On group-theoretical methods applied to music: some compositional and implementational aspects

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Abstract

This paper focuses on the group-theoretical approach to music theory and composition. In particular we concentrate on a family of groups which seem to be very interesting for a ›mathemusical‹ research: the non-Hajós groups.

This family of groups will be considered in relationships with Anatol Vieru's »Theory of modes« as it has been formalised and generalised to the rhythmic domain by the Roumanian mathematician Dan Tudor Vuza. They represent the general framework where one can formalize the construction of a special family of tiling canons called the »Regular Unending Complementary Canons of Maximal Category« (*RCMC-canons*). This model has been implemented in Ircam's visual programming language *OpenMusic*. Canons which are constructible through the Vuza's algorithm are called *Vuza Canons*. The implementation of Vuza's model in *OpenMusic* enables to give the complete list of such canons and offers to composers an useful tool to manipulate complex global musical structures. The implementation shows many interesting mathematical properties of the compositional process which could be taken as a point of departure for a computational-oriented musicological discussion.

1 Introductory remarks on the role of group theory in music

»The question can be asked: is there any sense talking about symmetry in music? The answer is yes« (Varga (1996), p. 86). By paraphrasing Iannis Xenakis previous statement, one could pose a similar question about groups and music: is there any sense talking about mathematical groups in music? With the assumption of the relevance of symmetry in music the answer follows as a logical consequence of this universal sentence: »Wherever symmetry occurs groups describe it« (Budden (1972)).

As Guerino Mazzola's *Mathematical Music Theory* suggests, there are many reasons for trying to generalise some questions about symmetry in music. But the question needs to be asked as to whether new results could be musically relevant, or whether they represent purely mathematical speculations. A concept of »musical relevance« in a mathematical theory of music is one of the most difficult to define precisely. Inevitably there is a »tension« between mathematics and music which has, as a practical consequence, the »mystical aura of pure form« (Roeder (1993)) of some mathematical theorems in contrast to the »mundanity« of their application to music . Criticism could be levelled against the potential competence of a mathematician expressing »in a very general way relations that only have musical meaning when highly constrained« (Roeder (1993), p. 307). This essay is an attempt to discuss some general abstract group-theoretical properties of a compositional process based on a double preliminary assumption: the algebraic formalization of the equal-tempered division of the octave and the isomorphism between pitch space and musical time.

Historically there have been different approaches from Zalewski's »Theory of Structures« (Zalewski (1972)) and Vieru's »Modal Theory« (Vieru (1980)), to the American *Set-Theory* (Forte (1973), Rahn (1980), Morris (1987)), whose special case is the so-called *diatonic theory*, an algebraic-oriented ramification of *Set-Theory* which is usually associated with the so-called »Buffalo School« at New York (Cf. (Clough J. (1986)) and (Clough (1994))). See (Agmon (1996)) for a recent summary in the theory of diatonicism.[1]

The common starting point is that every tempered division of the octave in a given number n of equal parts is completely described by the algebraic structure of the cyclic group $\mathbb{Z}_n$ of order n which is usually represented by the so-called 'musical clock'. Three theorists/composers are responsable for this crucial achievement: Iannis Xenakis, Milton Babbitt and Anatol Vieru. They form what we could call a »Trinity« of composers for they all share the interest towards the concept of group in music.[2] In Babbitt's words, »the totality of twelve transposed sets associated with a given [twelve-tone set] S constitutes a permutational group of order 12« (Babbitt (1960), p. 249). In other words, the Twelve-Tone pitch-class system is a mathematical structure i.e. a collection of »elements, relations between them and operations upon them« (Babbitt (1946), p. viii). Iannis Xenakis is sometimes more emphatic, as in the following sentence: »Today, we can state that after the Twenty-five centuries of musical evolution, we have reached the universal formulation for what concerns pitch perception: the set of melodic intervals has a group structure with respect to the law of addition« (Xenakis (1965), p. 69-70).

But unlike Babbitt's and Vieru's theoretical preference for the division of the octave in 12 parts, Xenakis' approach to the formalisation of musical scales uses a

1 A detailed bibliography on Set-Theory, diatonic theory and Neo-Riemannian theory is available *online* on:
http://www.ircam.fr/equipes/repmus/OutilsAnalyse/BiblioPCSMoreno.html

2 We could easily add some further references to the history of group-theoretical methods applied to music by also including music theorists as W. Graeser (Graeser (1924)), A.D. Fokker (Fokker (January 1947)), P. Barbaud (Barbaud (1968)), M. Philippot (Philippot (1976)), A. Riotte (Riotte (1979)), Y. Hellegouarch (Hellegouarch (1987))). We chose to concentrate on Babbitt, Xenakis and Vieru because of the great emphasis on compositional aspects inside of an algebraic approach. For a more general discussion on algebraic methods in XXth Century music and musicology see my thesis (Andreatta (2003)). For a detailed presentation of the algebraic concepts in music informatics see Chemillier (1989).

different philosophy. He considers the keyboard as a line with a referential zero-point which is represented by a given musical pitch and a unit step which is, in general, any well-tempered interval. Algebraically, the chromatic collection of the notes of keyboard could be indicated in such a way

$$1_0 = \{... - 3, -2, -1, 0, 1, 2, 3, ...\}$$

The symbol 1_0 means that the referential point is the 0 (usually 0 =C4 $= 261.6Hz$) and the unit distance is a given well-tempered interval (usually the semitone). Using the operations of union ($\cup$), intersection ($\cap$) and complementation ($\mathcal{C}$), it is also possible to formalise the diatonic collection in such a way:

$$(\mathcal{C}3_{n+2} \cap 4_n) \cup (\mathcal{C}3_{n+1} \cap 4_{n+1}) \cup (3_{n+2} \cap 4_{n+2}) \cup (\mathcal{C}3_n \cap 4_{n+3})$$

where $n = 0, 1, 2, ..., 11$ and $a_x = a_{n+i}$ if $x \equiv (n + i) mod(a)$ (cf.Orcalli (1993),p. 139). In a similar way one can formalise some other well-known music-theoretical constructions, like Messiaen limited transposition modes.[3] But Sieve-Theory could also be useful to construct (and formalise) musical scales which are not restricted to a single octave or which are not necessarely applied to the pitch domain.[4]

Another music-theoretically important sort of groups that we have to mention here[5] is the family of the *dihedral groups*. Historically they have been introduced by Milton Babbitt in a compositional perspective aiming at generalising Arnold Schoenberg's Twelve-Tone System to other musical parameters than the pitch parameter. This generalisation of the Twelve-Tone technique is usually called »integral serialism« and it represents an example of a remarkable convergence of two slightly different serial strategies. We will not discuss this point from a musicological perspective, although one would be tempted to say that a critical revision of some apparently well-established historical achievements will be soon necessary. European musicologists do not seem to have been particularly interested to seriously analyse Milton Babbitt's contribution in the field of the generalised serial technique. On the other hand, American musicologists consider M. Babbitt as the first total serialist, thanks to pieces like *Three Compositions for piano* (1947), *Compositions for Four Instruments* (1948), *Compositions for Twelve Instruments* (1948). Moreover M. Babbitt widely discussed this isomorphism between pitch and rhythmic domain in some crucial theoretical contributions, starting from his already quoted PhD thesis of 1946 (accepted by the Princeton Music Departement

3 The following quotation shows how the problem of expressing Messiaen's modes of limited transpositions in sieve-theoretical way was a central concern in Xenakis' theoretical speculation during the 60s: »I prepared a new interpretation of Messiaen's modes of limited transpositions which was to have been published in a collection of 1966, but which has not yet appeared« (Xenakis (1991), p. 377).

4 Following Xenakis' original idea, André Riotte gave the formalisation of Messiaen's modes in sieve-theoretical terms (Riotte (1979)) and suggested how to use Sieve-Theory as a general tool for a computer-aided music analysis. This approach has been developed in collaboration with Marcel Mesnage in a series of articles which have been collected in a two-volumes forthcoming book (Riotte and Mesnage (2003)).

5 A forthcoming article is dedicated to the sieve-theoretical and transformational strategies underlying Xenakis' piece *Nomos Alpha* involving generalized Fibonnacci sequences taking values in the group of rotations of the cube, see Agon and al. (2003). For more generalized investigations into the role of Coxeter groups in music see Andreatta (1997).

almost 50 years later!) and particularly in (Babbitt (1962)) where he introduced the concept of *Time-Point System*. We argue that there is a possibility to better understand the developments of integral serialism by »transgressing the (geographical) boundaries«, to seriously quote the title of Sokal's famous hoax, and by analysing how some ideas could have freely moved from Europe to USA and vice-versa. The French theorist and composer Olivier Messiaen has probably played a crucial role for what concerns the European assimilation of some Babbittian original intuitions. A piece like *Mode de valeurs et d'intensités*, written during the period Messiaen spent teaching composition at Tanglewood, is clearly influenced by his contacts with Babbitt's integral serialism. Musicologists usually stress the influence of this very particular piece on composers like P. Boulez and K. Stockhausen, but they seems to forget to pay attention to Babbitt's possible role in Messiaen's combinatorial attitude.[6] To come back to dihedral groups applied to music, one of the first examples which have been discussed by many theorists/composers is that of the Klein four-group $\mathcal{D}_2 \simeq \mathcal{C}_2 \times \mathcal{C}_2$. It may be realised geometrically as the group of symmetries of the rectangle (or, equivalently, of the rhombus or ellipse). Musically it represents the theoretical basis of Arnold Schoenberg's »Dodecaphonic System«, as pointed out in many writings by Milton Babbitt, Anatol Vieru and Iannis Xenakis. Xenakis discusses this relation in such a way (Xenakis (1991), p. 169).
Let $\mathbb{C}$ be the complex plane which is naturally isomorphic to the two dimensional Euclidean space $\mathbb{R}^2$. A musical sound of pitch y and time attack x can be represented by a point $z = x + iy \in \mathbb{C}$.
The four elements of $\mathcal{D}_2$ can be seen as the following operations on $\mathbb{C}$:

$$\begin{array}{lcl} f_1 & : & z \rightarrow z \\ f_2 & : & z \rightarrow \overline{z} \\ f_3 & : & z \rightarrow -z \\ f_4 & : & z \rightarrow -\overline{z} \end{array}$$

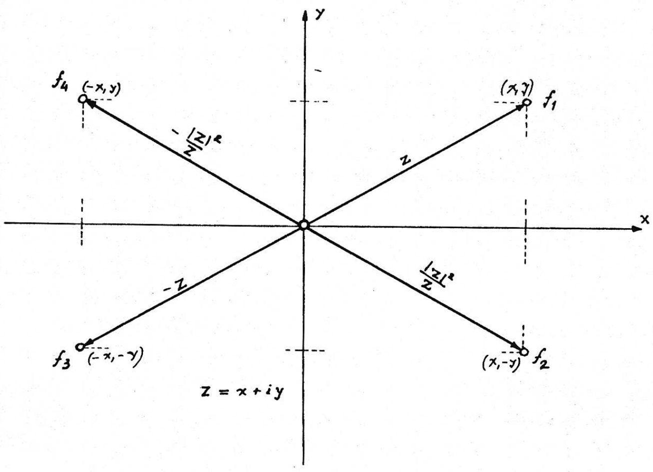
These correspond to the four forms of a twelve-tone row which are respectively, the original (or »prime«) form, the inversion, the inverted retrogradation (or retrograde inversion) and the retrogradation (See Fig. 1).
Note that $\mathcal{D}_2$ is generated by the operations f_2 and f_3, i.e.

$$\mathcal{D}_2 =< f_2, f_3' \mid {f_2}^2 = {f_3}^2 = 1 >$$

More generally one can show that the dihedral group $\mathcal{D}_p$ is generated by the complex mappings given by

$$\begin{array}{lcl} z & \longrightarrow & z \\ z & \longrightarrow & \omega z \text{ where } \omega = e^{\frac{2\pi i}{p}} \end{array}$$

6 See in particular the Tome III of his *Traité de Rythme, de Couleur et d'Ornithologie* (Messiaen (1992)) for Messiaen's detailed discussion on some combinatorial aspects of his compositional technique.



$Z = x + yi$

$f_1 = Z = x + yi = Z = f_1(Z) =$ original form

$f_2 = x - yi = |Z|^2/Z = f_2(Z) =$ inversion

$f_3 = -x - yi = -Z = f_3(Z) =$ inverted retrogradation

$f_4 = -x + yi = -(|Z|^2/Z) = f_4(Z) =$ retrogradation

Figure 1: The four forms of a twelve-tone row as transformations in the complex plane (From Xenakis (1991))

This is a simple formalization of Xenakis's original intuition that the four symmetries of the twelve-tone system are but a special case of a more general compositional construction. In the composer's words: »Let us assume that we have such a tree in the pitch versus time domain. We can rotate (transform) it; the rotation can be treated as groups. We can use traditional transformations of the melodic pattern: we can take the inverse of the basic melody, its retrograde and its retrograde inverse. There are of course many more possible transformations because we can rotate the object at any angle« (Varga (1996), p. 89). And, more recently: »This is the Klein group. But we can imagine different kinds of transformations, as a continuous or non continuous rotation of any angle. This gives new phenomena, new evenements, even by starting with a melody, for a simple melody becomes a polyphony« (Delalande (1997), p. 93).

In section 2 we will show in detail how an old problem in number theory has recently taken shape in an algebraic theory of musical canons. The theory has been developed by the Rumanian mathematician Dan Tudor Vuza independently from the solution of Minkowski's conjecture by the Hungarian mathematician G. Hajós. As far as I know my own work on *Hajós Groups, Canons and Compositions*

(Andreatta (1996)) was the first attempt to dicuss Vuza's theory from the perspective of the Minkowski/Hajós problem. For although Hajós Groups had been previously referred in connection with music, the context were completely different (Halsey and Hewitt (1978) and Bazelow and Brickle (1976)). For example Halsey and Hewitt's algebraic study on enumeration only concerns the interpretation of the cyclic group in the pitch domain. The 11th paragraph is dedicated to the discussion of »Parkettierung« (Tessellation or factorisation) of finite abelian groups. The underlying philosophy consists of considering such groups »die auch nur die geringste Chance haben, jemals in der Theorie der musikalischen Komposition eine Rolle zu spielen«. [7]
First of all, the family of finite abelian groups is restricted to that of cyclic groups. Non cyclic abelian groups have, in fact, »keinerlei Beziehung zur Musiktheorie im derzeit üblichen Sinne.«[8] The problem is that music theory, as discussed in Halsey and Hewitt's article, is concerned with chords inside a n-tempered System and the restriction $n \leq 24$ »schliesst alle Fälle ein, die in absehbarer Zukunft für das Komponieren von Musik in Frage kommen zu können scheinen«. [9]

From this perspective, non Hajós cyclic groups play no role in the discussion since the smallest group which does not have the Hajós property has order equal to 72. Again, there are many reasons for trying to describe Vuza's results on Canons by means of a more generalised algebraic theory, as we started in (Andreatta (1996) and Andreatta (1999)). For a technical presentation of the problem of classification of rational rhythms and canons by means of Mazzola's mathematical music theory see the section 16.2.3 of (Mazzola (2003)).[10]
With regard to the benefits which music and mathematics could gain from each other, one seems to have to agree with Olivier Revault d'Allonnes that »the sciences can bring infinitely more services, more illuminations, more fecundations to the arts and particularly to music than music can bring to scientific knowledge« (Xenakis (1985), p. 15). And that not only »musical thinking has not yet sufficiently utilized all the mathematical resources it could« but also that »given the relatively elementary level of mathematics [in the concepts employed] I would say that the interest is null for mathematics« (Xenakis (1985), p. 15).

This study is an attempt to describe some advanced algebraic appoaches to music compositions which are particularly interesting from a mathematical perspective.

7 i.e. that have the chance of playing a role in musical composition (Halsey and Hewitt (1978), p.190)

8 i.e. no relationship with music theory, as it appears today (Halsey and Hewitt (1978), p.200)

9 i.e. closes all the possibilities that appeare to be important for music composition in a near future (Halsey and Hewitt (1978), p.200). This prediction has been largely refuted, as we'll see in the following, by Vuza's model of periodic rhythm, which does not impose any limitation to the order of the cyclic group. This shows that the problem of »tension« between a mathematical construction and a possible musical application is, sometimes, very difficult to define and to predict. For a different example consider Lewin's *GIS* construction (Lewin (1987)) which offers many examples of musically relevant non commutative groups.

10 See (Fripertinger (2001)) for a different approach on the enumeration problem of non isomorphic classes of canons.

2 The Minkowski-Hajós Problem

As mentioned in the introduction, the family of Hajós groups originated by an old problem of number theory which Hermann Minkowski raised in 1896 (Minkowski (1896)). Recalling the story of the last theorem of Fermat, which was solved more than century and a half after its first formulation, we could call this problem the last theorem of Minkowski. For, although he was determined to furnish a proof in a short time, the problem »turned out to be unexpectedly difficult« (Robinson (1979)). So difficult, in fact, that Hajós' solution to Minkowski's problem has been described as »the most dramatic work in factoring« (Stein (1974)). I will not describe in details the transition from the original number-theoretical conjecture to Hajós' final formulation (and solution) in terms of the tiling of finite abelian groups. Rather, I will look at it »as the metamorphosis of a caterpillar to a butterfly« (Stein (1974)), from an advanced geometrical state concerned with tiling the n-dimensional Euclidean space $\mathbb{R}^n$ with a family of congruent cubes (i.e. cubes which are translated of each other). Some preliminary definitions are necessary. By *lattice tiling* (or lattice tessellation) of the n-dimensional Euclidean space, we mean a collection of congruent cubes that cover the space in such a way that the cubes do not have interior intersection and that the translation vectors form a lattice. This kind of lattice is sometimes called »simple«, to distinguish it from multiple tilings in which cubes can intersect in such a way that they are such that every point of the Euclidean space (which does not belong to the boundary of one cube) lies in exactly k cubes ($k < \infty$). In this case we speak of a k-fold tiling (or a tiling of multiplicity k).[11] The first geometric formulation of the last theorem of Minkowski, which requires the »lattice property« is the following:

Minkowski's Conjecture: in a simple lattice tiling of $\mathbb{R}^n$ by unit cubes, some pairs of cubes must share a complete (n-1)-dimensional face.[12]

Let us consider Hajós' translation of Minkowski's conjecture in algebraic terms Hajós (1942) as it appears, for example, in Stein (1974). For an exposition of Hajós proof the reader should refer to Fuchs (1960), Robinson (1979), Rédei (1967) or Stein and Szabó (1994).

Hajós Theorem:

let G be a finite abelian group and let $a_1, a_2, ..., a_n$ be n elements of G. Assume that G can be factored into n sets:

$$A_1 = \{1, a_1, ..., a_1^{m_1-1}\}, A_2 = \{1, a_2, ..., a_2^{m_2-1}\}, ..., A_n = \{1, a_n, ..., a_n^{m_n-1}\}$$

11 For a detailed account on the algebraic and geometric properties of cube tilings with respect to Minkowski's conjecture and some possible generalisations see Szabó (1993) and Stein and Szabó (1994).

12 Such cubes are sometimes called »twin« Szabó (1993). Note that in general the latticity condition cannot be removed. Minkowski's condition without latticity is historically due to O. H. Keller. He conjectured (Keller (1930)) that in any tiling of the n-dimensional Euclidean space by unit cubes there exists at least a pair of cubes sharing a complete $(n-1)$-dimensional face. J. C. Lagarias and P. W. Shor proved that this conjecture is false for all $n \geq 10$ (Lagarias and Shor (1992)). The conjecture is true for $n \leq 6$, as it was already known since 1940 Perron (1940) but the case $6 < n < 10$ remains open.

where $m_i > 0, i = 1, 2, ..., n$ in such a way that each element g of G is uniquely expressible in the form:

$$g = a_1^{e_1} \cdot a_2^{e_2} \cdots a_n^{e_n},$$

$0 \leq e_i < m_i (i = 1, ..., n, a_i \in A_i)$. Then at least one of the factors is a group (i.e. there is at least one integer i such that $a_i^{m_i} = 1$).[13]

Hajós' Theorem is also called the »Second main theorem for finite abelian groups« (Rédei (1967)) and it has been shown that there is a logical duality between it and another important result in group theory, known as Frobenius-Stickenberger's Theorem. Hajós' Theorem can be formulated in an equivalent way by using the concept of periodic subset of a groupe, i.e. a subset A such that there exists an element g in G (other than the identity element) such that $gA = A$. It is not difficult to see that Hajós' main Theorem is equivalent to the statement that an integer i exists such that A_i is periodic. An expression in which, as I have shown before, a group G has been factored in a direct sum (or, equivalently, using multiplicative notation, in a direct product of k subsets) is often called a k-Hajós factorisation. I will use the expression »Hajós factorisation« in the case where k=2, which is the most interesting aspect of the Theory that we will discuss.

A group G is said to possess the k-Hajós property [14] (or to be a k-Hajós group) if in every k-Hajós factorisation at least one factor is periodic. In a similar way to that which was noted before, a Hajós group is a group with the 2-Hajós property. Historically the first research related to the Hajós property were attempts to study the Hajós abelian finite groups. However the distinction between Hajós groups and groups with the k-Hajós property ($k > 2$) is very important, for there are results which apply to the case $k > 2$ only, as shown in Fuchs (1964). Whilst it is true that results of musical interest have only been obtained, for the moment, for $k = 2$, I will summarise all the more general results, because I think that this could help us to see new interesting applications in the musical domain.

3 List of Hajós groups

Firstly we give the complete list of groups which have the Hajós-property. We also provide a short account for the case of k-Hajós groups G with some particular assumption for what concerns the cardinality of every factor.

`List of Hajós groups:`

1. Finite abelian groups (see Sands (1962)).
   Let $(a_1, a_2, ..., a_n)$ be the short notation for the direct product $\mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times ... \times \mathbb{Z}_{a_n}$ and let p, q, r, s be distinct primes, $\alpha > 0$ integer.
   The finite abelian groups are then all (and only) these groups (and all subgroups of them):

13 For a proof of Hajós' Theorem, in the generalisation proposed by L. Rédei see Stein and Szabó (1994). A new axiomatic approach has been recently suggested by K. Corrádi and S. Szabó (Corrádi and Szabó (1997)).

14 In Stein and Szabó (1994) different terminology has been used!

(p^α, q)	(p^2, q^2)	(p, q, r)
(p^2, q, r)	(p, q, r, s)	$(p^3, 2, 2)$
$(p, 2^2, 2)$	$(p^2, 2, 2, 2)$	$(p, 2, 2, 2, 2)$
$(p, q, 2, 2)$	$(p, 3, 3)$	$(3^2, 3)$
$(2^\alpha, 2)$	$(2^2, 2^2)$	(p, p)

In particular it follows that cyclic finite Hajós groups are all $\mathbb{Z}_n$, with $n \in \aleph = \{p^\alpha, p^\alpha q, pq, p^2q, p^2q^2, p^2qr, pqr, pqrs\}$ and where p, q, r, s are distinct prime numbers, $\alpha > 0$ integer.

2. Infinite groups.
   This case doesn't appear to be completely solved (see Fuchs (1964)) because the restriction has been imposed such that the order of one factor is finite.
   Let $\mathbb{Z}(p^\infty)$ be the Pruferian group, i.e. the abelian infinite group in which every element has finite order $p^\alpha, \alpha > 0$ integer. Three cases are distinguishable:

   (a) The torsion free case: $\mathbb{Z}$ (see Hajós (1950)) and $\mathbb{Q}$ (see Sands (1962)).

   (b) The torsion case (see Sands (1959)): $\mathbb{Z}(p^\infty) + \mathbb{Z}_q$(i.e. direct sum), and all subgroups of these, where p and q are prime distinct numbers (except for the case $p = q = 2$, which is admitted).

   (c) The mixed case:
   $\mathbb{Q} + \mathbb{Z}_p$, $\mathbb{Z} + \mathbb{Z}_p$ and all subgroups of these, where p is prime.

`List of` k`-Hajós groups:`
Sands has studied the case of the Hajós k-property for finite cyclic groups G with the assumption that "every factor has a prime power of elements" (Cfr.Fuchs (1964), p.139). The following cyclic groups are shown to be k-Hajós groups:

1. $\mathbb{Z}_{p^n}$

2. $\mathbb{Z}_{p^n q}$

3. $\mathbb{Z}_m$, where the exponential sum $e(m)$ of m is k (where $m = p_1^{n_1} \cdots p_s^{n_s}$ and $e(m) = n_1 + ... + n_s$, p_i distinct primes).

Sands shows that a cyclic group which does not have the Hajós k-property $(k > 2)$ is $\mathbb{Z}_n$, where $n = p^\alpha q^2$. But this restriction can be relaxed by affirming that the previous cyclic group is not $(k-1)$-Hajós group for each $k > 2$.

We now follow the metamorphosis of Vuza's modal theory into a rhythmic domain which naturally brings us to a new perspective about some groups relevant to musical composition. In Vuza's theory of periodic rhythmic canons (as discussed in Vuza (1985), Vuza (1986) or Vuza (1991-93)), cyclic groups which do not have Hajós property are fundamental in the formalisation of particular rhythmic canons. There is apparently no limitation in the order of the cyclic group $\mathbb{Z}_n$ for a musically relevant application because the cyclic group itself applies to Vuza's definition of rhythm and, more generally, of rhythmic class. As pointed out in

Mazzola (1990) in considering Olivier Messiaen's theoretical and compositional ideas about »modes of limited transposition« and »non invertible rhythms«, the analogy between the two concepts is far from »adequate«. Vuza's new algebraic constructions, based on the notion of action of a group G on an ensemble principal homogéne S (or $PH\ G$-set, Cf. Vuza (1988)), furnishes a complete analogy (i.e. isomorphism) between the rhythmic world and the pitch domain. By means of the action of a (commutative) group G on a $PH\ G$-set S, the collection of orbits determined by the action of G on the set of all subsets of S, which are also called »modal classes« or »transposition classes«) has the algebraic structure of a commutative semigroup with a unit element.
Different notations have been used by Vuza in the more than 10 years that separate the original collection of papers on the mathematical aspects of Vieru's Modal Theory Vuza (1982-83) from the most recent article on Supplementary Sets and Rhythmic Canons Vuza (1991-93). In summarising the principal results of Vuza's rhythmic model we shall use concepts and definitions given in Vuza (1985), Vuza (1988) and Vuza (1991-93), and attempt to make uniform the notation of the latter.
By definition, a *periodic rhythm* is a periodic and locally finite subset R of $\mathbb{Q}$. This means:

1. $\exists t \in \mathbb{Q}_+$ such that $t + R = R$.
2. $\forall a, b \in \mathbb{Q}, a < b, \#\{r \in R : a \leq r < b\} < \infty$.

This second property differs slightly from that of Vuza (1985) and Vuza (1986). The least positive rational number satisfying the first condition is called the *period* of R whereas the greatest positive rational number dividing all differences $r_1 - r_2$ with $r_i \in R$ is called the *minimal division* of R.
As pointed out in Vuza (1986) this definition implies that the collection of rhythms is a *ring* of sets. Moreover, one may consider the translation class of a given rhythm with respect to the group $\mathbb{Q}$. More formally one can consider the action of $\mathbb{Q}$ on the set of periodic rhythms as defined by the map $(t, R) \longmapsto t + R$ and call a "rhythmic class" an orbit of R under this action. Following Vuza's notation I shall indicate with $[R]$ the rhythmic class of a given rhythm R and with $\mathcal{R}hyth$ the collection of all rhythmic classes. More generally, I shall indicate with $\mathcal{T}(G)$ the set of translation classes of a given group G, i.e. the set of equivalence classes determined by the relation $\sim$ between subsets of G which are the translate of each other through an element x in G (see Vuza (1991-93)). The sum of subsets M and N of G are defined in the following way:

$$M + N = \{x + y, x \in M, y \in N\}$$

It has been shown that $\mathcal{T}(G)$ is a commutative semigroup with unit element under the following law (called »composition« and indicated with +):

$$M + N = [M + N]$$

where $[M]$ signifies the translation class of a given subset M of G.
Taking $G = \mathbb{Z}_{12}$ we obtain the 352 »modal classes« (including the null class) which

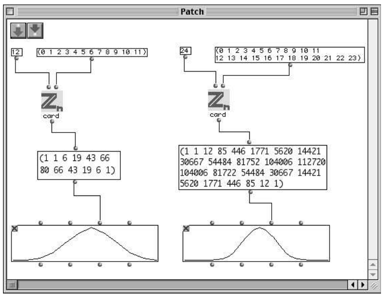


Figure 2: Number of transposition chords for the Twelve-Tone and quarter-tone temperament

have been originally studied by Zalewski in (Zalewski (1972)). Figure 10 shows the symmetric distribution of transposition classes of chords in $\mathbb{Z}_{12}$ and $\mathbb{Z}_{24}$. Note that, using dihedral symmetry, as usually applied in American Set-Theory, the number of equivalent classes of chords in a given well-tempered division of the octave (i.e. in a given cyclic group of order n) drammatically decreases[15], as it is clear from the following figure: We started to implement in *OpenMusic* some

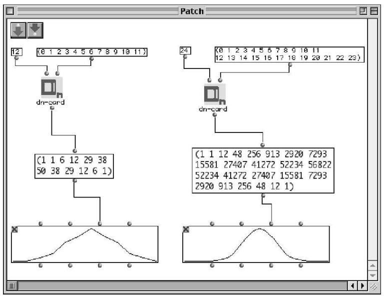


Figure 3: Number of orbits under the action of the dihedral group for the twelve-tone and for the quarter-tone temperament

some special families of translation classes of chords which have been considered, by some theorists and composers, as particularly relevant from a music-theoretical perspective. The families of equivalent classes of chords which are entirely implemented are the following :

1. Limited transposition chords

2. Self-complementary chords

15 From the perspective of Mazzola's *Mathematical Music Theory* these elementary musical structures can be considered as zero-addressed local compositions in the module $\mathbb{Z}_{12}$. T. Noll's concept of self-addressed chords, as it has been introduced in Noll (1995), has been one of the first steps towards a generalised theory of chords classification. For a presentation of the theory of classification of self-addressed local structures in the framework of denotators see section 11.3.7 of Mazzola (2003).

3. Auto-inverse chords
4. Inverse-complementary chords
5. All-combinatorial chords
6. Idempotent chords
7. Partitioning chords

In the present context we will concentrate basically on families which are central to the formalisation of tiling rhythmic canons. According to Zalewski's and Vieru's original idea, a translation class $\mathcal{M}$ will be indicated by the intervallic structure $(m_1, m_2, ..., m_i)$ which counts the number of unit steps between successive notes. $\#\mathcal{M}$ is called the order of the translation class (i.e. of the intervallic structure or, more simply, of the structure). It follows that $\sum_{j=1}^{i} m_j = n$ where n is the order of the cyclic group $\mathbb{Z}_n$.
The three families which are particularly relevant in the context of the present paper are:

1. *Idempotent classes*.
   Given two classes $\mathcal{A}, \mathcal{B} \in \mathcal{T}(\mathbb{Z}_n)$ we introduced, following Vuza (1982-83), a law of composition (+) which is by definition:
   $$\mathcal{A} + \mathcal{B} = [M + N]$$
   where $M \in \mathcal{A}, N \in \mathcal{B}$. This operation is formally equivalent to what the American music-theoretical tradition calls »transpositional combination« between chords. It represents a generalisation of Boulez' technique of »multiplication d'accords«, as it has been initially introduced in Boulez (1963) and Boulez (1966). For a discussion of this compositional technique from an american music-theoretical perspective see Cohn (1986).
   In the case of $\mathbb{Z}_{12}$ there are exactly 6 special classes $\mathcal{A}$ for which $\mathcal{A} + \mathcal{A} = \mathcal{A}$. These are called idempotent classes and their collection is indicated by $\mathcal{T}_{id}$. All of these are well known to musicians, from the unison $\mathcal{U} = (0)$ to the total chromatic $\mathcal{TC}$. They correspond to Zalewski's »monomorphic structures«, and mathematically speaking there are simply all subgroups of a given cyclic group.

2. *Limited transposition classes*.
   A modal class $\mathcal{A} \in \mathcal{T}(\mathbb{Z}_n)$ is a limited transposition structure if $\#\mathcal{A} < n$ i.e. its subgroup of stability
   $$S_{\mathcal{A}} := \{t \in \mathbb{Z}_n : t + M = M, M \in \mathcal{A}\}$$
   is not trivial. I shall indicate with $\mathcal{T}_{tl}$ the family of all limited transposition structures. There is a strong connection between transposition limited modes and idempotent classes thanks to the previous concept of »composition« between intervallic structures. It is easy to see that a class belongs to $\mathcal{T}_{tl}$ iff it is the transpositional combination of two classes, where at least one

of these classes must be idempotent. This enables to calculate very easily the transposition limited classes for any well-tempered division of the octave in a given number n of equal parts. For example, in the quarter-tone system there are 85 transposition limited chords whereas they are 1062 in Wyschnegradsky's division of the octave in 72 equal parts.

3. *Partitioning classes*
   The family $\mathcal{T}_p$ of »partitioning classes« has been introduced in Vuza (1982-83) and described in more detail in Vuza (1991-93). By definition, a partitioning class (or {Parkettierer}) in $\mathcal{T}(\mathbb{Z}_{12})$ is a translation class with the property that »there is a partition of the set of all twelve pitch classes into subsets belonging to that class« Vuza (1991-93). Obviously the order of a partitioning class must be an integer k such that $k \mid 12$, i.e. $k \in \{1, 2, 3, 4, 6\}$. Note that, by convention, the unison is a partioning class (even if trivial) while the chromatic total $\mathcal{TC}$ does not have such a property. It can be shown (see Vuza (1982-83)) that all 2-chords are partitioning classes with the exception of the dyad (8,4). The concept of partitioning classes, together with that of »supplementary sets« (see Vuza (1991-93)) is one of the most remarkable aspects of Vuza's theory, as is clear in relation to the Minkowski-Hajós problem outlined earlier.
   Partition problems in relation to music have been posed by different authors in a number of different ways. Some are more »combinatorial« (like Milton Babbitt's partition problem described in Bazelow and Brickle (1976)), or Halsey and Hewitt's algorithmic strategy (which enables to compute the number of partitioning classes once it is given a group G and a positive integer j dividing the cardinality of G). Others are more »structural«, in the sense that they deal with the abstract, or categorical, background out of which combinatorial problems emerge, »illuminat[ing] the setting within which one wishes to deal with more concrete compositional and theoretical issues«(Bazelow and Brickle (1976), p.281). David Lewin's problem of interval function, as posed in Lewin (1987), is an example of an essentially structural problem. Using concepts such as convolution and Fourier Transform, the problem »may be generalized to questions about the interrelation, in a locally compact group, among the characteristic functions of compact subsets«(Lewin (1987), p.103).

Before showing how partitioning classes are related to Messiaen's transposition limited concept in the formalisation and construction of tiling rhythmic canons, we need a further preliminary definition:
Definition: Given a finite group G, two intervallic classes $\mathcal{A}, \mathcal{B} \in \mathcal{T}(G)$ are *supplementary* if there exists $M \in \mathcal{A}, N \in \mathcal{B}$ such that the group G can be written in a unique way as $G = M + N$, i.e. G can be factored in the direct sum of the subsets M and N. By putting together Vuza (1991-93) and Halsey and Hewitt (1978) we have the following theorem of a characterisation of supplementary classes:
Theorem: let G be a finite abelian group and let M,N be subsets of G. The following statements are thus equivalent:

1. M and N are supplementary

2. $M + N = G$ and $(\#M)(\#N) = \#G$
3. $M + N = G$ and $(M - M) \cap (N - N) = \{0\}$
4. $1_A \star 1_B = \frac{1}{\#G} \cdot 1_G$ where 1_A (equivalently 1_B and 1_G) means the characteristic function of A and the convolution product is defined in the following way:

$$1_A \star 1_B(x) = \frac{1}{\#G} \sum_{y \in G} 1_A(x - y) 1_B(y).$$

For a proof of the previous theorem we refer to the quoted works. The following figure shows an axample of supplementary sets. They are respectively $A = \{0, 8, 10\}$ and $B = \{0, 5, 6, 11\}$. Partitioning and supplementary translation classes

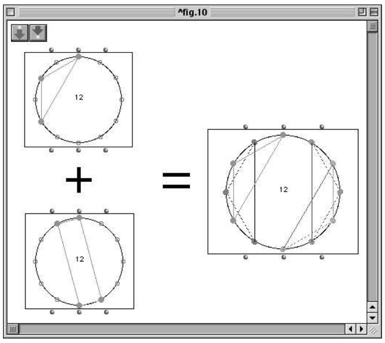


Figure 4: Factorization of the cyclic group of order n in two subsets.

are related to each other by the fact that a given translation class $\mathcal{M} \in \mathcal{T}_p(G)$ (i.e. is a partitioning class) iff a class $\mathcal{N} \in \mathcal{T}(G)$ such that $\mathcal{M}$ and $\mathcal{N}$ are supplementary exists. This leads immediately to the "canonic" interpretation of the partitioning concept and supplementary classes (See Figure 13) It is a canon in 4 voices ob-

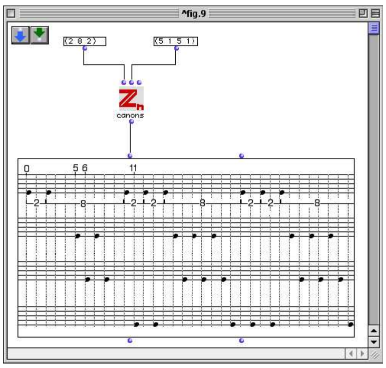


Figure 5: A tiling rhythmic canon.

tained by the time translation of the pattern $R = [2, 8, 2]$ in the onset-times 0, 5, 6,

11. The previous intervallic notation means that, given a minimal division u, two successive beats r_i, r_{i+1} of $R = [r_1, r_2, ..., r_k]$ are separeted by $r_{i+1} - r_i$, u-temporal units.
The first rhythmic pattern is also called *inner rhythm*, whereas the pattern of coming in of voices is called *outer rhythm* (Andreatta M. (2001) and Andreatta M. (2002)). Inner and outer rhythms replace Vuza's original *ground* and *metric classes* Vuza (1991-93), a terminology that could give rise to some confusions for what concerns the characterisation of rhythmic and metric properties of such global musical structures. This tiling condition implies that time axis is provided with a minimal division which holds as well for the inner and for the outer rhythm. Rhythmic canons verifying the tiling condition are also called, in Vuza's terminology, »Regular Complementary Canons«. In fact, voices are all complementary (there is no intersection between them) and once the last voice has come in, one hears only a regular pulsation (there are no holes in the time axis). It is now easy to show that the construction of the so-called »regular complementary of maximal category« (shortly RCMC-Canons) is equivalent to the problem of providing a pair of non periodic supplementary sets of some cyclic group $\mathbb{Z}_n$. Note that Vuza's Theorem 2.2 (Vuza (1991-93), p. 33) is formally equivalent to Hajós' theorem. Its special case, Theorem 0.1 provides a connection between the theory of supplementary sets and that of limited transpositional structures (i.e. translation classes with transpositional symmetry). It affirms that »given any pair of supplementary sets (in $\mathbb{Z}_{12}$) at least one set in the pair has transpositional symmetry«. As previously noted, this result remains true when $\mathbb{Z}_{12}$ is replaced by any cyclic Hajós group $\mathbb{Z}_n$. But the theory developed by Vuza suggests something further: given a cyclic group which does not have the Hajós property, it provides a method of constructing (all)[16] non periodic supplementary sets of such a group. Besides the musical relevance it seems to me that this theory could also be interesting from a purely mathematical perspective, for it concerns some structures which are far from being completely classified.[17]
In a previous study (Andreatta (1996)) we considered the case $\mathbb{Z}_n$, where $n = 108$, which Vuza seemed to forget to consider as belonging to the family of groups which do not satisfy the Hajós property. By following a slightly different method

16 We stress the fact that Vuza's algorhythm produces all factorisations of a non-Hajós cyclic group by two non-periodic subsets althought it has been shown that there are supplementary sets which are apparently not a direct solution of Vuza's algorithm. This question was originary raised by the composer George Bloch who observed that a RCMC-Canon of period p could be »interpreted« as a RCMC-Canon of period $2p$ simply by replacing each minimal division with two attacks holding a minimal division each. For example this process enables to construct a partitioning set of period $n = 144$ as a simple transformation of a partitioning set of period $n = 72$. This new set does not belong to the catalogue of solutions provided by Vuza's algorhythm. In fact it is, in some sens, »redundant«, since it is not maximally compacted. H. Fripertinger did the same discovery independently from the intuitions of the French composer and implemented the algorhythm which includes all those Vuza-Canons which do not have the maximal compactness property.

17 In presenting his generalised Hajós property in the Hungarian Colloquium on Abelian Groups (September 1963), Sands admitted that "the problem of obtaining the factorisations of those groups which do not possess this Hajós property remains". A recent paper on the k-factorisation of abelian groups (Amin (1999)) seems to suggest that the problem is a still interesting problem in mathematics.

for the construction of a pair of non-periodic supplementary sets of $\mathbb{Z}_{108}$ (see (Andreatta (1996), pp. 25-27) we found the following factorisation:

$$\begin{aligned} M &= \{39, 51, 63, 66, 78, 90\} \\ N &= \{16, 18, 20, 22, 26, 27, 36, 52, 56, 58, 62, 72, 81, 88, 90, 92, 94, 98\} \end{aligned}$$

Making use of the standard intervallic notation for a rhythmic class, we find that the rhythmic classes associated with the couple of non periodic supplementary sets M and N are, respectively:

$$\begin{aligned} \mathcal{S} &= [12, 12, 3, 12, 12, 57] \\ \mathcal{R} &= [2, 2, 2, 4, 1, 9, 16, 4, 2, 4, 10, 9, 7, 2, 2, 2, 4, 26] \end{aligned}$$

If we define the *power* of a canon as the number p of its voices, p obviously is equal to $\#\mathcal{S}$. We observe that the power p alone is far from being the most important parameter. For example, take the smallest cyclic group $\mathbb{Z}_{72}$ that does not have the Hajós property. This case has been taken as an example of remarquable number by F. Le Lionnais (Lionnais (1989)) who quoted the following decomposition of $\mathbb{Z}_{72}$ in two non-periodic subsets by L. Fuchs (Fuchs (1960), p. 316):

$$\begin{aligned} M &= \{0, 8, 16, 18, 26, 34\} \\ N &= \{0, 1, 5, 6, 12, 25, 29, 36, 42, 48, 49, 53\} \end{aligned}$$

These non periodic subsets of $\mathbb{Z}_{72}$ are isomorphically associated with the following rhythmic classes

$$\begin{aligned} \mathcal{S} &= [8, 8, 2, 8, 8, 38] \\ \mathcal{R} &= [1, 4, 1, 6, 13, 4, 7, 6, 6, 1, 4, 19] \end{aligned}$$

Note that the power of this canon is still 6, as in the case of $\mathbb{Z}_{108}$ and that in both cases the set $\mathcal{S}$ has a stronger symmetrical character than the set $\mathcal{R}$.
Looking at the metric class $\mathcal{S}$ of all these regular complementary canons of maximal category, we observe that it has a property which could be called of »partial non invertibility«. Writing $\mathcal{S}$ as $[s_1, s_2, ..., s_n]$, either $[s_1, s_2, ..., s_{n-1}]$ or $[s_2, s_3, ..., s_n]$ is a non- invertible rhythmic class (in the sense of Messiaen). By definition we say that the rhythmic class $[s_1, s_2, ..., s_n]$ is non invertible if it coincides with its inverse $[s_n, s_{n-1}, ..., s_1]$. The sense of the adjective "partial" in the definition of the previous property is clearly intuitive: $\mathcal{S}$ is non invertible with the exception of the first or the last temporal interval. The fact that every metric class of a given regular complementary canon of maximal category has the property of »partial non invertibility« seems not to be a direct consequence of the theory here under discussion. It seems appropriate, therefore, to class this fact as a conjecture which, because of its analogy with some of Messiaen's ideas, has been called the »M-Conjecture« (Andreatta (1997)). Moreover, a comparison between this theory and some of Messiaen's music- theoretical ideas, such as those contained in Messiaen's recent *Traité* (Messiaen (1992)) shows that the problem of formalisation of canons was very central to the French composer. An interesting example is given by the

Figure 6: A Messiaen's three-voices canon in the piece *Harawi*

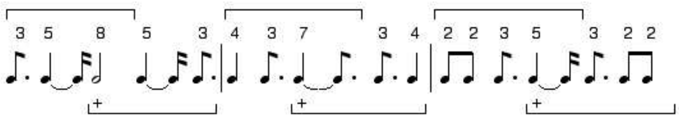


Figure 7: Rhythmic pattern of *Harawi*

piece *Harawi* (part no. 7, *Adieu*). From a rhythmic point of view, the previous example realises a canon in three voices, each voice being the concatenation of three non-retrogradable rhythms, as it is shown in figure 15: In Messiaen's words, this global musical structure is an example of an »organised chaos« (Messiaen (1992), p. 105), for the attacks of the three voices seem to be almost complementary. This is only partially true, as it is clear from the following representation of the canon in a grid in which points correspond to the onset-times of the voices. There are instants of time in which no voice is playing and, conversely, there are moments in which two or more voices are playing together (See Figure 16). To be noticed that the same grid has also been used by Messiaen in *Visions de l'Amen: Amen des anges, des saints, du chant des oiseaux*. The only difference concerns the minimal division of the rhythm, which is now equal to a 32th note. Figure 17 shows the formal rhythmic structure of this new canon.

4 Conclusion

There are basically two possible answers to the question we asked at the beginning of this essay. Talking about mathematical groups, as talking about symmetry or other algebraic concepts in music, could have either a mathematical or a musical sense. In the first case, mathematicians may consider, for example, that some group-theoretical problems do have something interesting from a purely mathematical perspective. Despite Olivier Revault d'Allones' already quoted pessimistic position, stressing the fact that the sciences, and mathematics in particu-

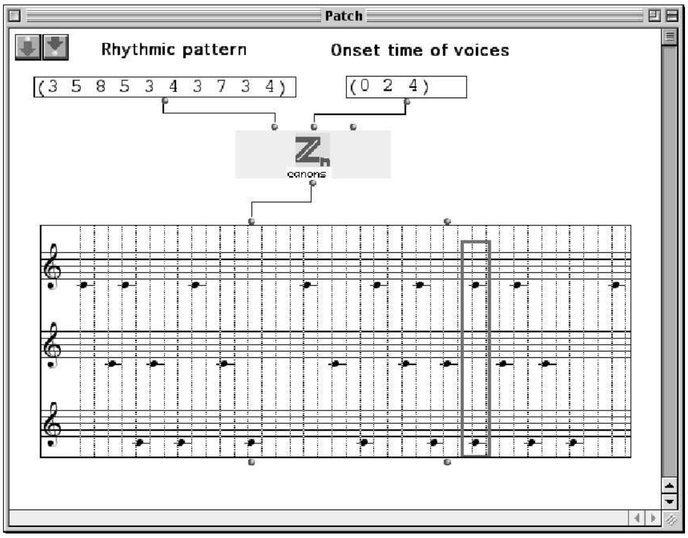


Figure 8: Grid representation of Harawi.

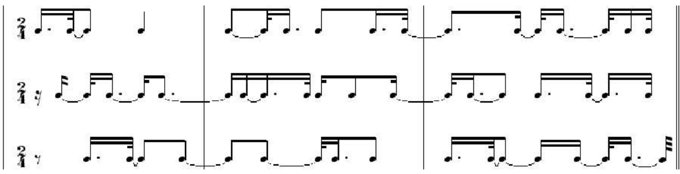


Figure 9: A three-voices canon in *Visions de l'Amen*

lar, »can bring infinitely more services [...] to music than music can bring to the scientific knowledge«, there are cases for which music could be the starting point for the mathematical research itself. But in order to give to the initial question a complete answer we also have to take into account the (sometimes unexpected) musical ramifications of a mathematical research.

Since Vuza's original paper on tiling canons and my personal contribution in revising such mathematical structures with the help of the concept of Hajós groups and with some more general MaMu-Theoretical constructions[18] many people have been fascinated by these remarquable structures. The implementation realised in collaboration with Carlos Agon and Thomas Noll made available the complete list of Regular Complementary Canons of Maximal Category for any given non-Hajós group $\mathbb{Z}_n$. Fig.18 gives the order n of non-Hajós cyclic groups with $72 \leq n \leq 500$. Figure 19 shows all possible *inner* and *outer* rhytms for Vuza-

18 I would like to express my thanks to Guerino Mazzola for stressing the necessity of revising most of the algebraic concepts introduced by Vieru and Vuza within the framework of the local/global theory. The generalisation of Vieru and Vuza's modal theory to more sophisticated modules will enable the theorist/composer to work not only in the pitch or rhythmic domain but in both domains, including a parametrised space for intensities and other relevant musical properties.

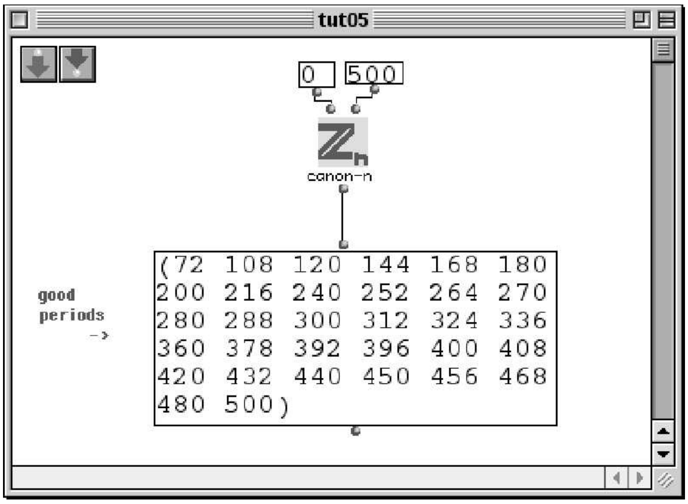


Figure 10: Non-Hajós cyclic groups with $72 \leq n \leq 500$

Canons of period 72. The case of the construction of special tiling rhythmic canons

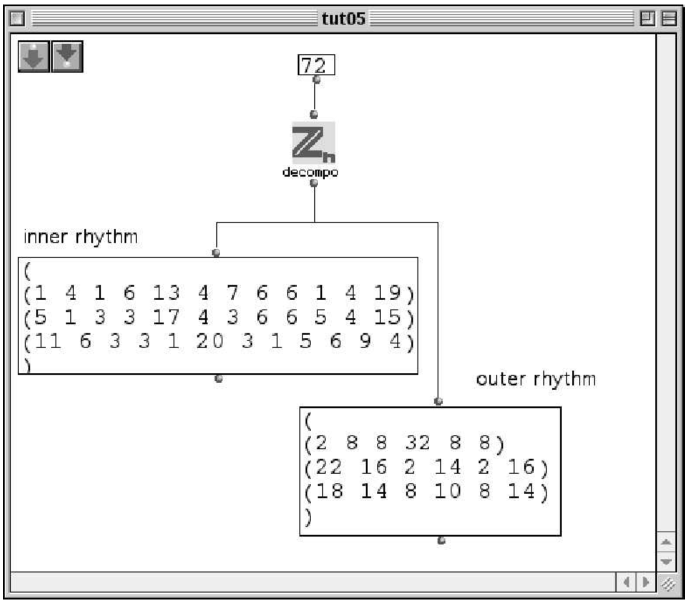


Figure 11: All possible *inner* and *outer* rhytms for Vuza-Canons of period 72

suggests that there are musical problems whose mathematical ramifications could be sometimes very unexpected.[19] The catalogue of all possible factorisations of

19 Emmanuel Amiot stresses the fact that these music-theoretical constructions could help mathe-

cyclic non-Hajós groups into two non-periodic subsets has been taken by some theoretically-inclined composers as the starting point for further speculations on tiling problems on music.[20] The case of composer Georges Bloch is particularly interesting for some unexpected musical ramifications of Vuza's original theory. Starting from two practical problems, he suggested how the theory could be generalised in order to take into account some crucial compositional questions. For example, how to use 6-voices canons for a composition which is based on, e.g., five players but in such a way that the 6-voices counterpoint will be perceptible. Another interesting problem is that of finding relationships between different canons having different numbers of voices. This opens the problem of the modulation process between global structures, a problem which has been only solved for some special cases.

Another open question concerning the Minkowski-Hajós problem and the construction of rhythmic tiling canons is whether one can find a general algorithm providing all possible solutions for RCMC-canons with respect to a given cyclic non-Hajós group.[21] We conjecture that Vuza's alogorithm can eventually be transfomed into a general one. However—in contrast with Fermat and Minkowski—I am not suggesting that the space around this written text were not long enough to provide evidence for a formal proof, nor that we will shortly be able to provide such a proof. Even if a solution will be found 50 years later, as in the case of the Minkowski-Hajós problem, I prefer to think that any attempt to find a solution to this ›mathemusical‹ problem will be a source of inspiration for both, mathematicians and musicians.

maticians to approach some still open mathematical conjectures (Amiot (2003)). Another famous example of a musical result which appeared to be connected with an old mathematical conjecture is the so-called Babbitt's Theorem of Hexachord, stating that two complementary hexachords do have the same interval content. The first knot-theoretical proof of this theorem by Ralph Fox was published as a new way of solving Waring's problem, one of the old standing problems in number theory (See Babbitt (1987), p. 105).

20 The pieces using some of these solutions, althought in a very different way, are *Coincïdences* for orchestra by Fabien Lévy and a piece for small ensemble by Georges Bloch called *Fondation Beyeler: une empreinte sonore.* Starting from Vuza's formalisation of tiling rhythmic canons and from the *OpenMusic* implementation, composer Tom Johnson posed the problem of the construction of tiling canons by augmentation. This problem, as in the case of Minkowski's Conjecture, turned out to be unexpectedly interesting from a mathematical point of view. One solution has been proposed by E. Amiot (Amiot (February 2002)) by using the polynomial representation of rhythmic canons as initially introduced by A. Tangian (Tangian (2001)) and by applying some advanced algebraic concepts from Galois Theory to music. The solution to Johnson-Tangian Conjecture by E. Amiot together with the generalisation proposed by H. Fripertinger is available online at the following address:
http://www.ircam.fr/equipes/repmus/documents/MaMuXtiling.html
For a different perspective on Galois Theory of concepts in music see Mazzola's contribution (Mazzola (2002) in the Fourth Diderot Mathematical Forum (Assayag and al. (2002)).

21 See Amiot (2003) for the most recent discussion in the subject.

5 Aknowledgements

This paper is a condensed and updated version of my Independent Study Dissertation »Group Theorectical Methods in Music« (c.f Andreatta (1997)). I would like to express my sincere acknowledgements to the editors, who gave me the possibility to revise and finally publish a study which seems to have progressively become a reference in the Minkowski/Hajós problem applied to the construction of tiling rhythmic canons. Many things happened since the period I was Visiting Student at the University of Sussex and the present article reflects the different ramifications of my present interests. Firstly the contact with the *Music Representation Team* at Ircam, expecially with Gérard Assayag and Carlos Agon, who trusted me in my algebraic approach to the formalisation of musical structures. The implementation of all these methods in *OpenMusic* is the most evident consequence of a very exciting collaboration between a mathematically-inclined musician, as I am, and a computer-music scientist, Carlos Agon, whose continuous encouragement and interest in this approach finally enabled to make the theory more suitable for compositional application. I would like to thank Thomas Noll and Guerino Mazzola for their great help in some mathematical aspects of Vuza Theory which has been progressively integrated in the more general framework of the MaMuTh-approach. Many thanks to Emmanuel Amiot and Harald Fripertinger for all the comments they provided on the mathematical strategies in the enumeration and construction of tiling rhythmic canons.

This paper is dedicated to Dan Tudor Vuza, with great respect and profound admiration.

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