Musical Experiences with Block Designs

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Abstract. Since the pioneer works of composer Tom Johnson, many questions arise about block designs. The aim of this paper is to propose some new graphical representations suitable for composers and analysts, and to study the relationship between pcsets and small t-designs. After a short introduction on the combinatorial aspects of t-designs, we emphasize the musical perspectives open by these mathematical objects.

1 t-Designs: A Brief Survey

A *t-design* t-(v, k, λ) is a pair $D = (X, \mathcal{B})$ where X is a set of v elements (i.e. a v-set) and $\mathcal{B}$ a set of k-subsets of X called blocks such that every t-subset of X is contained in exactly λ blocks. D is simple if it has no repeated block.

The 2-design is called a *Balanced Incomplete Block Design* (BIBD) or simply a *Block Design* and denoted (v, k, λ). If the index $\lambda = 1$, t-designs are called *Steiner Systems*. For $k = 3$, t-$(v, 3, 1)$ are *Triple Systems* (TS), 2-$(v, 3, 1)$ are *Steiner Triple Systems* (STS) and 2-$(v, 4, 1)$ are *Steiner Quadruple System* (SQS). A *symmetric design* is a BIBD (v, k, λ) such that the number of blocks is equal to the cardinality of the set ($b = v$). There are no known examples of non trivial t-designs with $t \geq 6$ and $\lambda = 1$. But it is known that 5-$(24, 8, 1)$ is a Steiner System. Two t-designs $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ are said to be *isomorphic* if there is a bijection $\varphi : X_1 \to X_2$ such that $\varphi(\mathcal{B}_1) = \mathcal{B}_2$. One of the simplest block design is *Fano plane*. It is a 2-$(7, 3, 1)$ design whose blocks are written (vertically) by this matrix:

$$\begin{pmatrix} 0\,0\,0\,1\,1\,2\,3 \\ 1\,2\,4\,2\,5\,3\,4 \\ 3\,6\,5\,4\,6\,5\,6 \end{pmatrix}$$

If the set X of a $(X, \mathcal{B})$ design is identified with musical objects such as pitch classes, modes, rhythms, etc. the combinatorial structure of blocks can be used to create a path through this musical material, linking blocks by their common objects. Composer Tom Johnson has explored these properties in *Block Design for piano* built on the 4-(12, 6,10) design defined by 30 base blocks and one automorphism of the permutation group over 12 elements, namely, in cyclic notation, $\sigma = (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10)(11)$. In *Kirkman's ladies,* he uses a *large* (15, 3, 1) design with 13×35 blocks. In *Vermont Rhythms*, he uses 42×11

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rhythms based on the (11,6,3) design, a system worked out by Jeffery Dinitz and his student Susan Janiszewski. Another example is the mapping of Messiaen's modes on the set X: Mode 2 with (6,3,2) 10 blocks, Mode 3 with (9,3,1) 12 blocks, Mode 4 with (8,4,3) 14 blocks, Mode 5 with (8,4,6) 28 blocks Mode 6 with (8,3,6) 56 blocks, Mode 7 with (10,4,2) 15 blocks.

As we have previously remarked, a t-design has only four parameters t-(v, k, λ). From these quantities, we can easily derive some combinatorial properties. For example, the number of blocks that contain any i-set is given by

$$b_i = \lambda \binom{v-i}{t-i} / \binom{k-i}{t-i}, \quad i = 0, 1, ..., t \tag{1}$$

where $\binom{a}{b} = a!/b!(a-b)!$ indicates the binomial coefficient. In particular, the number of blocks of a t-design is

$$b = \lambda \frac{v!}{(v-t)!} \frac{(k-t)!}{k!} \tag{2}$$

And by setting

$$r = \lambda \frac{(v-1)!}{(v-t)!} \frac{(k-t)!}{(k-1)!} \tag{3}$$

we get the following relation

$$bk = vr \tag{4}$$

As we have seen, two t-designs are isomorphic if there is a bijection between there blocks, and this reduces the research of representative. From a set theoretical perspective, the knowledge of a t-design $D = (X, \mathcal{B})$ leads to the knowledge of its complement $D^c = (X, X \backslash \mathcal{B})$ where $X \backslash \mathcal{B}$ is the set of blocks

$$X \backslash \mathcal{B} = \{B^c, \quad B \in \mathcal{B}\}$$

The complement of $t - (v, k, \lambda)$ design is the $t - (v, v-k, \mu)$ design with

$$\mu = \lambda \binom{v-t}{k} / \binom{v-t}{k-t} = \lambda \frac{(v-k)!}{(v-t-k)!} \frac{(k-t)!}{k!} \tag{5}$$

Remark that D and D^c have the same number of blocks, and for $t = 2$, the block design D with b blocks

$$b = \frac{v(v-1)\lambda}{k(k-1)}, \quad r = \lambda \frac{(v-1)}{(k-1)}, \quad bk = vr \tag{6}$$

has a complement D^c with b blocks and $(v, v-k, b-2r+\lambda)$. For example, the complement of the Fano Plane $(7, 3, 1)$ is $(7, 4, 2)$ with blocks $\{0, 1, 2\}^c = \{3, 4, 5, 6\}$, etc.

An *automorphism* of a design D is a permutation of the point set that preserves the blocks. The group of all automorphims of D will be indicated by $Aut(D)$.

For example, the $D = (7, 3, 1)$ design has an automorphism group $Aut(D)$ equal to the goup $L_3(2)$ of 168 elements with presentation

$$L_3(2) = \langle a, b \mid a^2 = b^3 = (ab)^7 = [a, b]^4 = 1 \rangle \tag{7}$$

where a and b are the permutations (in cyclic notation) of seven elements

$$a = (0\ 3\ 4\ 1\ 2\ 5\ 6), \quad b = (1\ 2\ 0\ 3\ 5\ 6\ 4) \tag{8}$$

We end this section by a characterisation of Steiner Systems. The proof of these theorems can be found in [2] and [7].

Theorem 1 (Wilson). *Let p^m be a prime power. If* 3-$(v + 1, p^m + 1, 1)$ *and* 3-$(w + 1, p^m + 1, 1)$ *are Steiner Systems then* 3-$(vw + 1, p^m + 1, 1)$ *is a Steiner System.*

Theorem 2 (Kirkman, 1847). *A Steiner Triple System of order v exists if and only if $v \equiv 1, 3$ (mod 6), (i.e. $v = 6n + 1$ or $v = 6n + 3$, i.e. for 7, 9, 13, 15, etc.)*

Examples of Steiner Systems: Let $q = p^m$ be a prime power

- $2 - (q^n, q, 1)$, $n \geq 2$
- $3 - (q^n + 1, q + 1, 1)$, $n \geq 2$
- $2 - (q^n + ... + q + 1, q + 1, 1)$, $n \geq 2$
- $2 - (q^3, q + 1, 1)$,
- $2 - (2^{r+s} + 2^r - 2^s, 2^r, 1)$, $2 \leq r < s$ (Denniston systems)

2 Drawing t-Designs

Until now t-designs have rarely been used for musical purposes. Moreover, there exists no canonical way to draw a t-design. Usually, musical transformations are not considered in the mathematical litterature of t-designs. We will restrict to the most common musical transformations, namely

1. Transpositions:
$$T_n(x) = x + n \pmod{v} \tag{9}$$
2. Inversions
$$I_n(x) = -x + n \pmod{v} \tag{10}$$
3. Affine transformations
$$M_{m,n}(x) = mx + n \pmod{v} \tag{11}$$

In *Kirkman's Ladies*, a strong unity of the score is obtained by considering *parallel classes,* i.e. sets of blocks that partition the point set. A design (v, k, λ) is *resolvable* if its blocks can be partitioned into parallel classes. For example, the (9,3,1) design is resolvable, as shown on table 1.

Table 1. The (9,3,1) design

0,1,2	0,3,6	0,4,8	0,5,7
3,4,5	1,4,7	1,5,6	1,3,8
6,7,8	2,5,8	2,3,7	2,4,6

The Kirkman problem (see also [8]) has been stated in 1850 by Thomas P. Kirkman: *Fifteen young ladies in a school walk out abreast for seven days in succession : it is required to arrange them daily, so that no two walk twice abreast.* Since that time, we define a *Kirkman Triple System* (KTS) as a resolvable Steiner Triple System, also called the *social golfer problem* in computer science. The following theorem limits the cardinality of the point set.

Theorem 3. *A* Kirkman Triple System *of order v exists if and only if $v \equiv 3$ (mod 6)*

For $v = 15$, it has been shown that there are eighty Steiner Triple Systems (15,3,1). One solution is given in table 2:

Table 2. The Kirkman Triple system $(15, 3, 1)$

Monday	0,1,2	3,9,11	4,7,13	5,8,14	6,10,12
Tuesday	0,3,4	1,8,10	2,10,14	5,7,11	6,9,13
Wednesday	0,5,6	1,7,9	2,11,13	3,12,14	4,8,10
Thursday	1,3,5	0,10,13	2,7,12	4,9,14	6,8,11
Friday	1,4,6	0,11,14	2,8,9	3,7,10	5,12,13
Saturday	2,3,6	0,7,8	1,13,14	4,11,12	5,9,10
Sunday	2,4,5	0,9,12	1,10,11	3,8,13	6,7,14

The musical question is: how to draw this solution showing each parallel class and considering musical transformations between them? Reinhard Laue [9] studied some visualizations of Steiner Systems which make resolvability obvious, and Tom Johnson [6] gave some drawings considering sub-networks in t-designs. For a simplier design such as (6,3,2), which is the best representation? Is it a graph where the set of vertices is the point set, or a graph where vertices are blocks ? (fig. 1).

In figure 1, the opposite borders are supposed to be glued together in the sense of the arrows, in such a way that if you leave the bottom through the line [3, 4] of the triangle $\{2, 3, 4\}$, you enter by the top through the same line in the triangle $\{1, 3, 4\}$. Each triangle has three neighbours. A compositional problem would be to find *Hamiltonian paths* (i.e. paths that visit each vertex exactly once) or *Hamiltonian circuits* (i.e. cycles that visit each vertex exactly once and return to the starting vertex), when vertices are blocks of a t-design. In figure 1, it corresponds to the second graph (on the right) or to the dual graph of the first graph (on the left).

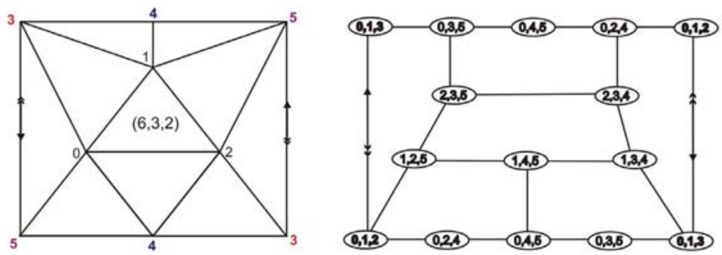

Fig. 1. Two dual representations of the (6,3,2) design

Another point of view is to consider affine transformations between blocks. In the previous example (6,3,2), it leads to a graph with two connected components (fig. 2).

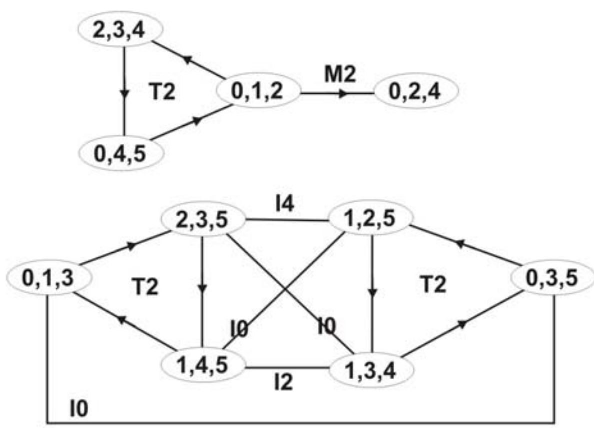

Fig. 2. The affine transformations in the (6,3,2) design

The graph shows all affine transformations modulo 6. The translation T_2 (adding 2 modulo 6) has an inverse T_4 and each inverse transformation I_n acts both ways. This kind of graph is suitable for neo-riemannian analysis. In the next section, we will see another type of graph, where the musical transformations are permutations on base blocks.

3 Cyclic Representations

In some cases, blocks can be constructed from generators under the action of a group. This is the case when $q = p^\alpha$ is a prime power, and the action is the translation T_1 of the cyclic group. The Steiner Triple Systems 2-($q^2 + q +$

$1, q+1, 1)$ are examples of projective geometries and denoted by $PG(2,q)$. More generally, *symmetric designs* are projective geometries with parameters $PG(m-1,q)$ corresponding to block designs

$$2-\left(\frac{q^m-1}{q-1}, \frac{q^{m-1}-1}{q-1}, \frac{q^{m-1}-1}{q-1}\right) \tag{12}$$

The following table (table 3) shows the first designs. Observe that there is no Steiner Triple System for $PG(2,6)$, since 6 is not a prime power.

Table 3. Generators of $PG(2,p)$

$(7,3,1)$	$PG(2,2)$	$(0,1,3)$
$(13,4,1)$	$PG(2,3)$	$(0,1,3,9)$
$(21,5,1)$	$PG(2,4)$	$(0,1,4,14,16)$
$(31,6,1)$	$PG(2,5)$	$(0,1,3,8,12,18)$
$(57,8,1)$	$PG(2,7)$	$(0,1,3,13,32,36,43,52)$
$(73,9,1)$	$PG(2,8)$	$(0,1,3,7,15,31,36,54,63)$
$(91,10,1)$	$PG(2,9)$	$(0,1,3,9,27,49,56,61,77,81)$

The parameters of designs are given in the first column, the second column gives the prime power p^α written $PG(2,p^\alpha)$ and the last column gives a generator. The action of the translation T_1 in $\mathbb{Z}_{p^2+p+1}$ yields to the set of blocks. Namely for (7,3,1), the blocks are $B=\{0,1,3\}$, $T_1(B)=\{1,2,4\}$, $T_1^2(B)$, etc. Can this construction be generalized for $(7,3,n)$ design with $n>1$? Unfortunately not. Look at the first values of n. For $n=1$, the design $(7,3,1)$ is generated by $B=\{0,1,3\}$ and the translation $T_1(x)=x+1 \bmod 7$, which is also the permutation in cyclic notation $\sigma=(0\ 1\ 2\ 3\ 4\ 5\ 6)$. This design is represented by a heptagone with outer triangles, corresponding to the blocks. For $n=2$, the design (7,3,2) is not generated by one block and a translation. However, it is generated by two blocks and two actions: the block $B_1=\{0,1,2\}$ and the permutation $\sigma=(0\ 1\ 5\ 3\ 4\ 2\ 6)$ and the block $B_2=\{0,1,3\}$ and the permutation $\sigma^2=(0\ 5\ 4\ 6\ 1\ 3\ 2)$ which is the square of the previous permutation. The drawing of (7,3,2) is a triangulation of two concentric heptagones, the vertices of each heptagone are labelled by the cyclic notation of the previous permutations. For $n=3$, the design (7,3,3) is generated by the action of $\sigma=(0\ 1\ 3\ 5\ 2\ 6\ 4)$ on the blocks $B_1=\{0,1,3\}$ and $B_2=\{0,1,2\}$, and the action of $\sigma^4=(0\ 2\ 1\ 6\ 3\ 4\ 5)$ on the block $B_3=\{0,2,3\}$. The drawing (fig. 3) shows the design $(7,3,1)$.

Another question is to determine the generators of a t-design. We sumarize now some results: Netto Theorem and Singer Difference Sets.

Theorem 4 (Netto, 1893). *Let p prime, $n\geq 1$, $p^n\equiv 1$ (mod 6). Let $\mathbb{F}_{p^n}$ be a finite field on X of size $p^n=6t+1$ with 0 as its zero element and α a primitive root of unity. The sets*

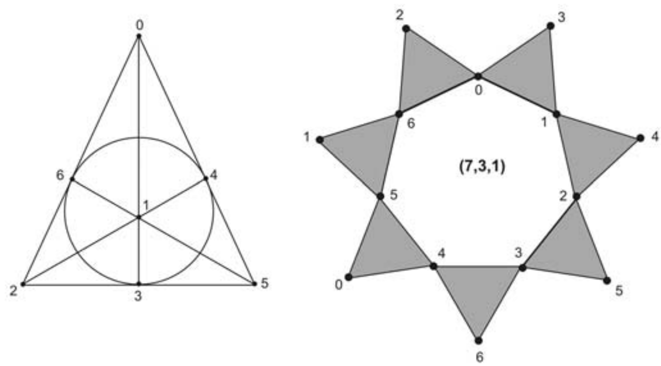

Fig. 3. The (7,3,1) design

$$B_i = \{\alpha^i, \alpha^{i+2t}, \alpha^{i+4t}\} \qquad \bmod\ p^n \tag{13}$$

for $i = 1, 2, ..., t-1$ *are generators* $(T_j(B) = j + B \ mod\ p^n)$ *of the set blocks of an* $STS(p^n)$ *on* X*.*

The proof of this theorem is given in [3]. As an example of how the theorem works, consider the (7,3,1) design. As $p = 7$, $n = 1$, $t = 1$, and $\alpha = 3$ is a generator of $\mathbb{F}_7$, then the set $\{1, \alpha^2, \alpha^4\}$ mod $7 = \{1, 2, 4\} \simeq \{0, 1, 3\}$ up to transposition, is a cyclic generator of $\mathcal{B}$.

Singer Difference Sets are introduced in [4]. Let p be a prime, and m a non-negative integer. Let $f(x)$ be a primitive polynomial of degree m in $\mathbb{F}_p$.

$$f(x) = x^m + a_1 x^{m-1} + ... + a_{m-1}x + a_m \tag{14}$$

Consider the recurrence relation

$$\begin{aligned} u_0 &= 1,\ u_1 = \cdots = u_{m-1} = 0 \\ u_n &= -(a_1 u_{n-1} + \cdots + a_{m-1}u_1 + a_m) \end{aligned} \tag{15}$$

Theorem 5. *The* Singer Difference Set

$$B = \left\{ i,\ 0 \leq i < \frac{p^m - 1}{p - 1},\ u_i = 0 \right\} \tag{16}$$

is a generator of the set of blocks.

Example. For $p = 7$, $m = 3$, $f(x) = x^3 + 3x + 2$ is a primitive polynomial of $\mathbb{F}_7$. The sequence defined by the relations $u_0 = 1,\ u_1 = 0,\ u_2 = 0$,

$$u_n = -3u_{n-2} - 2u_{n-3} = 4u_{n-2} + 5u_{n-3} \quad \bmod\ 7 \tag{17}$$

leads to the first values $u_3 = 5,\ u_4 = 0,\ u_5 = 6,\ u_6 = 4$. The index i such that $u_i = 0$ determines the base block of the cycle $B = \{1, 2, 4\}$.

4 Pcsets and Designs

We would like now to study the relationship between Forte' pcsets and t-designs. Precisely, we would like to investigate the question: is there a t-design t-(v, k, λ) with $v \leq 12$ such that k-blocks include all k-pcsets in Forte classification ? Such a design is called a *Forte design*. If the point set is identified with pitch classes ($v \leq 12$), each block can be considered a chord. If all chords are described by the design, the design is a Forte design. A computer program analyzing the 2-designs given by the *Encyclopedia of t-designs* shows that the 2-designs do not lead to a Forte design. In table 4, the first column gives the parameters of the design, the second column the number of blocks b in the design, the third column gives the complement of the design. In the fourth column is the Forte name of at least a missing k-chord. Stars indicate autocomplementation, and n a positive integer. To have a complete pcset of k-chords, at least two sets of blocks are required. For example, the $(9, 3, 1)$-design has under the action of $\sigma_1 = (2\ 6)(3\ 8)(4\ 7)(0)(1)(5)$

$$\mathcal{B}_1 = \begin{pmatrix} 0\,0\,0\,0\,1\,1\,1\,4\,2\,3\,5\,2 \\ 1\,2\,3\,4\,3\,4\,2\,6\,6\,5\,7\,3 \\ 6\,8\,7\,5\,8\,7\,5\,8\,7\,6\,8\,4 \end{pmatrix} \tag{18}$$

all Forte's trichords except 3-7 and 3-12. And under the action of $\sigma_2 = (2\ 7\ 8\ 6\ 5\ 4\ 3)(0)(1)$

$$\mathcal{B}_2 = \begin{pmatrix} 0\,0\,0\,0\,1\,1\,1\,2\,3\,4\,2\,5 \\ 1\,2\,3\,4\,2\,3\,4\,7\,5\,6\,3\,6 \\ 7\,5\,6\,8\,6\,8\,5\,8\,7\,7\,4\,8 \end{pmatrix} \tag{19}$$

it contains all Forte's trichords except 3-8 and 3-11. That way, using two sets of blocks of the same design, a composer can use the whole set of trichords.

To conclude this section, we would like to mention the link of t-design with Mathieu Groups. First, as it has been underlined in [5] Olivier Messiaen's *Ile de feu 2* use two permutations in cyclic notation

Table 4. Missing at least a Forte chord

(v, k, λ)	b	$(v, k, \lambda)^c$	*Missing*
$(6, 3, 2n)$	$10n$	$(6, 3, 2n)^*$	3-5
$(7, 3, n)$	$7n$	$(7, 4, 2n)$	3-1
$(8, 4, 3n)$	$14n$	$(8, 4, 3n)$	4-5
$(9, 3, n)$	$12n$	$(9, 6, 5n)$	3-2
$(9, 4, 3n)$	$18n$	$(9, 5, 5n)$	4-3
$(10, 4, 2n)$	$15n$	$(10, 6, 5n)$	4-2
$(10, 5, 4n)$	$18n$	$(10, 5, 4n)^*$	5-1
$(11, 5, 2n)$	$11n$	$(11, 6, 3n)$	5-2
$(12, 3, 2n)$	$44n$	$(12, 9, 24n)$	3-2
$(12, 4, 3n)$	$33n$	$(12, 8, 14n)$	4-1
$(12, 6, 5n)$	$22n$	$(12, 6, 5n)^*$	6-1

$$a = (1\ 7\ 10\ 2\ 6\ 4\ 5\ 9\ 11\ 12)(3\ 8) \tag{20}$$
$$b = (1\ 6\ 9\ 2\ 7\ 3\ 5\ 4\ 8\ 10\ 11)(1\ 2)$$

which generate Mathieu's group M_{12} of order 95040. In the same way, *Les yeux dans les roues* (O. Messiaen, *Livre d'orgue VI*) is built on six permutations (a permutation and five *actions*): $\sigma_0 = (1\ 11\ 6\ 2\ 9\ 4\ 8\ 10\ 3\ 5)$ and for $j = 1, ..., 5$, $\sigma_j = A_j\sigma_0$, the actions are defined by: *Extremes au centre*: $A_1 = (2\ 12\ 7\ 4\ 11\ 6\ 10\ 8\ 9\ 5\ 3)$, *Centre aux extrêmes*: $A_2 = (1\ 6\ 9\ 2\ 7\ 3\ 5\ 4\ 8\ 10\ 11)$; *Rétrograde*: $A_3 = (1\ 12)(2\ 11)(3\ 10)(4\ 9)(5\ 8)(6\ 7)$, *Extrêmes au centre, rétrograde:* $A_4 = A_1A_3$ and *Centre aux extrêmes, rétrograde*: $A_5 = A_2A_3$. If we set $a = A_2^{-1}A_1$ and $b = A_2^3A_1A_2^2A_1$ these permutation generate the Mathieu group M_{12} of presentation

$$M_{12} = \left\langle a, b \mid a^2 = b^3 = (ab)^{11} = [a, b]^6 = (ababab^{-1})^6 = 1 \right\rangle \tag{21}$$

Table 5 gives the links between Mathieu Groups and t-designs.

Table 5. Mathieu groups and t-designs

Groups	*Order*	*t-design*	*# blocks*
M_{11}	7 920	4-(11,5,1)	66
M_{12}	95 040	5-(12,6,1)	132
M_{22}	443 520	3-(22,6,1)	77
M_{23}	10 200 960	4-(23,7,1)	253
M_{24}	244 823 040	5-(24,8,1)	759

The two first Mathieu groups are built with eleven or twelve points. Neither M_{11}, nor M_{12} are Forte designs. In M_{11} 11 pcsets are missing (5-1, 5-3, 5-5, etc.), and in M_{12} 12 pcsets are missing (6-1, 6-4, 6-7, etc.). In M_{12} if we take only three notes in each block, we get neither 3-12, nor 3-1.

5 A Compositional Application

To show a specific compositional application for all this, and also to summarize the combinations and graphs that come together in block designs, we offer a brief analysis of the third movement of Johnson's *Twelve for Piano* (2008) (the score is reproduced in Annexe). This one-page piece uses the precise four-note chords produced by one of the over 17 million possible solutions of the (12,4,3) design, where 12 elements (notes) are partitioned into 33 subsets (chords) of four elements (notes), such that each pair of notes appears in exactly three of the chords.

To write this music, the composer needed to map the system, so as to see how the 33 chords related to one another, and to do this he drew a graph

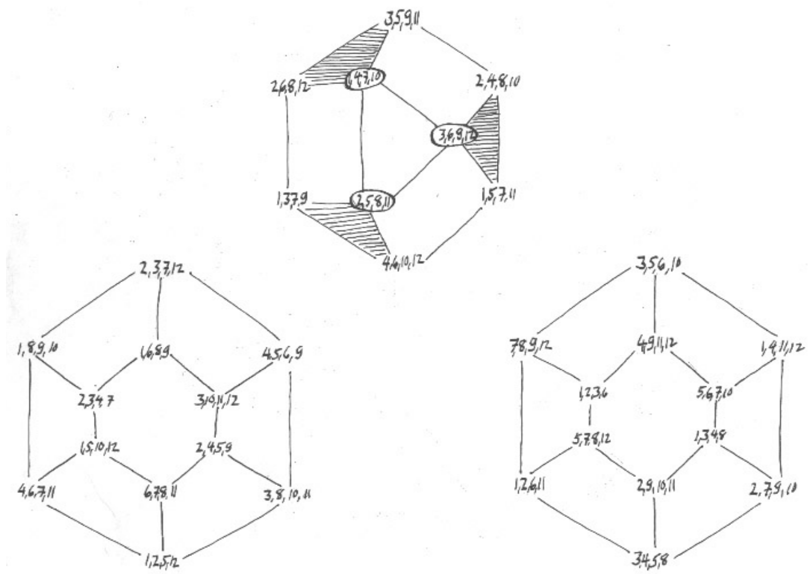

Fig. 4. Graph for *Twelve for piano*

by connecting chords when they had no notes in common. The graph would be different for each of the 17 million solutions, but In this case it takes the shape of the three hexagonal formations shown here. The three shaded triangles represent three parallel classes, three cases where chords with no notes in common come together as a collection of all 12 notes. These nine chords form the central section of the piece, beginning with (1,3,7,9) (4,6,10,12) (2,5,8,11). The remaining 24 chords, those in the other two hexagons, form the opening and closing sections.

The first four phrases of the piece, the first 12 chords, come from the hexagon at the lower right, beginning with the inner ring: (5,7,8,12) (1,2,3,6) (4,9,11,12) and (5,6,7,10) (1,3,4,8) (2,9,10,11) followed by the outer ring: (3,4,5,8) (1,2,6,11) (7,8,9,12) and (3,5,6,10) (1,4,11,12) (2,7,9,10). The final four phrases follow the hexagon at the lower left in this same manner. We have not shown the numbers on the accompanying score, but this is rather easy to decipher, since 1 is the lowest note of the scale and 12 is the highest.

Simply following the connections in this way produces a number of remarkable symmetries, symmetries that are difficult to imagine in a rigorous 12-tone music, and surely impossible in any non-rigorous music.

- Consider first of all the cadences. The first and second phrases both end on D-F-sharp, and the third and fourth phrases both end on B-flat-D. The final four phrases in the piece rhyme in this same way.
- The notes marked “a” appear twice in the same phrase. These same notes are omitted either in the phrase just before or in the phrase just after. Each of the 12 notes appears exactly 11 times in the piece.
- The intervals marked “b” occur at the same place in two subsequent phrases.

• The "c" interval of the first phrase appears again in the third phrase, and there is a similar pair of "c" intervals in the last section of the piece.

• In the middle section, containing three complete sets of 12 notes, one finds three "d" intervals, three "e" intervals and three "f" intervals, though it is difficult to explain why they fall as they do. But then, it is difficult to explain all these other symmetries as well. The music produced by this block design simply does not behave like music we already know.

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