pairs together
4. Just keep the structure

This structure is called a chord diagram.
Proposition

The 48 forms of the series are represented by the same chord diagram. i.e. Chord diagrams are invariant under the action of the dihedral group.

Example:
Retrogradation is obtained by mirror symmetry and rotation.
Theorem (A. Khruzin)

Under the action of the dihedral group, the number of chord diagrams for the equal temperament with $2n$ degrees is

$$d_n = \frac{1}{2}(c_n + \frac{1}{2} (\kappa_{n-1} + \kappa_n))$$

$$c_n = \frac{1}{2n} \sum_{i|2n} \varphi(i) \nu_n(i), \quad \kappa_n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n!}{k!(n-2k)!}$$

$\varphi(i)$ is the Euler totient function and $\nu_n$ is given for all divisors of $2n$ by

$$\nu_n(i) = \begin{cases} 
  \frac{i^n}{i}(2n/i - 1)!! & \text{if } i \text{ is odd} \\
  \sum_{k=0}^{\left\lfloor \frac{n}{i} \right\rfloor} \left( \frac{2n/i}{2k} \right)^i k(2k - 1)!! & \text{if } i \text{ is even}
\end{cases}$$
For the first $2n$-equal temperaments,

<table>
<thead>
<tr>
<th>$2n$</th>
<th>$c_n$</th>
<th>$d_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>8</td>
<td>18</td>
<td>17</td>
</tr>
<tr>
<td>10</td>
<td>105</td>
<td>79</td>
</tr>
<tr>
<td>12</td>
<td>902</td>
<td>554</td>
</tr>
<tr>
<td>14</td>
<td>9 749</td>
<td>5 283</td>
</tr>
<tr>
<td>16</td>
<td>127 072</td>
<td>65 346</td>
</tr>
<tr>
<td>18</td>
<td>1 915 951</td>
<td>966 156</td>
</tr>
<tr>
<td>20</td>
<td>32 743 182</td>
<td>16 411 700</td>
</tr>
<tr>
<td>22</td>
<td>625 002 933</td>
<td>312 702 217</td>
</tr>
<tr>
<td>24</td>
<td>13 176 573 910</td>
<td>6 589 356 711</td>
</tr>
</tbody>
</table>

Interest: about 10 millions rows are represented by only 554 diagrams.
Knots with six double points

Oriented Knots with six double points represent rows and chord diagrams

Gauss word is the smallest integer of the representation of the six chords pairs

121324563654

Chord diagrams are classified by Gauss words.
Adjacent matrix $A_{ij} = 1$ iff $(i, j)$ are connected (0 otherwise). The rank of $A$ is an invariant.

$$\#\text{faces}(D) = \text{corank}(A) + 1$$

Example Superscripto :

$$rk(A) = \det(A) = 6, \#\text{faces}(D) = 7 - rk(A) = 1$$
Vassiliev’s invariants $v_0 = 0$ and

$$v(\overleftarrow{\overrightarrow{\text{)}}} := v(\overleftarrow{\overrightarrow{\text{)}}} - v(\overrightarrow{\overleftarrow{\text{)}}),$$

are related to Jones polynomials $J_K(t)$

$$t^{-1}J_\overleftarrow{\overrightarrow{\text{)}}} - tJ_\overrightarrow{\overleftarrow{\text{)}}} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) J^{\uparrow\uparrow}_{\overrightarrow{\overleftarrow{\text{)}}}}$$

Let $t = e^h$ and $n$ the number of crossings, we have

$$J_K(h) = \sum_{n=0}^{\infty} v_n(K) h^n$$

Coefficients $v_n \in \mathcal{V}_n$ with $\mathcal{V}_0 \subset \mathcal{V}_1 \subset ... \mathcal{V}_n \subset ...$

Finite type invariants = Vassiliev’s invariants $= \mathcal{V} = \bigcup_n \mathcal{V}_n$
Vassiliev’s invariants

From chord diagram $D$

$$v(D) = \sum_s \left( \prod_c s(c) \right) \left( -2 \right)^{|s| - 1}$$

$c$ is the label of the chord ($c = 1, \ldots, 6$)
$s$ is a state obtained from $D$ by deleting chords.
$s(c) = 1$ if $c$ is kept
$s(c) = 2$ if $c$ is deleted
The sum is taken over all $2^6 = 64$ states of $D$
(removing $p$ chords, $p = 0, \ldots, 6$).
$|s|$ is the number of components of state $s$. 
Example: For $n = 6$, the knot $K = 6_2$

$$J_K(t) = \frac{3}{t} - \frac{2}{t^2} + \frac{2}{t^3} - \frac{2}{t^4} + \frac{1}{t^5}$$

$$J_K(e^h) = 2 - 2h + 3h^2 - \frac{19}{3}h^3 + \frac{41}{4}h^4 - \frac{751}{60}h^5 + \frac{487}{40}h^6 + O(h^7)$$

The Vassiliev's invariant is $v_6 = 487/40$. 
Artin braid groups

Definition

$B_n$ is the group generated by $(n - 1)$ generators $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ and the braid relations

\[ \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for all } i = 1, 2, ..., n - 1 \text{ and } |j - i| \geq 2 \]

and

\[ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{for all } i = 1, 2, ..., n - 2 \]

$B_1 = \{1\}$, $B_2 = \{\sigma_1\}$ has no braid relations.

$B_n$ is an infinite cyclic group.
Burau representations

Linear representations by matrices over the ring of Laurent polynomials $\mathbb{Z}[t^{-1}, t]$

For $i = 1, \ldots, n - 1$

$$A_i = \begin{pmatrix}
  I_{i-1} & 0 & 0 & 0 \\
  0 & 1 - t & t & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & I_{n-i-1}
\end{pmatrix}$$

The Burau representation $\Psi_n$ is faithful for $n \leq 3$ (i.e. $\ker \Psi_n = \{1\}$) and non faithful for $n \geq 5$. It is unknown if $\Psi_4$ is faithful.
Braids in Messiaen’s *Etudes de rythmes*

From *Modes de valeurs et d’intensités*, define

\[
\begin{align*}
\sigma_1 &= \text{Motive } (1, 2, 3) \text{ of series I} \\
\sigma_1^{-1} &= \text{Motive } (10, 11, 12) \text{ of series I} \\
\sigma_2 &= \text{Motive } (1, 2, 3) \text{ of series II} \\
\sigma_2^{-1} &= \text{Motive } (10, 11, 12) \text{ of series II} \\
\sigma_3 &= \text{Motive } (1, 2, 3) \text{ of series III}
\end{align*}
\]

On the first pages

\[
\sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_3 \sigma_1^{-1} \sigma_2 \sigma_1
\]
Each chord diagram $D_n$ is associated with a permutation $P_n$
Luigi Nono *Canto Sospeso* $D_{358}$ is associated with (in cyclic notation)

$$(0, 1)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)$$

The generating power of this row is measured by the order of the *Serial group* defined by

$$G_n = \langle P_n, T, I \rangle$$

$T = \text{permutation associated with transposition}$

$$T = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$$

$I = \text{permutation associated with inversion}$

$$I = (1, 11)(2, 10)(3, 9)(4, 8)(5, 7)$$
Examples of serial groups

- A. Webern *Concert for nine instruments*, opus 24. 
  \( S = 11, 10, 2, 3, 7, 6, 8, 4, 5, 0, 1, 9 \)

\[
G_{549} = \left\langle a, b, c \mid a^2 = c^2 = (b^{-1}c)^2 = (acb)^2 = (ab^2c)^2 = (ab^{-1})^4 = (abab^{-1})^3 = 1 \right\rangle
\]

This group has 192 elements.

- Schoenberg's *Ode to Napoleon*, Kees van Baaren's *Variations in Isometrical Series*

  \( 5, 10, 6, 9, 7, 8, 2, 1, 3, 0, 4, 11 \)

is associated with the diagram \( D_{538} \). The associated group

\[
G_{538} = \langle P_{538}, T, I \rangle
\]

has 7680 elements.
Examples of serial groups

- Dallapiccola’s *Quaderno Musicale di Annalibera*

  
  10, 11, 3, 6, 8, 2, 7, 1, 5, 9, 0, 4

  
  is associated with the diagram $D_{56}$. The group generated by the three operators

  
  $$G_{56} = \langle P_{56}, T, I \rangle$$

  
  has presentation

  
  $$G_{56} = \left\langle a, b, c \mid a^2 = c^2 = (b^{-1}c)^2 = (ac)^4 = acb^{-1}(abc)^2 = (acb^3)^2 = (ab^2ab^{-2})^2 = b^{12} = (acab^{-2})^4 \right\rangle$$

  
  It is a group of order 28800.

- Many diagrams have a maximum generating power

  
  $$|\mathcal{S}_{12}| = 479\,001\,600$$
Thermodynamics of braids

Partition function is linked with Jones polynomial. For a braid
\[ k = \sigma_{i_1}^{\varepsilon_1} \ldots \sigma_{i_m}^{\varepsilon_m} \]
\[ Z_k = (1 + e^\beta) e^{-\beta \frac{1+w}{2}} J_k(e^\beta) \]

Tait number = writhe = \( w = \varepsilon_1 + \ldots + \varepsilon_m \)

Energy and entropy are defined by
\[ U = -\frac{\partial \log Z}{\partial \beta} \quad S = \beta U + \log Z \]

Example: The entropy of the knot
\( 6_2 = \sigma_1^{-1} c \sigma_1^{-1} \sigma_2^3 \)
has a discontinuity at \( \beta = 0.492 \ldots \)
Braiding the rows lattice

\[ S = \text{Basic row}, \ T_n = \text{transpositions}, \ I_n = \text{inversions}. \]

\[ \Phi : T_n(S) \rightarrow \sigma_n, \ I_n(S) \rightarrow \sigma_n^{-1} \]

Jean Barraqué, *Sonata for piano,*

\[ S_1 l_9 S_2 l_8 S_6 l_{12} \ldots \rightarrow \sigma_1 \sigma_9^{-1} \sigma_2 \sigma_8^{-1} \sigma_6 \sigma_{12}^{-1} \ldots \]
Maps and constellations

**Definition**

A *topological map* $M$ is a graph $\Gamma$ embedded into a surface $X$ such that
- the vertices are distinct points of the surface
- the edges are curves on the surfaces that intersect only at vertices
- the set $X \setminus \Gamma$ is a disjoint union of connected components called faces such that each face is homeomorphic to an open disk.

**Definition**

A *$k$-constellation* of degree $n$ is a $k$-uplet $[\sigma_1, \ldots, \sigma_k]$ with $\sigma_j \in S_n$ such that the monodromy group $G = \langle \sigma_1, \ldots, \sigma_k \rangle$ acts transitively on the set on $n$ points and

$$\sigma_1 \ldots \sigma_k = id$$
A (combinatorial) map $[\sigma, \alpha, \varphi]$ is a 3-constellation (= hypermap) in which $\alpha$ is an involution without fixed points.

$\sigma = \text{vertices}$, $\alpha = \text{edges}$, $\varphi = \alpha^{-1} \sigma^{-1} = \text{faces}$.

The group $\langle \sigma, \alpha \rangle$ is transitive = the map is connected.

Automorphism group of a map $M = \text{centralizer of the monodromy group } G$

$$Aut(M) = Z(G) = \{h, hg = gh, \forall g \in G\}$$

Euler’s Formula

$$\chi(M) = c(\sigma) + c(\alpha) + c(\varphi) = 2 - 2g$$

$c(x) = \text{number of cycles of the permutation } x$
Example of a 3-constellation

\[ \alpha = (1, 2)(3, 4)(5, 6) = \text{permutation of edges} \]
\[ \sigma = (1, 3, 5, 6)(2)(4) = \text{permutation of vertices} \]
\[ \varphi = (1, 2, 6, 3, 4)(5) = \text{permutation of faces} \]

Another representation as binary graph with black and white vertices
Let $G \subset \mathfrak{S}_n$ be a transitive group. A group $G \subset \mathfrak{S}_n$ is called transitive if its group action is transitive. In other words, if the group orbit $G(x)$ is equal to the set $\{1, 2, \ldots, n\}$ for some $x \in G$. A block system $\mathcal{B} = \{b_1, \ldots, b_m\}$ for $G$ is a partition of $\{1, \ldots, n\}$ invariant under $G$, (i.e. $\forall g \in G, \ g(b_i) = b_j$).

Example: $G = \langle (1, 2, 3, 4), (1, 3) \rangle$, $\mathcal{B} = \{\{1, 3\}, \{2, 4\}\}$

**Definition**

$G$ is primitive if the only block systems are trivial i.e. $\{1, 2, \ldots, n\}$ and $\{\{1\}, \{2\}, \ldots, \{n\}\}$. $G$ is imprimitive if $G$ is not primitive.

**Theorem**

*For $n = 12$, there are 301 transitive groups, but only 6 imprimitive groups.*

Blocks act as attractors for serial fragments, that is why primitive groups are important for serial music.
Olivier Messiaen’s *Ile de feu 2*

*Ile de feu 2* use two permutations in cyclic notation

\[ a = (1, 7, 10, 2, 6, 4, 5, 9, 11, 12)(3, 8) \]

and

\[ b = (1, 6, 9, 2, 7, 3, 5, 4, 8, 10, 11)(1, 2) \]

These permutations generate Mathieu’s group \( M_{12} \)

\[ \text{Order}(M_{12}) = 95040 \]
O. Messiaen : Livre d’orgue VI - Les yeux dans les roues

Built on six permutations (a permutation and five actions):

$$\sigma_0 = (1, 11, 6, 2, 9, 4, 8, 10, 3, 5)$$

and for $$j = 1, \ldots, 5$$, $$\sigma_j = A_j \sigma_0$$

Actions are defined by:

- $$A_1$$ = “Extremes au centre”
  $$= (2, 12, 7, 4, 11, 6, 10, 8, 9, 5, 3)$$

- $$A_2$$ = “Centre aux extrêmes”
  $$= (1, 6, 9, 2, 7, 3, 5, 4, 8, 10, 11)$$

- $$A_3$$ = “Rétrograde”
  $$= (1, 12)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)$$

- $$A_4$$ = “Extremes au centre, rétrograde” = $$A_1 A_3$$
- $$A_5$$ = “Centre aux extrêmes, rétrograde” = $$A_2 A_3$$

with $$a = A_2^{-1} A_1$$ and $$b = A_2^3 A_1 A_2^2 A_1$$

Mathieu group $$M_{12}$$ of order 95040

$$M_{12} = \langle a, b | a^2 = b^3 = (ab)^{11} = [a, b]^6 = (ababab^{-1})^6 = 1 \rangle$$
Given a chord diagram, how to build a 3-constellation? Ean Barraqué, ... au delà du hasard: 0, 8, 7, 1, 4, 2, 10, 3, 11, 5, 6, 9

\[ \alpha = (0, 6)(1, 7)(2, 8)(3, 9)(4, 10)(5, 11) \]

Read pitch classes along the diagram with doubling chords
0, 8, 2, 10, 4, 2, 8, 7, 1, 4, 10, 3, 9, 0, 6, 9, 3, 11, 5, ...

Delete doubles

\[ \varphi = (0, 8, 2, 10, 4, 7, 1, 3, 9, 6, 11, 5) \]

Compute the permutation of the vertices

\[ \sigma = \varphi^{-1} \alpha^{-1} = (0, 11)(3, 7, 10, 8, 6) \]
Using sets of limited transposition, we get a 3-constellation $[\alpha, \sigma, \varphi]$

- with $(0,6)$:
  it is the usual chord diagram $A : (0,6) \rightarrow (0,2)$
  $B : (1,7) \rightarrow (8,3)$ $C : (2,8) \rightarrow (4,1)$
  $D : $ etc. We associate the permutation
  $(edges) \alpha = (0,2)(1,4)(5,11)(3,8)(6,10)(7,9)$

- with $(0,4,8)$:
  $A : (0,4,8) \rightarrow (0,5,1)$
  $B : (1,5,9) \rightarrow (8,9,6)$
  $C : (2,6,10) \rightarrow (4,2,11)$
  $D : (3,7,11) \rightarrow (10,3,7)$
  We associate the permutation
  $(vertices) \sigma = (0,5,1)(8,9,6)(4,2,11)(10,3,7)$

- with $(0,2,4,6,8,10)$
  $A : (0,2,4,6,8,10) \rightarrow (0,4,5,2,1,11)$
  $B : (1,3,5,7,9,11) \rightarrow (8,101,9,3,6,7)$
  We associate the permutation
  $(faces) \varphi = (0,4,5,2,1,11)(8,10,9,3,6,7)$
Some other diagrams

- with \((0,3,6,9)\)
  - \(A : (0,3,6,9) \rightarrow (0,10,2,6)\)
  - \(B : (1,4,7,10) \rightarrow (4,9,1,7)\)
  - \(C : (2,5,8,11) \rightarrow (8,5,3,11)\)
  - We associate the permutation \(u = (0,10,2,6)(4,9,1,7)(8,5,3,11)\)

- with \((0,1,2,3,4,5,6,7,8,9,10,11)\)
  - \((0,1,2,3,4,5,6,7,8,9,10,11) \rightarrow (0,8,4,10,5,9,2,3,1,6,11,7)\)
  - We associate the permutation \(v = (0,8,4,10,5,9,2,3,1,6,11,7)\)

\([\alpha', \sigma', \varphi']\) is a 3-constellation with

\[\alpha' = \sigma, \sigma' = u^{-1}, \varphi' = v^{-1}, (uv = \sigma)\]
**Definition**

A Gauss diagram is a chord diagram with oriented chords. The orientation is determined by

\[ a \rightarrow b \iff a \leq b \]

**Turaev invariant polynomial**

\[ u(t) = \sum_{c,n(c) \neq 0} sgn(n(c)) t^{|n(c)|} \]

\[ n(1) = -2, \]
\[ n(2) = -1, n(3) = 0, n(4) = n(5) = n(6) = +1 \]

\[ u(t) = -t^2 + 2t \]