1 t-Designs

Definition:

$\mathcal{D} = (V, \mathcal{B})$ t-(v, k, λ) **design :**

$|V| = v$, $\mathcal{B} \subseteq \binom{V}{k}$,

$\forall T \in \binom{V}{t}$

$$|\{B \mid T \subset B \in \mathcal{B}\}| = \lambda.$$

Remark: We restrict to designs with no multiple blocks, i.e. *simple* designs, in our definition.

Find t-designs.

Determine isomorphism types.

Isomorphism "renames" elements of V: permutation

Isomorphism class = orbit of S_V.

Visualize t-designs.

Show properties: Symmetries, Intersection properties

Kramer-Mesner(1976):

- Prescribe group of automorphisms A.
- Condense incidence matrix $W(\binom{V}{t} \times \binom{V}{k})$
  to orbit incidence matrix $W^A(\binom{V}{t}/A \times \binom{V}{k}/A)$,
  $W^A(T^A, K^A) = |\{K' \mid T \subseteq K' \in K^A\}|$.
- Solve

$$W^A \cdot X = \lambda \cdot J$$

  with X a 0-1 vector, J all 1 vector.

Example

Largest known Steiner 5-design 5-$(244, 6, 1)$:

$6.916.056.048 \times 280.100.269.944$ matrix W
is condensed by $A = P\Sigma L(2, 3^5)$ to
196×7940 matrix W^A.

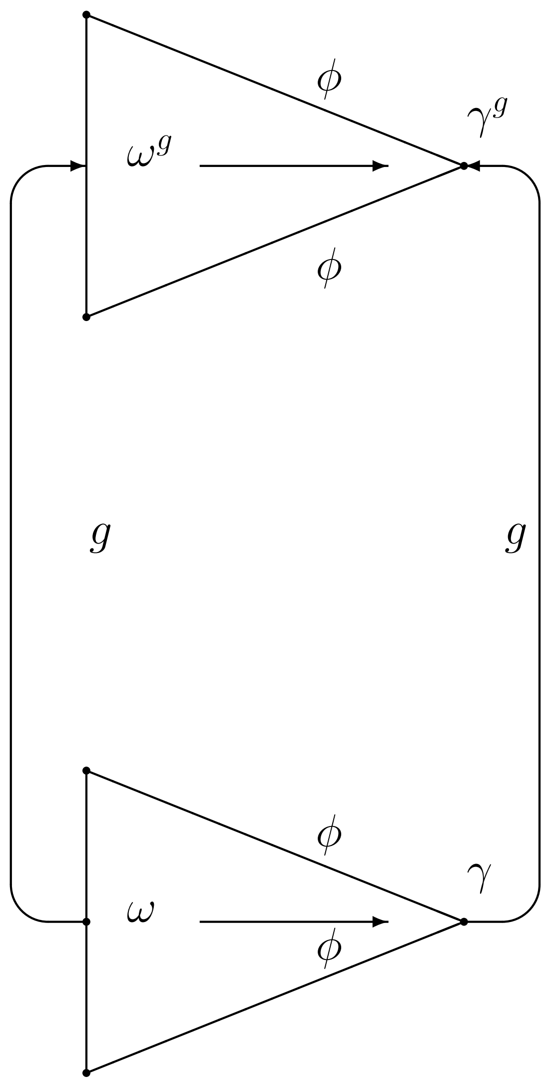
$$W^A \cdot X = 1 \cdot J$$

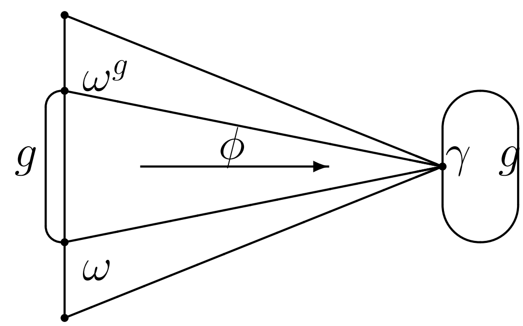
has more than 12900 solutions.

G-Homomorphism $\phi : \Omega_1 \longrightarrow \Omega_2$:

$\phi(\gamma)^{-1}$ and $\phi(\gamma^g)^{-1}$ intersect same orbits

$\phi(\omega) = \gamma = \phi(\omega^g) \Longrightarrow g \in G_\gamma$





$g \in G_\gamma$

G-Homomorphism $\phi : \Omega \longrightarrow \mathcal{L}(G)$,
lattice of subgroups of G.

$A \in \mathcal{L}(G)$.
Orbits of $N_G(A)$ on $\phi^{-1}(A)$ = G-orbits in $\phi^{-1}(A)$

Each orbit length: $|N_G(A)/A|$.

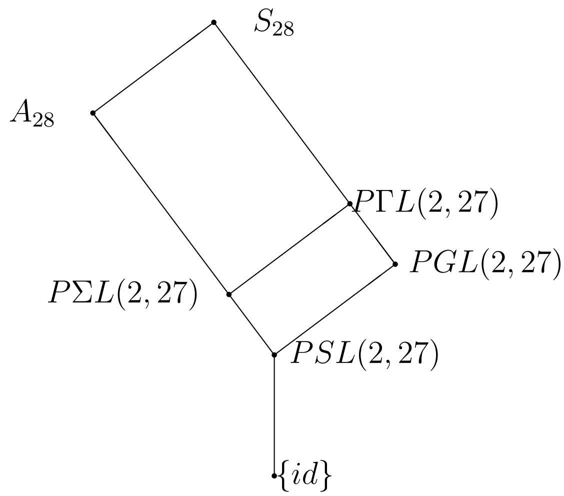
$\phi^{-1}(A)$:
Remove from **fixed points of** A all fixed points of B for $A < B \leq G$.

Counting: Moebius-Inversion.

Theorem (Faradsev(1976)):
$A <_{max} G$, not normal.
All $\omega \in \Omega_1$ fixed by A lie in pairwise different orbits.

Application:
$A = P\Gamma L(2, 32)$ automorphism group of
4.996.426
pairwise non-isomorphic 7-$(33, 8, 10)$ designs.

Figure 1: 5-(28, 6, 9) designs



Prescribed A:	$PSL(2,27)$	$PGL(2,27)$	$P\Sigma L(2,27)$	$P\Gamma L(2,27)$
fixed designs:	68976931	369	31	3

$$\begin{pmatrix} \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 68976931 \\ 369 \\ 31 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 11496089 \\ 122 \\ 14 \\ 3 \end{pmatrix}$$

Theorem (Jordan):

$A \leq G$, fixing ω_1, ω_2.
$\omega_1^g = \omega_2$ for some $g \in G$.
$P \in Syl(G_{\omega_2})$ for some $P \leq A$

$$\Longrightarrow$$

$\omega_1^n = \omega_2$ for some $n \in N_G(P)$.

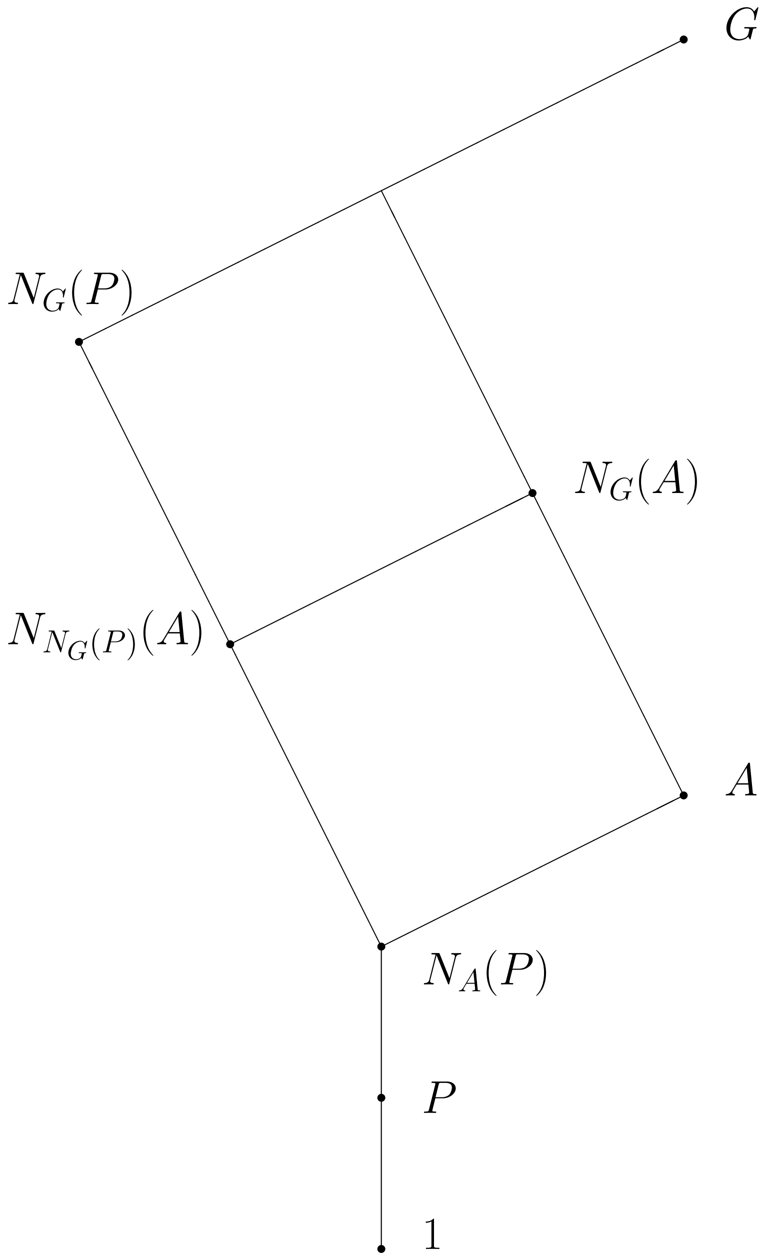
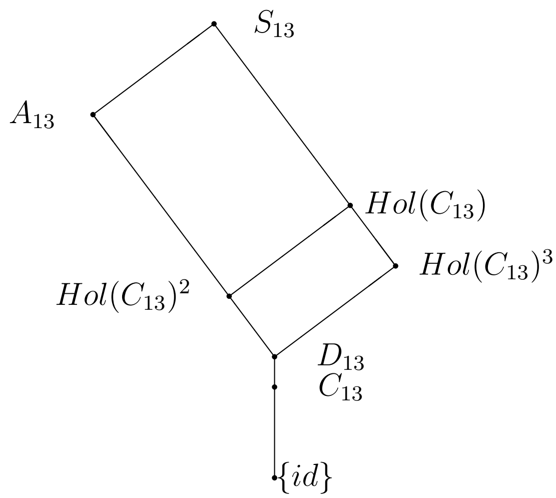

G
$N_G(P)$
$N_G(A)$
$N_{N_G(P)}(A)$
A
$N_A(P)$
P
1

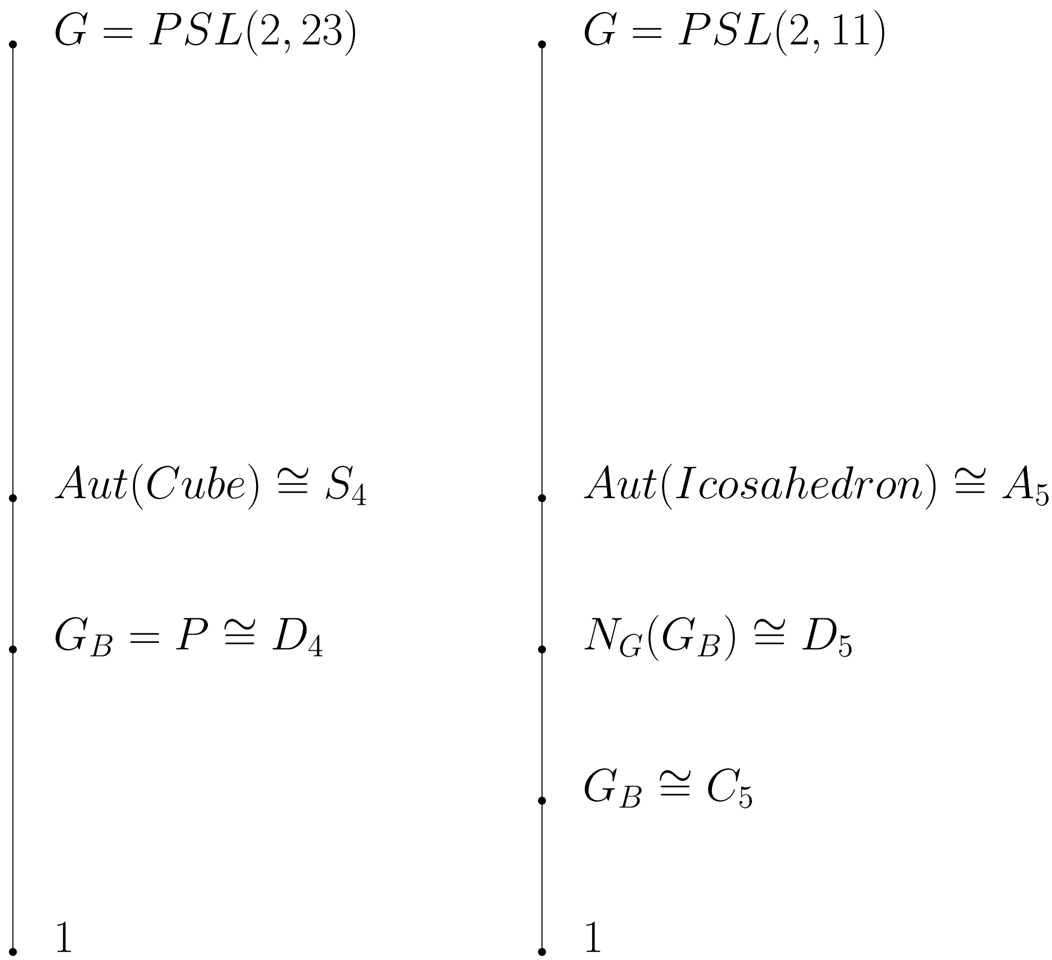
Figure 2: Kramer and Mesner's example

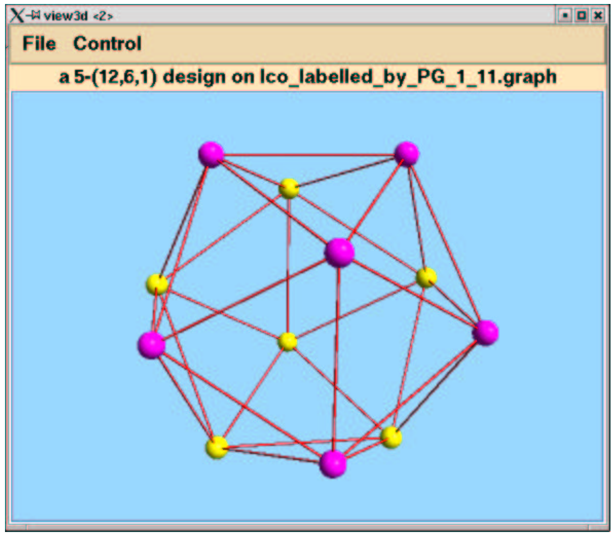


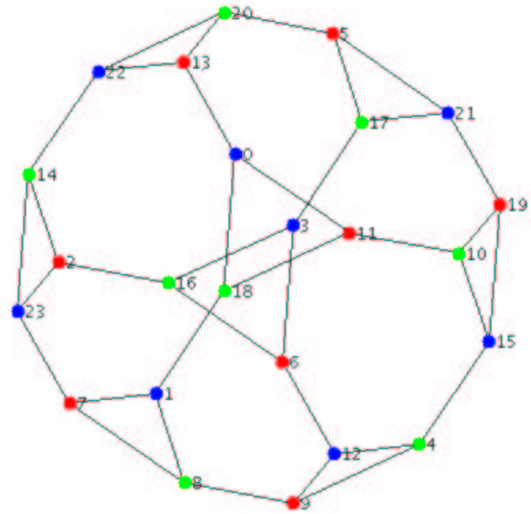
2-(13, 5, 45) Designs

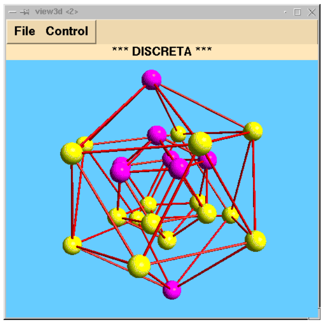
$$\begin{pmatrix} \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 136876801 \\ 24643 \\ 890 \\ 28 \end{pmatrix}$$

$$= \begin{pmatrix} 22825216 \\ 8205 \\ 431 \\ 28 \end{pmatrix} \text{ for } \begin{cases} D_{13} \\ Hol(C_{13})^3 \\ Hol(C_{13})^2 \\ Hol(C_{13}) \end{cases}$$


$G = PSL(2,23)$
$Aut(Cube) \cong S_4$
$G_B = P \cong D_4$
1
$G = PSL(2,11)$
$Aut(Icosahedron) \cong A_5$
$N_G(G_B) \cong D_5$
$G_B \cong C_5$
1


view3d <2>
File Control
a 5-(12,6,1) design on Ico_labelled_by_PG_1_11.graph


20
5
22
13
21
17
0
14
19
3
11
2
10
16
18
23
15
6
7
1
12
4
8
9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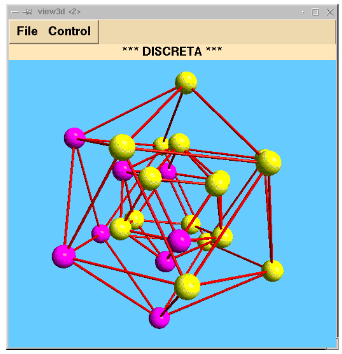
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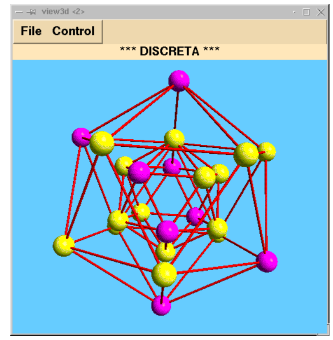
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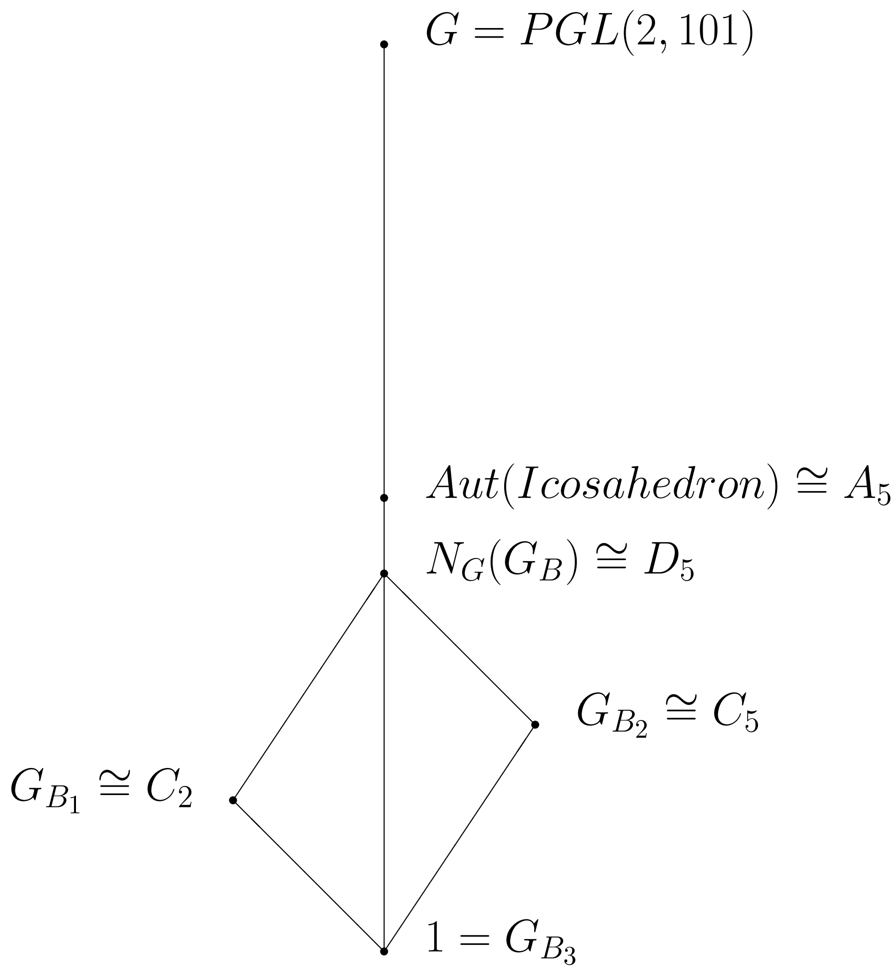
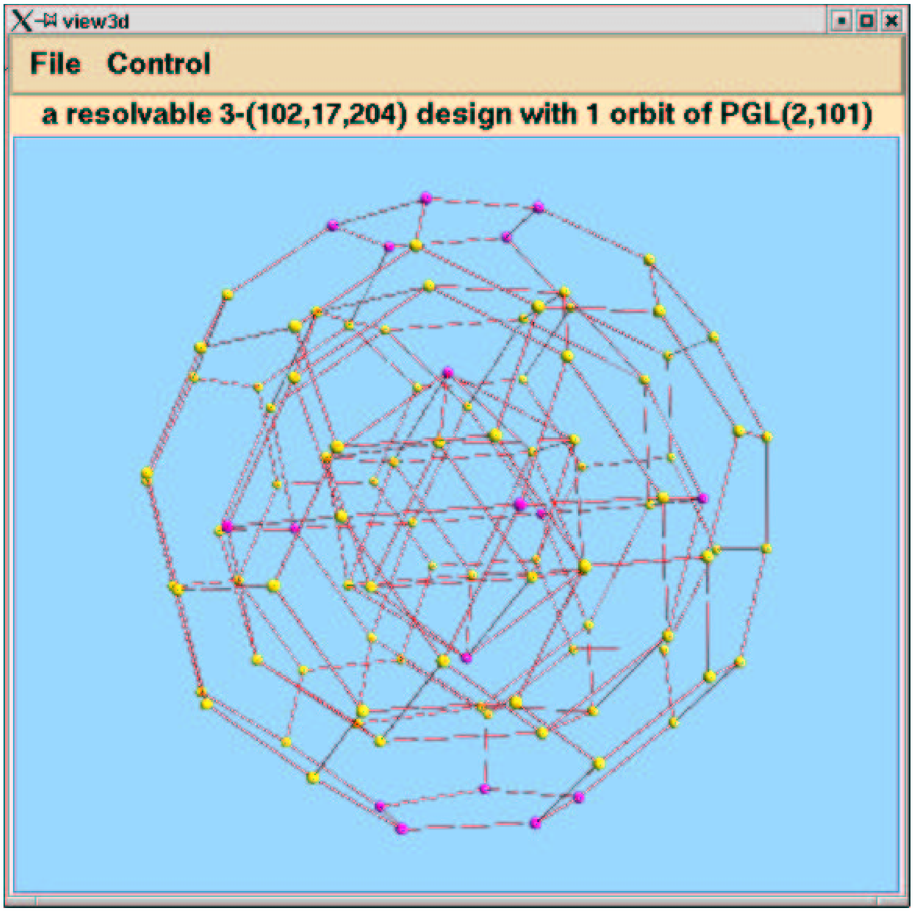

$G = PGL(2, 101)$
$Aut(Icosahedron) \cong A_5$
$N_G(G_B) \cong D_5$
$G_{B_2} \cong C_5$
$G_{B_1} \cong C_2$
$1 = G_{B_3}$


view3d
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a resolvable 3-(102,17,204) design with 1 orbit of PGL(2,101)

2 Intersection Numbers

Definition 2.1 *Let $\mathcal{D} = (V, \mathcal{B})$ be a t-(v, k, λ) design. For an s-subset S of V*

$$\alpha_i(S) = |\{B | B \in \mathcal{B}, |B \cap S| = i\}|$$

is the i-th intersection number of S for $1 \leq i \leq s$.

For S a block this was introduced by Mendelsohn [**?**], the general version is due to Köhler [**?**].

Theorem 2.1 *Let $\mathcal{D}$ be a t-(v, k, λ) design and S an s-element subset. Then for all $0 \leq j \leq min(s, t)$ we have*

$$\sum_{i=j}^{s} \binom{i}{j} \alpha_i(S) = \binom{s}{j} \lambda_j.$$

Example 2.1 **Intersection Numbers of a 5-$(36, 6, 1)$ Design**
b = 62832
r = 10472
The Mendelsohn *system:*

(1)

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 & 4 & 5 & 6 \\ & & 1 & 3 & 6 & 10 & 15 \\ & & & 1 & 4 & 10 & 20 \\ & & & & 1 & 5 & 15 \\ & & & & & 1 & 6 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} = \begin{pmatrix} 62832 \\ 62832 \\ 22440 \\ 3520 \\ 240 \\ 6 \end{pmatrix} = \begin{pmatrix} \binom{6}{0} & 62832 \\ \binom{6}{1} & 10472 \\ \binom{6}{2} & 1496 \\ \binom{6}{3} & 176 \\ \binom{6}{4} & 16 \\ \binom{6}{5} & 1 \end{pmatrix}$$

The (unique) solution:

$$\begin{pmatrix} 19155 \\ 27576 \\ 13275 \\ 2600 \\ 225 \\ 0 \\ 1 \end{pmatrix} \tag{2}$$

3 Extensions

Theorem "Glueing Designs"

n_1 t-$(v, k-1, \lambda)$ designs , n_2 t-(v, k, μ) designs,

pairwise non-isomorphic, $\mu = \lambda(v-k+1)/(k-t)$,

$\forall_D \ |Aut(D)| \leq a$

$\Longrightarrow$

$\exists \geq \frac{n_1 \cdot n_2 \cdot v!}{(v+1)a^2}$ isomorphism types of
t-$(v+1, k, \lambda+\mu)$ designs D, $|Aut(D)| \leq a(v+1)$.

Construction (Tran van Trung, van Leijenhorst) *Identify the points of a t-$(v, k-1, \lambda)$ design $\mathcal{D}_1$ and a t-(v, k, μ) design $\mathcal{D}_2$,*

add a new point to the blocks of size $k-1$.

The $v!$ identifications of v points fall into orbits under $Aut(\mathcal{D}_1) \times Aut(\mathcal{D}_2)$.

Correspond to **double cosets** *in S_V.*
Up to v isomorphisms may move the new point.

Theorem Family

Each parameter set of a non-trivial design

belongs to a finite family

of formally admissible parameter sets

with a **unique** ancestor.

Parameter Sets of 7-Designs in the Family of 15-$(32, 16, 6)$

7-(32,16,721050)
>= 1.5E4165

7-(31,15,259578)	7-(31,16,461472)
>= 4.0E2080	>= 4.0E2080

7-(30,14,86526)	7-(30,15,173052)	7-(30,16,288420)
>= 9.7E1035	>= 1.3E1039	>= 9.7E1035

7-(29,13,26334)	7-(29,14,60192)	7-(29,15,112860)	7-(29,16,175560)
>= 3.0E511	>= 3.8E514	>= 3.8E514	>= 3.0E511

7-(28,12,7182)	7-(28,13,19152)	7-(28,14,41040)	7-(28,15,71820)	7-(28,16,103740)
>= 3.9E252	>= 2.8E250	>= 5.0E255	>= 2.8E250	>= 3.9E252

7-(27,11,1710)	7-(27,12,5472)	7-(27,13,13680)	7-(27,14,27360)	7-(27,15,44460)	7-(27,16,59280)
>= 1.5E125	>= 6.2E117	>= 1.1E123	>= 1.1E123	>= 6.2E117	>= 1.5E125

7-(26,10,342)	7-(26,11,1368)	7-(26,12,4104)	7-(26,13,9576)	7-(26,14,17784)	7-(26,15,26676)	7-(26,16,32604)
>= 1E62	>= 6.6E51	>= 4.0E54	>= 1.2E57	>= 4.0E54	>= 6.6E51	>= 1E62

7-(25,9,54)	7-(25,10,288)	7-(25,11,1080)	7-(25,12,3024)	7-(25,13,6552)	7-(25,14,11232)	7-(25,15,15444)	7-(25,16,17160)
>= 1.2E20	>= 2.3E19	>= 4.5E19	>= 1.4E22	>= 1.4E22	>= 4.5E19	>= 2.3E19	>= 1.2E20

7-(24,8,6)	7-(24,9,48)	7-(24,10,240)	7-(24,11,840)	7-(24,12,2184)	7-(24,13,4368)	7-(24,14,6864)	7-(24,15,8580)	7-(24,16,8580)
>= 132	>= 2827	>= 91	>= 155	>= 28500	>= 155	>= 91	>= 2827	>= 132