Dodecaphonic Knots and Topology of Words

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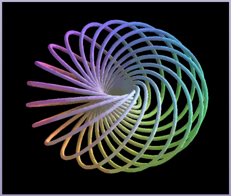

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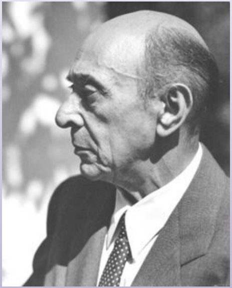
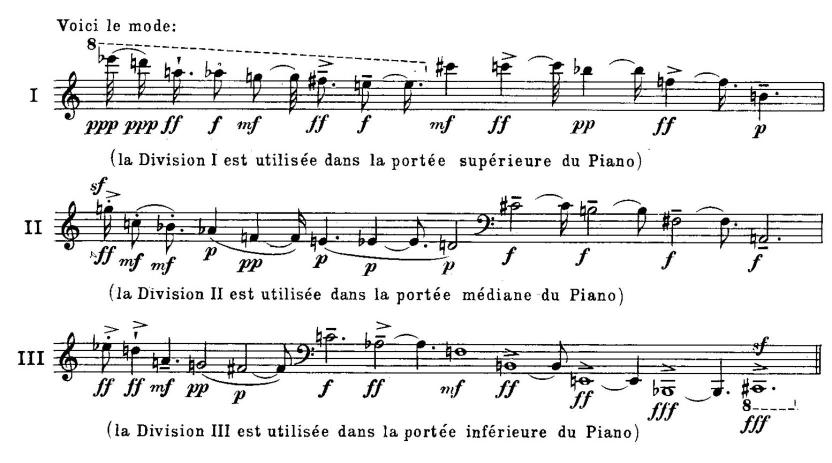
Dodecaphonism and Serialism

- Arnold Schoenberg Suite Op. 25 for piano (1923). Twelve-tone composition. "Dodecaphonism principle"
- Olivier Messiaen Modes de valeurs et d'intensité for piano (1949). Extension of the twelve-tone technique to all parameters (not only pitch classes but rhythms, articulation, dynamics and so on).
- Jean Barraqué, Le temps restitué (1957). Multiple rows (*Proliferation of series*)
- Serialism with Microtones (Alain Bancquart, Jean-Etienne Marie, Ben Johnston). Extension to N-tone rows in equal or non-equal temperaments.





O. Messiaen, Modes de valeurs et d'intensité

Twelve-tone Technique

Identify the 12 notes with $\mathbb{Z}_n$ i.e. 0=do, 1= do $\sharp$, etc.

- Choose a 12-tone row.
- Example : Jean Barraqué ...au-delà du hasard
- $P = (0, 8, 7, 1, 4, 2, 10, 3, 11, 5, 6, 9)$
- Apply the 4 basic following transformations
  - *Transpose* up or down : $T_n(x) = x + n \mod 12$
  - *Retrograde* = reverse in time $R(P) = 9,6,5,11$, etc.
  - *Inversion* = reverse in pitch $I_n(x) = -x + n \mod 12$
  - *Retrograde Inverse* = composition RI

  (P, I, R, RI) = Klein group = dihedral group D_2 with presentation

  $$D_2 = \langle a, b \mid a^2 = b^2 = (ab)^2 = 1 \rangle$$

  with $1 = P$, $a = I$, $b = R$
- Material for composition = at most $4 \times 12 = 48$ rows, called *derivative forms*

Action of the dihedral group $\mathcal{D}_n$ on the permutation group $\mathfrak{S}_n$

Theorem

In the n-tone equal temperament ($n \geq 3$), under the equivalence of the derived forms (i.e. under the action of the dihedral group), there are

$$\begin{cases} \frac{1}{4}(n-1)! + 2^{(n-4)/2}\frac{1}{n}\left(\frac{n}{2}+1\right)! & \text{if } n \text{ is even} \\ \frac{1}{4}(n-1)! + 2^{(n-5)/2}\left(\frac{n-1}{2}\right)! & \text{if } n \text{ is odd} \end{cases}$$

tone rows of n pitch classes.
In particular, for $n = 12$, $|\mathcal{S}| = 9,985,920$ rows.

The problem of counting the tone rows of $\mathcal{S}$ is equivalent to count the number of orbits of the group $G = D_{12} \times \mathbb{Z}_2$ generated by the transformations T, I and R on the series P. D_{12} denotes the dihedral group (of order 24). Burnside's lemma says that the number of orbits is the average number of fixed points.

Tone Rows with Limited Derivative Forms

Remark

$12!/48 = 479,001,600/48 = 9,979,200$ is not the number of 12-tone rows because some tone rows have less than 48 derivative series.

Example 1

$A = (0,4,8,11,3,7,1,9,5,2,10,6)$ verifies
$T_6(A) = R(A) = (6,10,2,5,9,1,7,3,11,8,4,0)$
Thus $T_n(A) = RT_{n+6}(A)$ and $I_n(A) = RI_{n+6}(A)$.
The series A has only 24 derivative forms.

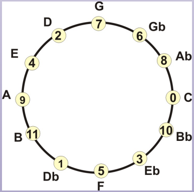
Example 2

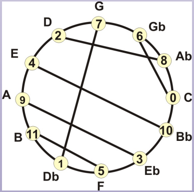
$A = (0,3,11,2,10,1,6,9,5,8,4,7)$ is equal to $RI_7(A)$.

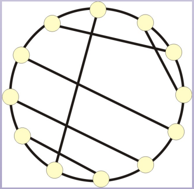
Chord Diagrams

How to build a Chord diagram ?

1. Take a row of twelve pitch classes : 0,8,6,7,2,4,9,11,1,5,3,10
2. Place the numbers on a circle
3. Join the tritone pairs together
4. Just keep the structure







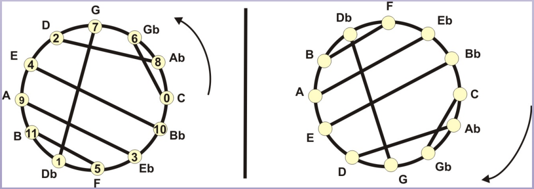
This structure is called a chord diagram.

Proposition

The 48 forms of the series are represented by the same chord diagram. i.e. Chord diagrams are invariant under the action of the dihedral group.

Example :

Retrogradation is obtained by mirror symmetry and rotation.



Theorem (A. Khruzin)

Under the action of the dihedral group, the number of chord diagrams for the equal temperament with $2n$ degrees is

$$d_n = \frac{1}{2}\left(c_n + \frac{1}{2}(\kappa_{n-1} + \kappa_n)\right)$$

$$c_n = \frac{1}{2n}\sum_{i|2n}\varphi(i)\nu_n(i), \quad \kappa_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\frac{n!}{k!(n-2k)!}$$

$\varphi(i)$ is the Euler totient function and ν_n is given for all divisors of $2n$ by

$$\nu_n(i) = \begin{cases} i^{n/i}(2n/i-1)!! & \text{if } i \text{ is odd} \\ \sum_{k=0}^{\lfloor \frac{n}{i} \rfloor}\binom{2n/i}{2k} i^k(2k-1)!! & \text{if } i \text{ is even} \end{cases}$$

Enumeration of Chord Diagrams

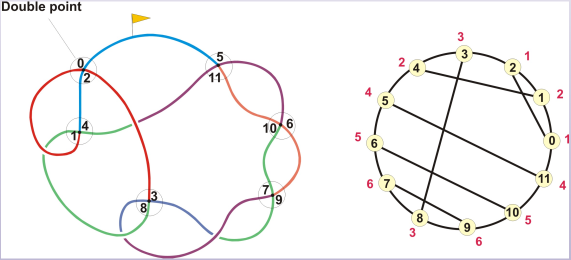
For the first $2n$-equal temperaments,

$2n$	c_n	d_n
6	5	5
8	18	17
10	105	79
12	**902**	**554**
14	9 749	5 283
16	127 072	65 346
18	1 915 951	966 156
20	32 743 182	16 411 700
22	625 002 933	312 702 217
24	13 176 573 910	6 589 356 711

Interest : about 10 millions rows are represented by only 554 diagrams.

Knots with six double points

Oriented Knots with six double points represent rows and chord diagrams



Gauss word is the smallest integer of the representation of the six chords pairs

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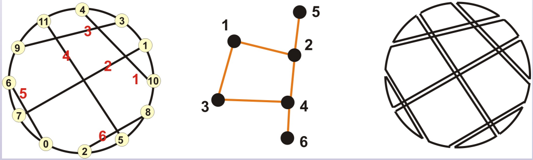
Chord diagrams are classified by Gauss words.

Brian Ferneyhough Superscripto : 10, 1, 3, 4, 11, 9, 6, 7, 0, 2, 5, 8



Adjacent matrice $A_{ij} = 1$ iff (i, j) are connected (0 otherwise).
The rank of A is an invariant.

$$\boxed{\#\text{faces}(D)\text{=corank}(A)\text{+}1}$$

Example Superscripto :

$$rk(A) = \det(A) = 6, \#faces(D) = 7 - rk(A) = 1$$

Vassiliev's invariants $v_O = 0$ and

$$v(\times\!\!\!\!\cdot\;) := v(\nwarrow\!\!\!\!\nearrow) - v(\nwarrow\!\!\!\!\nearrow),$$

are related to Jones polynomials $J_K(t)$

$$t^{-1} J_{\nwarrow\!\!\!\!\nearrow} - t J_{\nwarrow\!\!\!\!\nearrow} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) J_{\uparrow\uparrow}$$

Let $t = e^h$ and n the number of crossings, we have

$$J_K(h) = \sum_{n=0}^{\infty} v_n(K) h^n$$

Coefficients $v_n \in \mathcal{V}_n$ with $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \ldots \mathcal{V}_n \subset \ldots$
Finite type invariants = Vassiliev's invariants = $\mathcal{V} = \bigcup_n \mathcal{V}_n$

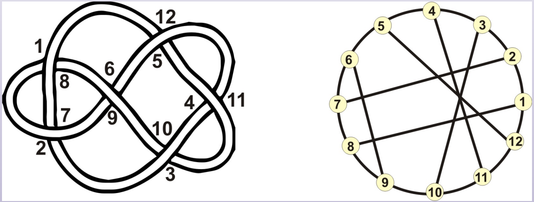
Vassiliev's invariants

From chord diagram D

$$v(D) = \sum_{s} \left(\prod_{c} s(c) \right) (-2)^{|s|-1}$$

c is the label of the chord ($c = 1, ..., 6$)
s is a state obtained from D by deleting chords.
$s(c) = 1$ if c is kept
$s(c) = 2$ if c is deleted
The sum is taken over all $2^6 = 64$ states of D
(removing p chords, $p = 0, ..., 6$).
$|s|$ is the number of components of state s.

Example : For $n = 6$, the knot $K = 6_2$



$$J_K(t) = \frac{3}{t} - \frac{2}{t^2} + \frac{2}{t^3} - \frac{2}{t^4} + \frac{1}{t^5}$$

$$J_K(e^h) = 2 - 2h + 3h^2 - \frac{19}{3}h^3 + \frac{41}{4}h^4 - \frac{751}{60}h^5 + \frac{487}{40}h^6 + O(h^7)$$

The Vassiliev's invariant is $v_6 = 487/40$.

Artin braid groups

Definition

B_n is the group generated by $(n-1)$ generators $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ and the braid relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{for all } i = 1, 2, ..., n-1 \text{ and } |j-i| \geq 2$$

and

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \qquad \text{for all } i = 1, 2, ..., n-2$$

$B_1 = \{1\}$, $B_2 = \{\sigma_1\}$ has no braid relations.
B_n is an infinite cyclic group.

Burau representations

Linear representations by matrices over the ring of Laurent polynomials $\mathbb{Z}[t^{-1}, t]$

For $i = 1, ..., n-1$

$$A_i = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix}$$

The Burau representation Ψ_n is faithful for $n \leq 3$ (i.e. $\ker \Psi_n = \{1\}$) and non faithful for $n \geq 5$.
It is unknown if Ψ_4 is faithful.

Braids in Messiaen's *Etudes de rythmes*

From *Modes de valeurs et d'intensités*, define

$$
\begin{aligned}
\sigma_1 &= \text{Motive } (1,2,3) \text{ of series I} \\
\sigma_1^{-1} &= \text{Motive } (10,11,12) \text{ of series I} \\
\sigma_2 &= \text{Motive } (1,2,3) \text{ of series II} \\
\sigma_2^{-1} &= \text{Motive } (10,11,12) \text{ of series II} \\
\sigma_3 &= \text{Motive } (1,2,3) \text{ of series III}
\end{aligned}
$$

On the first pages

$$\sigma_1\sigma_2\sigma_1^{-1}\sigma_2\sigma_3\sigma_1^{-1}\sigma_2\sigma_1$$

Each chord diagram D_n is associated with a permutation P_n
Luigi Nono *Canto Sospeso* D_{358} is associated with (in cyclic notation)

$$(0,1)(2,11)(3,10)(4,9)(5,8)(6,7)$$

The generating power of this row is measured by the order of the *Serial group* defined by

$$G_n = \langle P_n, T, I \rangle$$

T = permutation associated with transposition

$$T = (0,1,2,3,4,5,6,7,8,9,10,11)$$

I = permutation associted with inversion

$$I = (1,11)(2,10)(3,9)(4,8)(5,7)$$

Examples of serial groups

- A. Webern *Concert for nine instruments*, opus 24.
  $S = 11, 10, 2, 3, 7, 6, 8, 4, 5, 0, 1, 9$

$$G_{549} = \left\langle \begin{array}{c} a, b, c \mid a^2 = c^2 = (b^{-1}c)^2 = (acb)^2 \\ = (ab^2c)^2 = (ab^{-1})^4 = (abab^{-1})^3 = 1 \end{array} \right\rangle$$

  This group has 192 elements.

- Schoenberg's *Ode to Napoleon*, Kees van Baaren's *Variations in Isometrical Series*

$$5, 10, 6, 9, 7, 8, 2, 1, 3, 0, 4, 11$$

  is associated with the diagram D_{538}. The associated group

$$G_{538} = \langle P_{538}, T, I \rangle$$

  has 7680 elements.

Examples of serial groups

- Dallapiccola's *Quaderno Musicale di Annalibera*

$$10, 11, 3, 6, 8, 2, 7, 1, 5, 9, 0, 4$$

is associated with the diagram D_{56}. The group generated by the three operators

$$G_{56} = \langle P_{56}, T, I \rangle$$

has presentation

$$G_{56} = \left\langle \begin{array}{c} a, b, c \mid a^2 = c^2 = (b^{-1}c)^2 = (ac)^4 = acb^{-1}(abc)^2 \\ = (acb^3)^2 = (ab^2ab^{-2})^2 = b^{12} = (acab^{-2})^4 \end{array} \right\rangle$$

It is a group of order 28800.

- Many diagrams have a maximum generating power

$$|\mathfrak{S}_{12}| = 479\ 001\ 600$$

Partition function is linked with Jones polynomial. For a braid $k = \sigma_{i_1}^{\varepsilon_1} ... \sigma_{i_m}^{\varepsilon_m}$

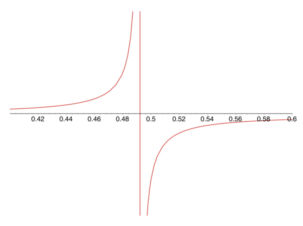
$$Z_k = (1 + e^{\beta}) e^{-\beta \frac{1+w}{2}} J_k(e^{\beta})$$

Tait number = writhe = $w = \varepsilon_1 + ... + \varepsilon_m$

Energy and entropy are defined by

$$U = -\frac{\partial \log Z}{\partial \beta} \qquad S = \beta U + \log Z$$

Example : The entropy of the knot $6_2 = \sigma_1^{-1} c \sigma_1^{-1} \sigma_2^3$ has a discontinuity at $\beta = 0.492...$



Braiding the rows lattice

S = Basic row, T_n = transpositions, I_n = inversions.

$$\Phi : T_n(S) \to \sigma_n, \quad I_n(S) \to \sigma_n^{-1}$$

Jean Barraqué, *Sonata for piano*,

$$S_1 I_9 S_2 I_8 S_6 I_{12} \ldots \to \sigma_1 \sigma_9^{-1} \sigma_2 \sigma_8^{-1} \sigma_6 \sigma_{12}^{-1} \ldots$$

Definition

A *topological map* M is a graph Γ embedded into a surface X such that

- the vertices are distinct points of the surface

- the edges are curves on the surfaces that intersect only at vertices

- the set $X \backslash \Gamma$ is a disjoint union of connected components called faces such that each face is homeomorphic to an open disk.

Definition

A *k-constellation* of degree n is a k-uplet $[\sigma_1, ..., \sigma_k]$ with $\sigma_j \in \mathfrak{S}_n$ such that the monodromy group $G = \langle \sigma_1, ..., \sigma_k \rangle$ acts transitively on the set on n points and

$$\sigma_1 ... \sigma_k = id$$

A (*combinatorial*) *map* $[\sigma, \alpha, \varphi]$ is a 3-constellation (= *hypermap*) in which α is an involution without fixed points.
σ = vertices, α = edges, $\varphi = \alpha^{-1}\sigma^{-1}$ = faces.
The group $\langle \sigma, \alpha \rangle$ is transitive = the map is connected.

Automorphism group of a map M = centralizer of the monodromy group G

$$Aut(M) = Z(G) = \{h, hg = gh, \ \forall g \in G\}$$

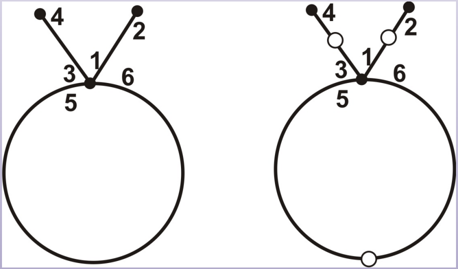
Euler's Formula

$$\chi(M) = c(\sigma) + c(\alpha) + c(\varphi) = 2 - 2g$$

$c(x)$ = number of cycles of the permutation x

Example of a 3-constellation

$\alpha = (1,2)(3,4)(5,6)$ = permutation of edges
$\sigma = (1,3,5,6)(2)(4)$ = permutation of vertices
$\varphi = (1,2,6,3,4)(5)$ = permutation of faces



Another representation as binary graph
with black and white vertices

Primitive groups

Let $G \subset \mathfrak{S}_n$ be a transitive group. A group $G \subset \mathfrak{S}_n$ is called *transitive* if its group action is transitive. In other words, if the group orbit $G(x)$ is equal to the set $\{1, 2, ..., n\}$ for some $x \in G$.
A *block system* $\mathcal{B} = \{b_1, ..., b_m\}$ for G is a partition of $\{1, ..., n\}$ invariant under G, (i.e. $\forall g \in G, \; g(b_i) = b_j$).
Example : $G = \langle (1,2,3,4), (1,3) \rangle$, $\mathcal{B} = \{\{1,3\}, \{2,4\}\}$

Definition

G is *primitive* if the only block systems are trivial i.e. $\{1, 2, ..., n\}$ and $\{\{1\}, \{2\}, ..., \{n\}\}$. G is *imprimitive* if G is not primitive.

Theorem

For $n = 12$, there are 301 transitive groups, but only 6 imprimitive groups.

Blocks act as attractors for serial fragments, that is why primitive groups are important for serial music.

Olivier Messiaen's *Ile de feu 2*

Ile de feu 2 use two permutations in cyclic notation

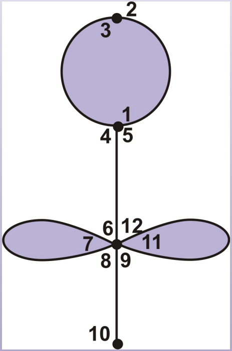
$$a = (1, 7, 10, 2, 6, 4, 5, 9, 11, 12)(3, 8)$$

and

$$b = (1, 6, 9, 2, 7, 3, 5, 4, 8, 10, 11)(1, 2)$$

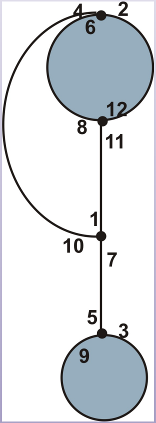
These permutations generate Mathieu's group M_{12}

$$Order(M_{12}) = 95040$$



Built on six permutations (a permutation and five *actions*) :

$$\sigma_0 = (1, 11, 6, 2, 9, 4, 8, 10, 3, 5)$$



and for $j = 1, ..., 5$, $\sigma_j = A_j \sigma_0$

Actions are defined by :

A_1 = "Extremes au centre"

$= (2, 12, 7, 4, 11, 6, 10, 8, 9, 5, 3)$

A_2 = "Centre aux extrêmes"

$= (1, 6, 9, 2, 7, 3, 5, 4, 8, 10, 11)$

A_3 = "Rétrograde"

$= (1, 12)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)$

A_4 = "Extrêmes au centre, rétrograde" $= A_1 A_3$

A_5 = "Centre aux extrêmes, rétrograde" $= A_2 A_3$

with $a = A_2^{-1} A_1$ and $b = A_2^3 A_1 A_2^2 A_1$

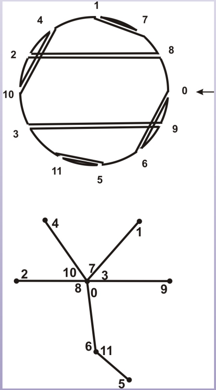
Mathieu group M_{12} of order 95040

$$M_{12} = \langle a, b \mid a^2 = b^3 = (ab)^{11} = [a, b]^6 = (ababab^{-1})^6 = 1 \rangle$$

From chord diagrams to hypermaps

Given a chord diagramm, how to build a 3-constellation ? ean Barraqué, ... au delà du hasard : $0, 8, 7, 1, 4, 2, 10, 3, 11, 5, 6, 9$

$\alpha = (0, 6)(1, 7)(2, 8)(3, 9)(4, 10)(5, 11)$



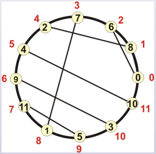
Read pitch classes along the diagram
with doubling chords
$0, 8, 2, 10, 4, 2, 8, 7, 1, 4, 10, 3, 9, 0, 6, 9, 3, 11, 5, \ldots$

Delete doubles
$\varphi = (0, 8, 2, 10, 4, 7, 1, 3, 9, 6, 11, 5)$

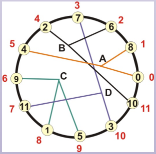
Compute the permutation of the vertices
$\sigma = \varphi^{-1}\alpha^{-1} = (0, 11)(3, 7, 10, 8, 6)$

Using sets of limited transposition, we get a 3-constellation $[\alpha, \sigma, \varphi]$

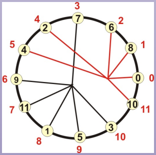
- **with (0,6)** :
  it is the usual chord diagram A : $(0,6) \to (0,2)$
  B : $(1,7) \to (8,3)$ C : $(2,8) \to (4,1)$
  D : etc. We associate the permutation
  (*edges*) $\alpha = (0,2)(1,4)(5,11)(3,8)(6,10)(7,9)$



- **with (0,4,8)** : A :
  $(0,4,8) \to (0,5,1)$ B : $(1,5,9) \to (8,9,6)$ C :
  $(2,6,10) \to (4,2,11)$ D : $(3,7,11) \to (10,3,7)$
  We associate the permutation
  (*vertices*) $\sigma = (0,5,1)(8,9,6)(4,2,11)(10,3,7)$

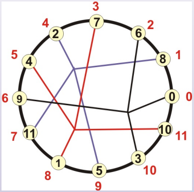


- **with (0,2,4,6,8,10)**
  A : $(0,2,4,6,8,10) \to (0,4,5,2,1,11)$
  B : $(1,3,5,7,9,11) \to (8,101,9,3,6,7)$
  We associate the permutation
  (*faces*) $\varphi = (0,4,5,2,1,11)(8,10,9,3,6,7)$

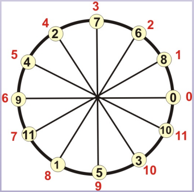


Some other diagrams

- **with (0,3,6,9)**
  A : $(0,3,6,9) \to (0,10,2,6)$
  B : $(1,4,7,10) \to (4,9,1,7)$
  C : $(2,5,8,11) \to (8,5,3,11)$
  We associate the permutation
  $u = (0,10,2,6)(4,9,1,7)(8,5,3,11)$



- **with (0,1,2,3,4,5,6,7,8,9,10,11)**
  $(0,1,2,3,4,5,6,7,8,9,10,11) \to$
  $(0,8,4,10,5,9,2,3,1,6,11,7)$
  We associate the permutation
  $v = (0,8,4,10,5,9,2,3,1,6,11,7)$



$[\alpha', \sigma', \varphi']$ is a 3-constellation with
$\alpha' = \sigma, \sigma' = u^{-1}, \varphi' = v^{-1}, (uv = \sigma)$

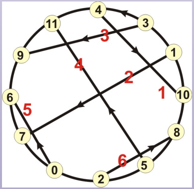
Definition

A Gauss diagram is a chord diagram with oriented chords.

The orientation is determined by

$$a \rightarrow b \Longleftrightarrow a \leq b$$

Turaev invariant polynomial



$$u(t) = \sum_{c, n(c) \neq 0} sgn(n(c)) t^{|n(c)|}$$

$n(1) = -2$,
$n(2) = -1$, $n(3) = 0$, $n(4) = n(5) = n(6) = +1$

$$u(t) = -t^2 + 2t$$