A Generalisation of Diatonicism and the Discrete Fourier Transform as a Mean for Classifying and Characterising Musical Scales

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Abstract. Two approaches for characterising scales are presented and compared in this paper. The first one was proposed three years ago by the musician and composer Pierre Audétat, who developed a numerical and graphical representation of the 66 heptatonic scales and their 462 modes, a new cartography called the *Diatonic Bell*. It allows sorting and classifying the scales according to their similarity to the diatonic scale.

The second approach uses the Discrete Fourier Transform (DFT) to investigate the geometry of scales in the chromatic circle. The study of its coefficients brings to light some scales, not necessarily the diatonic one, showing remarkable configurations. However, it does not lead to an evident classification, or linear ordering of scales.

1 Introduction

Over centuries, western musicians have extensively used half a dozen of heptatonic scales, but combinatorics teach us that they represent only a tenth of the totally available musical material. Many catalogues exist, but they often reduce to numerical tables, that may not be easy to handle for composers.

The musician and composer Pierre Audétat [2] developed a numerical and graphical representation of all 66 heptatonic scales and their 462 associated modes. Such a cartography, called the *Diatonic Bell*, opens a field of experiment equally relevant for composition and analysis, and presents interesting developments for teaching.

The first part of this paper deals with the classification and ordering of scales obtained with the diatonic bell, presenting a mathematical formulation of Audétat's original empirical work. The second part investigates scales in the chromatic circle using the Discrete Fourier Transform (DFT) in order to exhibit certain scales with remarkable properties.

David Lewin proposed this tool in 1958 for analysing intervallic relationships. The idea was pursued by Ian Quinn [7] for classifying chords and by Emmanuel Amiot [1] for redefining Clough and Douthett's maximal evenness [4]. Inspired by this work, we will see how DFT coefficients reflect the geometric configuration of a scale in the chromatic circle, and how they can be used to characterise scales.

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The two methods differ structurally, the former being tonal, the latter atonal. We will discuss in the conclusion some points of convergence between these two approaches.

2 The Diatonic Bell

Modes play a key role in Jazz. The first *diatonic bell* was produced by hand in an effort to investigate the 66 heptatonic scales and their 462 modes. The reader interested in how this system displays a network of musical relations offering new opportunities in composition and may facilitate the modal approach of improvisation, is invited to consult the online documentation.[1] We will focus on the step by step procedure, along with the mathematical formulas necessary for a complete construction.

The general idea is to consider every scale as an alteration of a reference, *natural* scale. We will call it *diatonic*, but it may be another maximally even scale. Scales are ordered according to their increasing degree of alteration, from the maximally even to its maximally compact counterpart.

Two different musical spaces are successively used in the process. We first enumerate scales in the finite chromatic circle before moving to the infinite *diatonic spiral* — a generalisation of the spiral of fifths to microtonal contexts — for the graphical representation. This is to avoid the ambiguity induced by enharmony: alterations of the diatonic scale such as $G\sharp = [5+1]_{12} = [6]_{12} = [7-1]_{12} = A\flat$ are not distinguishable in the chromatic circle, whereas they represent two different points on the diatonic spiral.

Two conditions need to be fulfilled before we can compare scales:

1. They need to be centred, or aligned on the symmetry axis of the diatonic scale.
2. We have to make sure that their representation on the diatonic spiral is as compact as possible.

2.1 Input Parameters

Only two parameters are necessary. The size c of the chromatic universe of pitch classes, and the size d of the scale. The procedure works under certain conditions:

1. (a) d must be odd. This is to avoid a hole at the origin (symmetry axis) in the diatonic spiral.
   (b) d must be prime. It guarantees the existence and the unicity of a centred scale in each transposition class, and we avoid scales with internal symmetries (e.g. Messiaen's modes with limited transposition), as a by-product.
2. The parameters must be coprime (i.e. $< c, d >= 1$). This guarantees existence and unicity of a reference scale.

[1] http://www.cloche-diatonique.ch/

2.2 Find All Scales

The chromatic gamut is modelled by the chromatic circle $\mathcal{C}_c = \mathbb{Z}/c\mathbb{Z}$, or cyclic group. The pitch classes are modular integers $[x]_c := x + c \cdot \mathbb{Z}$. A scale S is an unordered subset of $\mathcal{C}_c$. We define the set of all d-notes scales of $\mathcal{C}_c$ as

$$\mathcal{S}_c^d := \{S \subseteq \mathcal{C}_c | d = Card(S)\}. \tag{1}$$

This set has cardinality $\binom{c}{d}$ and contains all possible transpositions of a same scale, a c times redundant information. The cyclic group $\mathbb{Z}_c$ of transpositions acts on $\mathcal{S}_c^d$, and the quotient space will be indicated by $\mathcal{S}_c^d/\mathbb{Z}_c$. A transposition class contains all scales equivalent by translation $T_{[l]_c}$ where $[l]_c \in \mathbb{Z}_c$:

$$S' \sim_{\mathbb{Z}_c} S :\Leftrightarrow \exists [l]_c \in \mathbb{Z}_c : S' = T_{[l]_c}(S) \quad S, S' \in \mathcal{S}_c^d. \tag{2}$$

The most economical way to enumerate all transposition classes is to generate all intervallic structures that uniquely define each class. This can be done by searching for all integer partitions of c into d parts, see [6].

2.3 Find All Centred Scales

In each transposition class $[S]_{\mathbb{Z}_c} \in \mathcal{S}_c^d/\mathbb{Z}_c$, find the unique scale $S^\star$ centred around $[0]_c$: Its chromatic coordinates (pitch classes) sum to zero. The fact that d is coprime with c guarantees the existence and unicity of such a centred scale for each transposition class.

$$\mathcal{S}^{\star}{}_c^d := \{S \in \mathcal{S}_c^d | \sum_{[x]_c \in S} [x]_c = [0]_c\} \tag{3}$$

2.4 Find the Reference Scale

We search for the *maximally even* scale $S_0^\star$ [4]. Here again, the condition $< d, c >= 1$ guarantees existence and unicity of such a scale [1]. It will also be *generated* in the sense of [3], and the most compact in the diatonic spiral. We set it to be the reference scale in our representation. It can be found using the discrete Fourier transform $F\{S\}$ of a scale $S \in \mathcal{S}_c^d$

$$\begin{aligned} F\{S\} : \mathcal{C}_c &\longrightarrow \mathbb{C} \\ [k]_c &\longmapsto \sum_{[x]_c \in \mathcal{C}_c} \mathbb{1}_S([x]_c) \cdot e^{-i\frac{2\pi}{c} \cdot x \cdot k} \end{aligned} \tag{4}$$

where $\mathbb{1}_S$ is the indicator (or characteristic) function of the subset S. The maximally even scale will maximise the module of the d-th coefficient.

$$S_0^\star := argmax_{S^\star \in \mathcal{S}^{\star}{}_c^d} \left| F\{S^\star\}([d]_c) \right| \tag{5}$$

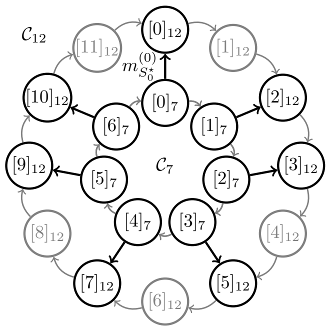

Fig. 1. The diatonic scale's dorian mode is the reference centred mode $m^{([0]_d)}_{S^\star_0}$ in the usual context ($c = 12$ and $d = 7$). It begins with a D ($[0]_{12}$).

2.5 Find the Reference Mode

Order makes the difference between scales and modes. While a scale is defined as an unordered subset, the cyclic ordering of its steps is essential to distinguish between its d modes. We define a mode m_S of a scale $S \in \mathcal{S}^d_c$ to be a function $m_S : \mathcal{C}_d \longrightarrow \mathcal{C}_c$ whose image is exactly the subset S

$$Im(m_S) = S \tag{6}$$

and which preserves the cyclic sequence of the element of the circles (consider them as cyclic oriented graphs G).

$$\begin{aligned} V(G) &= \{[0]_c, \ldots, [c-1]_c\} \\ ([x]_c, [x']_c) \in A(G) &\Leftrightarrow [x']_c = [x+1]_c. \end{aligned} \tag{7}$$

Since d was chosen to be prime, all d modes of a scale S are distinct (no limited transposition modes). A cyclic permutation $\pi = ([0]_d[1]_d \ldots [d-1]_d)$ of the diatonic circle $\mathcal{C}_d$ connects them altogether.

$$m^{([k]_d)}_S := m^{([0]_d)}_S \circ \pi^k \quad \forall k \in \mathbb{Z} \tag{8}$$

We choose the centred mode $m^{([0]_d)}_{S^\star_0}$, to be the one starting at $[0]_c$:

$$m^{([0]_d)}_{S^\star_0}([0]_d) = [0]_c. \tag{9}$$

Fig. 1 shows the example of the diatonic scale.

2.6 Find All Centred Modes

For every scale $S^\star \in {\mathcal{S}^\star}^d_c$, find the centred mode $m^{([0]_d)}_{S^\star}$ that implies the minimum amount of alterations of the reference centred mode $m^{([0]_d)}_{S^\star_0}$.

$$m^{(0)}_{S^\star} = argmin_{m_{S^\star}} \sum_{[k]_d \in \mathcal{C}_d} d_{\mathcal{C}_c}(m_S([k]_d), m^{(0)}_{S_0}([k]_d)) \tag{10}$$

where $d_{\mathcal{C}_c}$ is the circle distance:

$$d_{\mathcal{C}_c} : \mathcal{C}_c \times \mathcal{C}_c \rightarrow \mathbb{N}$$
$$([x]_c, [x']_c) \longmapsto argmin_{n \in [x]_c, n' \in [x']_c} |n - n'|_{\mathbb{Z}} \quad (11)$$

2.7 Construct All Representations

Once we have a centred mode $m_{S^\star}^{(0)}$, we can associate the chromatic coordinate $[x]_c = m_{S^\star}^{(0)}([k]_d)$ of each step $[k]_d$ with its original pitch classes $[x_0]_c$ in the reference mode $m_{S_0^\star}^{(0)}$ and compute the chromatic alteration $[a]_c$ necessary to obtain the new pitch classes

$$[x]_c = [x_0 + a]_c \quad (12)$$

a processes graphically depicted in Fig. 2.

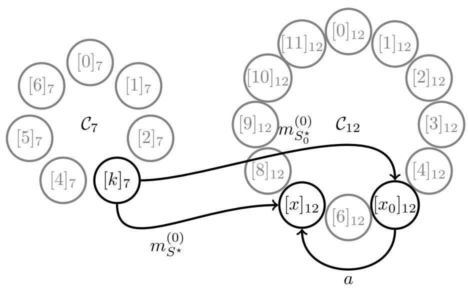

Fig. 2. Chromatic alteration. Two sharps ($[a]_{12} = [+2]_{12}$) alter a G ($[x_0]_{12} = [5]_{12}$).

Before changing our representation space for the diatonic spiral, modelled by the integers $\mathbb{Z}$, an unfolding operation of the chromatic circle is needed. We already defined a distance on $\mathcal{C}_c$; we still need to know the direction from one chromatic coordinate $[x]_c$ to another $[x']_c$.

$$sgn_{\mathcal{C}_c} : \mathcal{C}_c \times \mathcal{C}_c \rightarrow \{-1, +1\}$$
$$([x]_c, [x']_c) \longmapsto \begin{cases} +1 & [x' - x]_c \in \{[0]_c, \ldots, [\lfloor \frac{c-1}{2} \rfloor]_c\} \\ -1 & \text{otherwise} \end{cases} \quad (13)$$

Both functions combine into the unfolding operation $u_{\mathcal{C}_c}$.

$$u_{\mathcal{C}_c} : \mathcal{C}_c \longrightarrow \mathbb{Z}$$
$$[x]_c \longmapsto sgn_{\mathbb{Z}_c}([x]_c) \cdot d_{\mathbb{Z}_c}([x]_c) \quad (14)$$

It is now possible to compute the original diatonic coordinate ξ_0 and the diatonic alteration α on the diatonic spiral for every step $[k]_d \in \mathcal{C}_d$ of a mode.

$$\alpha := d \cdot u_{\mathcal{C}_c}([a]_c)$$
$$\xi_0 := u_{\mathcal{C}_c}([d]_c^{-1} \cdot [x_0]_c) \quad (15)$$

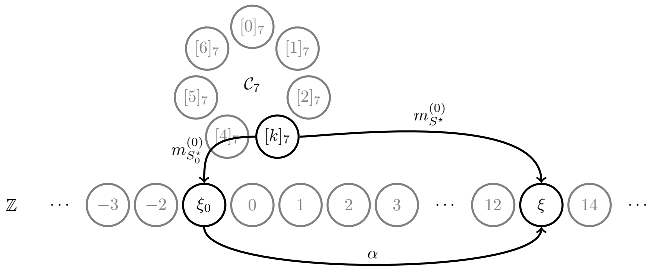

Fig. 3. The diatonic alteration corresponding to Fig. 2. The diatonic spiral is modelled by the discrete line of integers. $G\sharp\sharp$ is indexed by $13 = -1 + 7 \cdot 2$.

The same relation as in (12) holds for the diatonic space. The final diatonic coordinate is given by

$$\xi := \xi_0 + \alpha, \tag{16}$$

a process depicted in Fig. 3. Note that it is impossible for two different pairs (ξ_0, α) to correspond to a same ξ. On the diatonic spiral we have $G\sharp = -1+7 = +6 \neq -6 = +1-7 = A\flat$. This is due to the fact that $\mathbb{Z} = \{-3, \ldots, +3\} \oplus 7\mathbb{Z}$.

2.8 Order All Scales

The distribution of a centred mode's diatonic coordinates can be used to define a linear ordering on the set of centred scales, from the most compact (the diatonic) to the most widely spread (the chromatic). This order is preserved by inversion, and in case a scale is not symmetric, we need to distinguish between two members of an antisymmetric pair. Thus, each transposition class $[S]_{\mathbb{Z}_c}$ receives two indices: The first one designates the rank of the dihedral class $[S^\star]_{D_c}$ (equivalence through transposition and/or inversion) in the compactness order, whereas the second one tells if it is a palindrome (0), or which member of an antisymmetric pair (-1 and $+1$) it is.

We want to express a scale's compactness around the symmetry axis $0_{\mathbb{Z}}$. So we compare diatonic coordinates from the edge to the centre. The permutation $o : \mathcal{C}_d \to \mathcal{C}_d$ orders them by decreasing absolute value.

$$\left|\xi(o([0]_d))\right| \geq \left|\xi(o([1]_d))\right| \geq \ldots \left|\xi(o([d-1]_d))\right| \tag{17}$$

We define an ordering of scales by comparing these ordered vectors:

$$S > S' :\Leftrightarrow \exists k \in \mathbb{N} : \rho([k]_d) > \rho'([k]_d) \text{ and } \rho([\tilde{k}]_d) = \rho([\tilde{k}]_d), \forall \tilde{k} < k \tag{18}$$

where $\rho = \xi \circ o$. In case of an antisymmetric pair, the scale containing the greatest positive coordinate is given index $+1$, and the scale with the greatest negative coordinate index -1. Fig. 4 shows an example of his construction.

This procedure was first applied to the heptatonic scales, the result can be seen in Fig. 5.

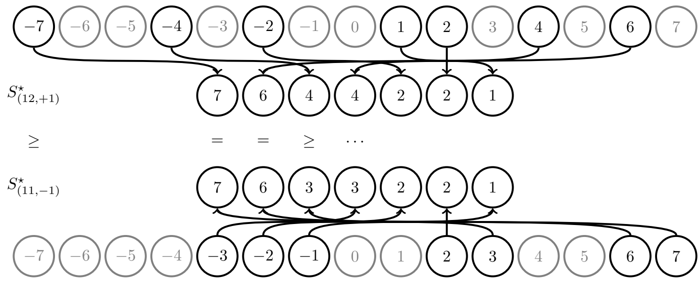

Fig. 4. Two successive classes, 11 and 12, of antisymmetric pairs −1 and +1 are being compared by testing for the spread of their diatonic distribution

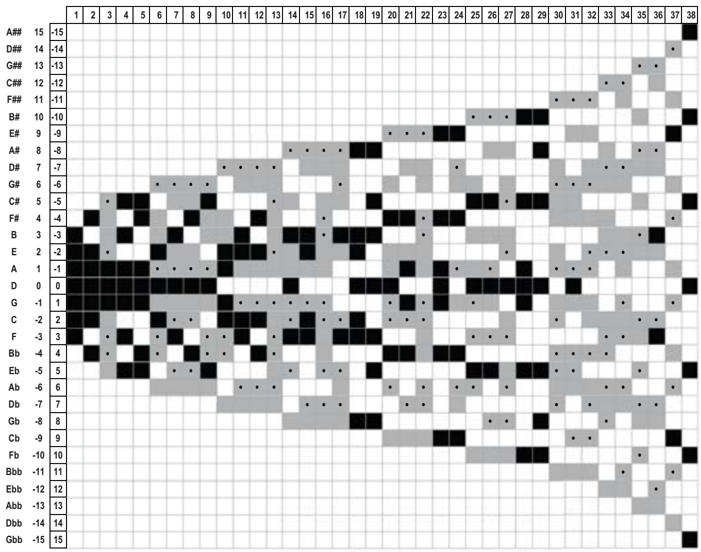

Fig. 5. ©2006 Pierre Audétat. His original diatonic bell for heptatonic scales, as proposed in [2]. Each cell represents a note and the mode corresponding to it. Each column contains a dihedral class, consisting either of a single symmetric scale or a pair of inverse scales. Alterations increase from the diatonic scale on the left to the maximally altered chromatic scale on the right. Each row represents a diatonic coordinate. The origin of the vertical axis is D, units are in steps of fifths. Black cells are symmetric notes, gray cells anti-symmetric notes, the bullet distinguishes the negative scale from the positive.

2.9 Modal Transposition

The online catalogue offers many musical examples of a same melody transformed into each of the 462 heptatonic modes. There are two possibilities to transform pitches in order to preserve their role from the diatonic to the target scale, depending on the presence or absence of notes foreign to the diatonic scale (black keys). In the first case, only the scale (along with its complement) can be mapped. Information about a possible mode gets lost. In the second case, it is possible to play with modes, and even to transpose a melody from one mode to the other within a same scale.

In the diatonic scale, we can identify every pitch class $[x_0]_c$ with a specific step $[k]_d$ of a given mode $m_{S_0^\star}^{([n_0]_c)}$ of the reference scale $S_0^\star$, and then map it to the same step of a given mode $m_{S^\star}^{([l]_c)}$ in the target scale $S^\star$.

$$\begin{array}{ccccccccc} \text{orig. pitch} & & \text{orig. pc} & & \text{mode's step} & & \text{new pc} & & \text{new pitch} \\ \mathbb{N} & \longrightarrow & \mathcal{C}_c & \xrightarrow{(m_{S_0^\star}^{([n_0]_c)})^{-1}} & \mathcal{C}_d & \xrightarrow{m_{S^\star}^{([l]_c)}} & \mathcal{C}_c & \longrightarrow & \mathbb{N} \\ x_0 & \longmapsto & [x_0]_c & \longmapsto & [k]_d & \longmapsto & [x]_c & \longmapsto & x \end{array}$$

Note that some freedom is left for converting the pitch classes back into integer pitches in the last step. The octave equivalence can be used to alter the melody as least as possible.

3 The DFT Analysis of Scales

The Discrete Fourier Transform (4) is a measure of periodicity. Traditionally, its modulus has been used in greater extend than its phase, because of its is greater ability to pinpoint some quantities invariant under transposition and inversion. Characteristics of scales or chords in music theory, energy in signal processing.

On the other hand, the phase may often be as complex and difficult to interpret as the original data. Making again an analogy with signal processing, phases are not perceptually relevant for stationary sounds, but are critical when it comes to transients. In our case, it depends on the particular transposition of a scale. This arbitrariness disappears when we use the centred representatives of each transposition class used in the diatonic bell. Hence, the phase provides information about the symmetric character of a scale.

In order to interpret the DFT coefficients, we first identify the chromatic circle with the unit circle in the complex plane, see Fig. 6.

$$\begin{aligned} \mathcal{C}_c &\longrightarrow \mathbb{C} \\ [k]_c &\longmapsto e^{i\frac{2\pi}{c}k} \end{aligned} \tag{19}$$

Computing the $[k]_c$-th DFT coefficient reduces to the vector addition of d unit vectors pointing to the (possibly multi-) set $[k]_c \cdot S^\star$, as shown by [1].

Is the index k coprime with c, the sum (4) will be computed on a shuffled regular c-polygon. Otherwise, it is computed on a polygon having fewer vertices, possibly populated with more than one pitch class. Such situations are described in [7]. They are called *balances*, because the DFT coefficients then point to a lack of equilibrium in the pitch class distribution.

If we display all pitch classes that accumulate in a given angle, we get stars with $\frac{c}{k}$ branches, as in Fig. 7. Pitch classes occupying symmetric positions at

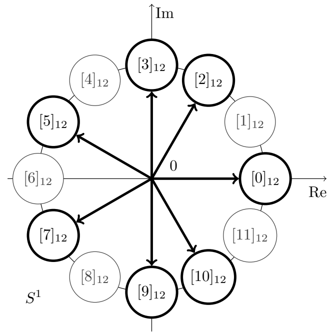

Fig. 6. The embedding of the diatonic scale $S^{\star}_{1,0}$ in the unit circle S^1 of the complex plane $\mathbb{C}$. A unit vector $e^{-i\frac{2\pi}{c}\cdot x}$ points to each chromatic coordinate $[x]_c$.

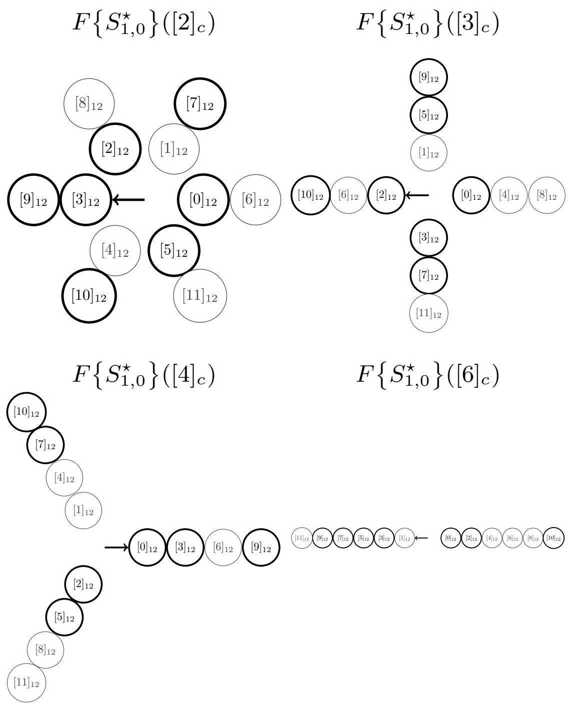

Fig. 7. The four DFT balances of the diatonic scale $S^{\star}_{1,0}$. The arrow represents the $[k]_{12}$-th coefficient, in this case a unit vector always pointing to a single unbalanced pitch class.

diameters, or regular triangles cancel each other out. The vector resulting from their sum points at the origin and yields a null DFT coefficient. Only pitch classes in "excess", that are not balanced by some other ones, contribute to the coefficient.

The choice of coprimes c and d has a direct consequence on the balance of the diatonic scale $S^{\star}_{1,0}$: there will always be at least one unbalanced pitch for a coefficient not coprime with c. The scale size d was also chosen to be odd, so that it is impossible to cancel all pitch classes out with opposite pairs.

$$F\{S\}([k]_c) \neq 0 \quad \forall [k]_c \in \mathcal{C}_c : k|c \tag{20}$$

But a triple cancellation is possible in the hexagonal $[2]_{12}$-th balance. This is achieved by the melodic minor, $S^{\star}_{2,0}$, as well as $S^{\star}_{12,\pm 1}$ and $S^{\star}_{29,0}$.

The coefficients of the DFT show a particularly nice behaviour for two operations common in music. Both preserve the dihedral class numbering of the diatonic bell.

1. Inversion. It is connected to the scales symmetry. The real part of the DFT coefficients is an even function, the imaginary part an odd one. In case of a palindromic (symmetric) scale, it hence must disappear. Corresponding phases of asymmetric pairs will have opposite signs.
2. Complementation. Moving from a pentatonic $S^{\star}$ to a heptatonic scale $S^{\star c}$ preserves DFT modules, and inverses phases of non null even coefficients This follows from the linearity of the DFT,

$$d \cdot \delta_{k,0} = F\{\mathcal{C}_c\}([k]_c) = F\{S\}([k]_c) + F\{S^c\}([k]_c) \quad \forall [k]_c \in \mathcal{C}_c \tag{21}$$

and the additional rotation of π radians necessary to centre the complement:

$$S^{\star c \star} = -S^{\star c}. \tag{22}$$

Also notice that the indicator function of a scale is a real function, so its DFT is symmetric: there are only $\lfloor \frac{c}{2} \rfloor + 1$ independent coefficients.

Having restated those general principles, we now turn to the interpretation of particular phases and modules. We keep coefficient $F\{S^{\star c}\}([0]_c)$ aside. It always points towards the positive real direction (null phase), and its length measures the (already given) scale's cardinality.

3.1 Phases

Coefficient $F\{S^{\star}\}([\frac{c}{2}]_c)$ tells if there is an excess of even or odd pitch classes. In the first case, the phase will be null, in the second case, the coefficient points to the negative region of the real axis, and the phases is $\pm\pi$.

For all other coefficients, the phase indicates the direction of the resulting unbalanced excess. Fig. 7, shows how class B ($[9]_{12}$) is unbalanced for the second coefficient: it is the famous tritonus B-F that populates twice one corner of the hexagon. The coefficient will thus point to -1 and the phase be equal to $\pm\pi$.

Since coefficients of palindromic scales are real, their phases will be either 0 (positive) or $\pm\pi$ (negative). For asymmetric scales, the phases will be opposites.

3.2 Modules

We will measure three different aspects of the geometric configuration of scales with help of the modules a DFT. They all have to deal with the idea of uniform distribution of pitch classes across the chromatic circle. The integers d and c are coprime, which prevents us from finding an absolutely regular d-polygon, where the three criteria would be confounded.

Symmetry. The first coefficient of the DFT becomes the sum of unit vectors pointing to each of the pitch classes.

$$\begin{aligned} \sigma : \mathcal{S}_c^d &\longrightarrow \mathbb{R} \\ S &\longmapsto \big|F\{S\}([1]_c)\big| \end{aligned} \tag{23}$$

A lower index indicates a higher degree of symmetry, the perfect case being achieved when the sum is null (all vectors cancel out). In $c = 12$, only the double harmonic scale ($S^{\star}_{5,0}$) shows a perfect balance (Fig. 8). The chromatic scale ($S^{\star}_{38}$) being compactly grouped on one side of the circle shows the worst results.

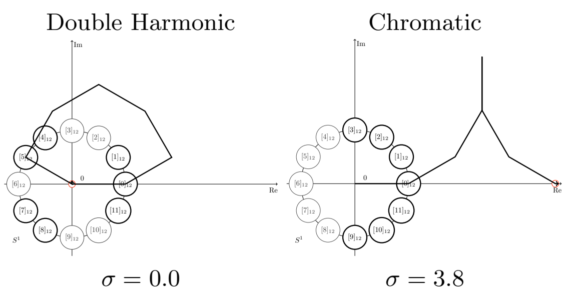

Fig. 8. Vector addition and symmetry index σ. The perfectly symmetrical double harmonic scale is built with an augmented triad $[0]_{12}, [4]_{12}, [8]_{12}$ that forms a regular triangle and two triton pairs $[1]_{12}, [7]_{12}$ and $[5]_{12}, [11]_{12}$. In the least symmetrical chromatic scale, only the triton $[3]_{12}, [9]_{12}$ is neutralised, leaving five unbalanced pitch classes.

3.3 Periodicity

It is well known that the DFT measures periodicity. The higher the modulus of the $[k]_c$-th coefficient, where $k|c$, the more $\frac{c}{k}$-periodic is the pitch class distribution. We define an index measuring the periodicity of a scale with:

$$\begin{aligned} \pi : \mathcal{S}_c^d &\longrightarrow \mathbb{R} \\ S &\longmapsto \max_{k|c} \big|F\{S\}([k]_c)\big|. \end{aligned} \tag{24}$$

A higher index shows higher periodicity. It is maximal for the unitonic scale ($S^{\star}_{4,0}$), see Fig. 9.

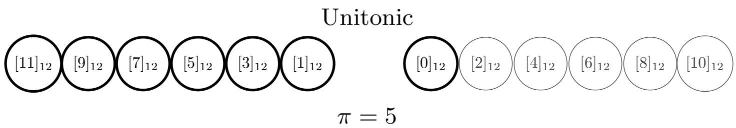

Fig. 9. Periodicity π and the $[6]_{12}$-th balance. The unitonic scale contains all odd pitch classes, that form one of the two whole tone scales, whose periodicity is $\frac{12}{6} = 2$. This achieves an excess of 5 odd pitch-classes, the maximum reachable in $c = 12$. The diatonic scale, whose maximal evenness ensures no excess greater than 1 on any coefficient, obtains the worst score, see Fig. 7.

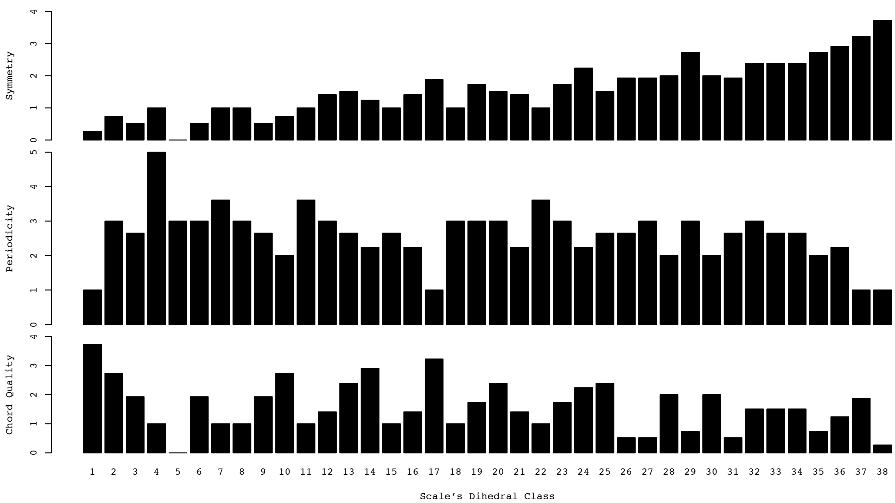

Fig. 10. Comparison of the three module based DFT indices σ, π and ε versus the diatonic bell's linear ordering of dihedral classes. Numbering of heptatonic scales goes from 1 on the left for the diatonic scale $S^{\star}_{1,0}$, towards the chromatic scale $S^{\star}_{38,0}$ on the right.

3.4 Chord Quality

As mentioned in Sec. 2.4, the $[d]_c$-th modulus called *chord quality* by Quinn [7] serves also for a new definition of maximall evenness.

$$\begin{aligned} \varepsilon : \mathcal{S}_c^d &\longrightarrow \mathbb{R} \\ S &\longmapsto \left|F\left\{S\right\}([d]_c)\right| \end{aligned} \tag{25}$$

It is related to the symmetry index σ through an affine permutation of the coefficients.

Despite some correlation appearing between the symmetry index σ and the diatonic bell's ordering, the three indices do not lead to a progressive classification from a diatonic to chromatic character, see Fig. 10.

4 Conclusion

The diatonic bell and the DFT differ in their structure. Whereas the underlying space of the former is infinite, the usual definition of a DFT requires finiteness. Nevertheless, both are constructed on an analogous principle: the *balance.* The idea of a physical balance lies behind the process of centring scales in the diatonic bell, and this image also helps for interpreting Fourier coefficients.

By lifting up scales from the chromatic circle to the spiral of fifths, the diatonic bell adds a tonal structure to the atonal combinatorics of musical set theory. Although the DFT is defined on the chromatic circle and, in this sense, is purely atonal, it shares some elements with the diatonic bell, namely the relevance of symmetry and the ability of pinpoint the diatonic flavour of some scales.

4.1 Symmetry

The role pitch class D plays as a symmetry axis in both the chromatic and diatonic worlds is clearly shown. This remarkable fact was already noticed by the french music theorist and composer Camille Durutte in his treatise of 1855 [5], where he described pitch classes with 31 integers, ranging from -15 to $+15$, centred around $D = 0$, and ordered by fifths. The diatonic bell's horizontal axis thus already appeared in the first historic attempt to formalise pitch classes algebraically.

The symmetry axis is also essential for the DFT, since it lies on the real axis of the complex plane. Inversion then corresponds to complex conjugation.

Using the centred and compact representatives of the diatonic bell has two advantages. The comparison between transpositional classes makes sense and interpreting phases of the DFT coefficients becomes more accessible: it eliminates a great amount of uninformative components that would have been induced by an arbitrary rotation. The most striking fact is that the coefficients of palindromic scales are purely real.

4.2 Measuring the Diatonic Character of a Scale

The diatonic bell displays scales as a deformation of the diatonic scale and arranges them according to the their degree of compactness on the spiral of fifths, ranging over all dihedral classes from the diatonic to the chromatic. Our initial intention was to use this linear ordering to define a measure of a scale's diatonic or chromatic character. The former being the maximally even scale, the latter the minimally even one, we expected to observe the same trend with DFT coefficients measuring regularity in the geometric configurations. As shown in Fig. 10, DFT-based indices did not confirm the bell's ordering. The chromatic or diatonic character of a scale does not reduce to a one-dimensional question, at least not this way.

On the other hand, almost all measures succeed in isolating the diatonic and chromatic scales as poles. What happens in between is less clear, but both approaches converge in distinguishing a group of particular scales, formed by the six first scales located on the left side of the diatonic bell. They correspond exactly to those used in the western tradition: diatonic ($S^{\star}_{1,0}$), melodic minor ($S^{\star}_{2,0}$), harmonic major ($S^{\star}_{3,-1}$) and minor ($S^{\star}_{3,+1}$), unitonic ($S^{\star}_{4,0}$) and double harmonic ($S^{\star}_{5,0}$).

One reason may be that they have to be the most compact, so that the tonic pitch class D, is surrounded with its dominant A and subdominant G, a feature essential for tonal music. Note that only three other scales show a similar behaviour: $S^{\star}_{22,\pm1}$ and $S^{\star}_{28,0}$. Optimums of the geometrical measurements defined with help of DFT modules in Sec. 3 systematically exhibit scales from this same *harmonic* block.

- Diatonic is the most even: $\varepsilon(S^{\star}_{1,0}) = 3.73$.
- Minor melodic is also one of the three balanced scales with regard to the triton periodicity: $F\{S^{\star}_{2,0}\}([2]_{12}) = 0$.
- Unitonic is the most periodic: $\pi(S^{\star}_{4,0}) = 5.00$.
- Double harmonic is the most symmetric: $\sigma(S^{\star}_{5,0}) = 0.00$.

In that case, the diatonic bell's requirement for compactness seems to agree with those of he DFT for regularity. This follows from the property of the diatonic scale to be generated by a succession of fifths, and that this sequence is not degraded too much for the first scales. Convergence between musical practice and mathematical interest as personified by the diatonic scale seems to extend also to the neighbour scales.

We are currently working on the implementation of these two approaches (the diatonic bell and the DFT) within OpenMusic visual programming language, as a package included in the MathTools environment. These new tools will allow the user to automatically generate diatonic bells and musical transpositions for the heptatonic and pentatonic scales. Divisions of the octave other than $c = 12$ will also be handled, as long as the requirements on the parameters are fulfilled (Sec. 2.1). One simply should take care about the exponential growth of the diatonic bell in microtonal context.

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