Towards Pedagogability of Mathematical Music Theory: Algebraic Models and Tiling Problems in computer-aided composition

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Abstract

The paper aims at clarifying the pedagogical relevance of an algebraic-oriented perspective in the foundation of a structural and formalized approach in contemporary computational musicology. After briefly discussing the historical emergence of the concept of algebraic structure in systematic musicology, we present some pedagogical aspects of our *MathTools* environment within *OpenMusic* graphical programming language. This environment makes use of some standard elementary algebraic structures and it enables the music theorist to visualize musical properties in a geometric way by also expressing their underlying combinatorial character. This could have a strong implication in the way at teaching mathematical music theory as we will suggest by discussing some tiling problems in computer-aided composition.

1. Introduction

In an interesting article on the notion of music theory's new pedagogability and the state of music theory as a research field [12], Richard Cohn suggests the opportunity to transgress the boundaries between teaching and research by showing that this opportunity is linked to the ways that music theorists have been recently crossing boundaries, particularly between fields of knowledge. One example of this necessity of transgressing the boundaries between disciplines is given by the difficulty to establish a strict separation between music theory, analysis and composition, in particular from the perspective of an algebraic-oriented approach towards a structural formalized computational musicology [6].

Historically, the emergence of algebraic methods in music has been a long process that has occurred, surprisingly, independently from stylistic considerations and geographical contexts. According to Iannis Xenakis, the structure of mathematical *group* enables an "universal formulation for what concerns pitch perception" [25]. In other words, any division of the octave in a given number n of equal parts can be represented as a group, the cyclic group of integers modulo n, with respect to the addition modulo n.

In a large study dedicated to the emergence and development of algebraic models in 20th century music and musicology [3], one of the authors of this paper proposed to try to transgress the boundaries between the American serial and set-theoretical tradition (Krenek, Babbitt, Perle, Forte, Lewin,…) and an European structural tradition (Xenakis, Vieru, Riotte, Mesnage, Mazzola,…). In fact, in the *set-theoretical* (Forte) and *transformational* (Lewin) approaches Milton Babbitt's initial formalization of the twelve-tone system has some very deep intersections with the music-theoretical constructions proposed around the 60s by some European theorists/composers, in particular Anatol Vieru and Iannis Xenakis. The system was conceived as a "collection of elements, relations between them and operations upon them"[8].

In this paper, we focus on a relatively elementary theoretical approach which is essentially based on the structure of mathematical *group*. In particular we will concentrate on three groups that are the basis of our structural approach to the enumeration, classification and computer-aided implementation of musical tiling structures: cyclic, dihedral and affine groups. Nevertheless, more abstract approaches that use the ring structure of polynomials [2] and category/topos theory [20] clearly show that the tiling constructions that we present in the following sections might be described by means of a more powerful theoretical and computational framework.

2. Some preliminary definitions

This section introduces the three basic families of groups that constitute the theoretical framework of our computational model of tiling musical structures. Let $(G, \cdot)$ be a group with inner binary operation "$\cdot$".

2.1. Definition and music-theoretical interpretation of cyclic groups

A cyclic group of n elements (i.e., of order n) is a group $(G, \cdot)$ in which there exists (at least) one element g such that each element of G is equal to $g \cdot g \cdot \ldots \cdot g$, where the group law is applied a finite number of times. In other words, G is generated by g. A cyclic group of order n can be represented by the set $\{0,1,\ldots,n\text{-}1\}$ of integers (modulo n), and it is usually indicated as $\mathbf{Z/nZ}$. In general a cyclic group of order n is generated by all integers d which are relatively primes to n and it can be represented by a circle where the integers 0, …, n-1 are distributed uniformly. In the usual twelve-tone clock, one may go from an integer to another simply by rotating the circle around his centre by an angle equal to a multiple of $\pi/6$. Musically speaking, rotations are equivalent to *transpositions*. Let T_k be the transposition of k minimal divisions of the octave (i.e., halftones in the case n=12). For any integer k relatively prime to n the transposition T_k generates the whole cyclic group.[1] As originally shown by Halsey and Hewitt in one of the first application of combinatorial algebra to music theory [16], musical transpositions define mathematical *actions* so that classifying transposition classes of chords is finally equivalent to study orbits under the action of the group $\mathbf{Z/nZ}$ on itself. Polya enumeration theory provides therefore the underlying framework that enables to approach tiling problems in computer-aided composition in all generality. We will not enter in the mathematical aspects of this theory that has also been generalized to the computations of orbits of more complex musical structures (melodic patterns, twelve-tone rows, mosaics, …). See [14] for a detailed discussion of this approach.

2.2. Definition and music-theoretical interpretation of dihedral groups

A dihedral group $(\mathbf{D_n}, \cdot)$ of order $2n$ is a group generated by two elements a, b with relations:

1. $a^n = b \cdot b = e$ where a^n means $a \cdot a \cdot \ldots \cdot a$ (n times)
2. $a^n \cdot b = b \cdot a^{n-1}$

In other words, the dihedral group $\mathbf{D_n}$ consists of all $2n$ products $a^i \cdot b^j$ for i=1,…,n, and j=1,2. Geometrically, the dihedral group corresponds to the group of symmetries in the plane of a regular

[1] This is a well-known music-theoretical statement that we find already in Babbitt's dissertation [8].

polygon of n sides. These symmetries are basically of two types: rotations and reflections (with respect to an axis). Musically speaking, reflections are inversions with respect to a given note that is taken as a fixed pole. As in the case of the cyclic group, the dihedral group can be considered as generated by transpositions and inversions. Orbits under the action of the dihedral group on the equal-tempered system are also called *pitch-class sets* in the American tradition [13].

2.3. Definition and music-theoretical interpretation of affine groups

The affine group $\mathbf{Aff_n}$ consists of the collection of affine transformations, i.e., the collection of function f from $\mathbf{Z/nZ}$ into itself which transforms a pitch-integer x into $ax+b$ (modulo n) where a is an integer relatively prime with n and b belongs to $\mathbf{Z/nZ}$. In the special case of n=12, the multiplicative factor a belongs to the set U={1,5,7,11} of units of $\mathbf{Z/12Z}$. Note that an affine transformation reduces to a simple transposition when a=1. On the other side, inversions are affine transformations with a=11. All musical structures which are equivalent up to transposition will also be in the same orbit under the action of the affine group on $\mathbf{Z/nZ}$. This means that affine orbits are a natural generalization of pitch-class sets, a music-theoretical statement which is shared by the American [21] and European traditions [19].

3. Tessellations and Tiling in music

Tiling problems in music theory, analysis and composition have a relatively old history in music theory. From a pitch-perspective, the study of some tiling problems in music is historically related to Hugo Riemann original representation of the tone space by means of a translation of deformed squares generated by oblique major and minor thirds axis (see Figure 1).

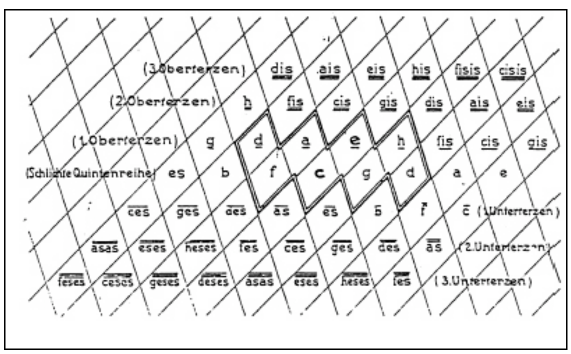

Figure 1: *Hugo Riemann's* Tonnetz [22].

Many attempts have been made in order to provide some alternative models of the tiling process of the tone space, particularly by the so-called Neo-Riemannian American and European tradition.[2]

Surprisingly, despite the well-known canonical equivalence (isomorphism) between a equal-tempered division of the octave and the cyclic character of any periodic rhythm [24] the study of some tiling properties of the timeline by means of translations of a given rhythmic tile (or some usual transformations of it) is a relatively new research area inside mathematical music theory. We have already sketched the history of the emergence of tiling rhythmic structures in music composition starting from Olivier Messiaen original attempt at defining musical canons independently of any considerations about pitch values [3]. In this paper we focus on different tessellations' canonic structures as they have been implemented in the "MathTools" environment in *OpenMusic*.

[2] See [11] for a survey and a historical perspective on Neo-Riemannian theory. See [18] for one of the first examples of analytical computational models based on the tiling properties of the tone space.

4. Tiling problems in *OpenMusic* new "MathTools" Environment

Algebraic models for computer-aided music theory, analysis and composition have been implemented as a package of mathematical tools in the last version of *OpenMusic*, a visual programming language developed at Ircam.[3] There are six main families of tools, which are: circle, sieves, groups, sequences, polynomials, canons. Recently [6] we presented some aspects of these mathematical tool strictly connected with the problem of paradigmatic classification of musical structures (circular representation, groups and polynomials). After briefly summarizing the potentialities of the circular representation of musical structures (in the pitch as well as in the time domain) we now focus on two families of tiling musical canons whose mutual intersection is at the present an active field of research for composers involved in computer-aided composition.[4] We will take, as a point of departure, a recent paper by R.W. Hall and P. Klingsberg [15] that makes the bridge between the combinatorial theory of asymmetric periodic rhythmic patterns and various models of tiling canons that have been proposed in the music-theoretical literature. This shows some theoretical potentialities and some pedagogical benefits of the *MathTools* environment.

4.1. Circular representation of tessellation structures and pitch-time isomorphisms

By using the geometrical representation of the equal-tempered system as a circle divided in 12 parts, a musical chord of m distinct notes can be represented as a m-polygon inscribed in a circle. To each chord one can associate a sequence of m integer numbers counting the successive intervals in the chord (*interval structure*). Such a structure is an invariant enabling us to identify, in a unique way, a given chord up to transpositions. Figure 2 shows two rhythmic patterns which are particularly interesting from a tiling perspective. They are called "3-asymmetric" in Hall and Klingsberg's terminology since they generalized the oddity property [9]. From a group-theoretical point of view these two rhythmic patterns are what mathematical music theorists usually call "self-inverse partitioning" musical structures. In fact they are "inversionally" related. This means that they are basically the same orbit under the action of the dihedral group on the twelve-tone system. Moreover, the twelve-tone system can be partitioned into a disjoint union of transposition classes of the two structures. We show their circular representation, the intervallic structure (1 2 3 6) and one possible rhythmic interpretation (with the free choice of a period and of a metric unitary step, in this case respectively the eight and the sixteenth note)

[3] See [1] for several analyses of recent musical pieces composed by means of *OpenMusic*.

[4] As pointed out by the French composer Georges Bloch in a recent article devoted to the compositional aspects of the tiling canon process, "this is a intuitively evident structure that has interested numerous composers, among them Messiaen. But the construction of such a canon was beyond the means of composers working without computers, and research on the characteristics of such objects requires a musical representation tool such as *OpenMusic*" [1].

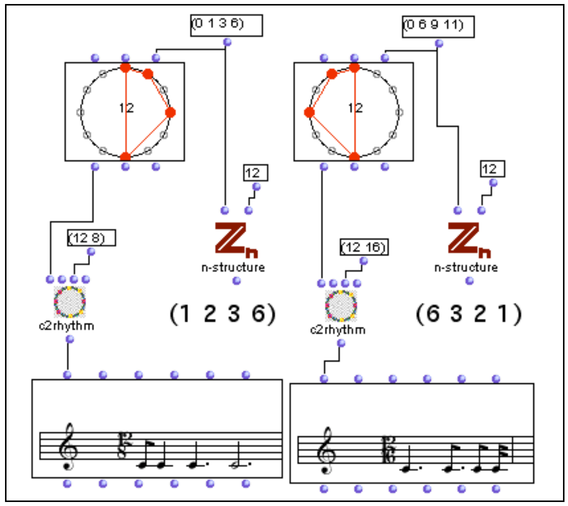

Figure 2: *Circular representation, intervallic structure and rhythmic interpretations of two inversionally related 3-asymmetric rhythmic patterns.*

By interpreting each rhythmic structure in the pitch domain, we obtain the tiling of the entire equal-tempered space with transposition classes of the same chord structure (Figure 3).

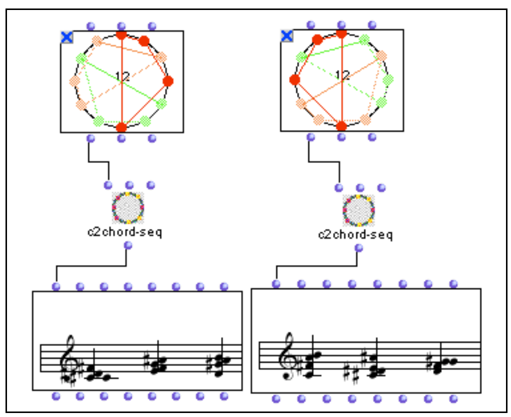

Figure 3: *Pitch-interpretation of the two previous 3-asymmetric rhythmic patterns*

4.2. Some canonical representation of tessellation structures

Instead of tiling the chromatic space with transposition classes of the same chord, as discussed previously, we can interpret the tiling process in the time domain. This corresponds to the construction of rhythmic canons in which each voice is the translation of a given rhythmic pattern, with the property that once the last voice appears, each instant of time is affected to one (and only one) rhythmic beat (*tiling rhythmic canons*). Figure 4 shows this first case.[5]

[5] The tiling property is made evident by revealing the underlying metric grid.

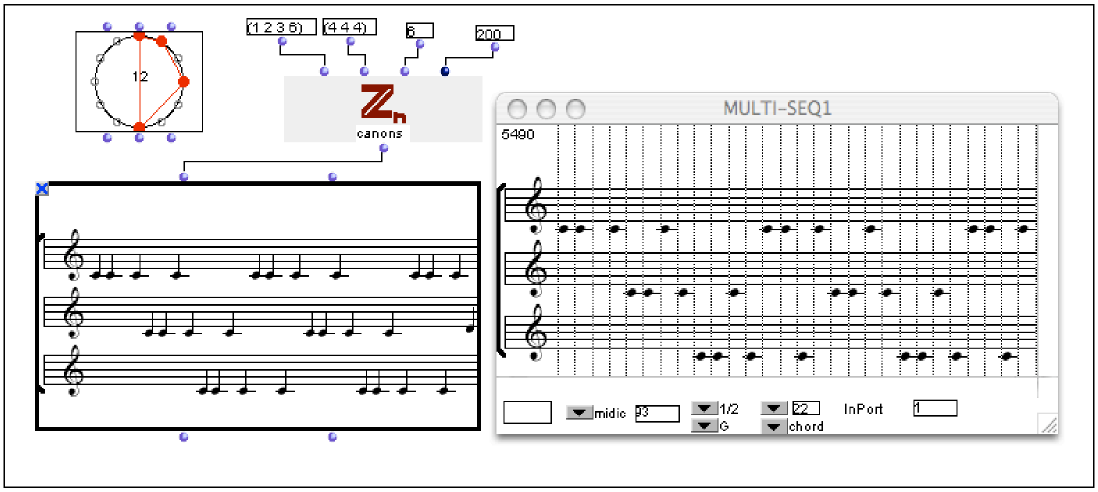

Figure 4: *A tiling rhythmic canon in the* “Math Tools” *environment*

By adding the pitch content to this global structure, it is possible to build melodic-rhythmic tiling canons, as shown in Figure 5. In this case the pitch dimension is isomorphic to the rhythmic content, as the circular representation clearly shows.

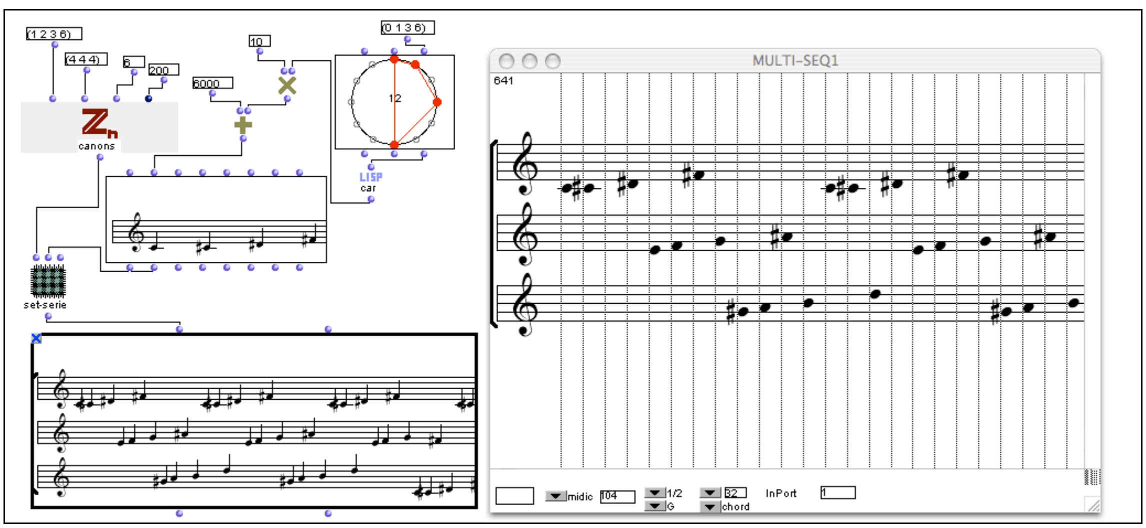

Figure 5: *A melodic-rhythmic tiling canon*

This model of canonic constructions makes use implicitly of the action of the cyclic group on the space of rhythmic structures. By changing the “paradigm” and by using for example the action of the affine group it is possible to build canonic structures in which the voices are augmentations (by a given factor) of an initial rhythmic pattern. This provides the basis of a second class of tiling rhythmic canons called “Augmented Canons” [4]. It is particularly interesting to apply this second model of tiling canons to the first of the two 3-asymmetric rhythmic patterns that we presented before. There are in fact many multiplicative factors that can be applied to this type of patterns in order to canonically tile the space. More precisely, we can easily see that there are three choices of multiplicative factors (Figure 6).

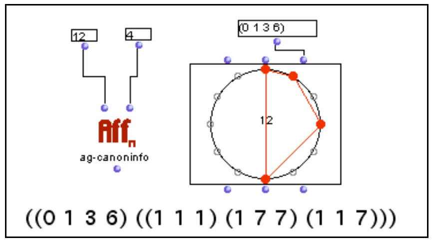

Figure 6: *Potential augmentation factors for a given rhythmic pattern*

Apart from the first solution, in which the multiplicative factors are equal to the identity (which means that the corresponding augmented canon simply reduces to the previous regular tiling canon of figure 7), there is one non trivial factor by which one could augment the initial rhythmic pattern in order to tile the time line. By taking the third solution, (1 1 7), we may construct an augmented canon in 9 voices: the first two voices are the identical patterns and the remaining 7 voices are 7 repetitions of the initial voice but augmented by a factor equal to 7.

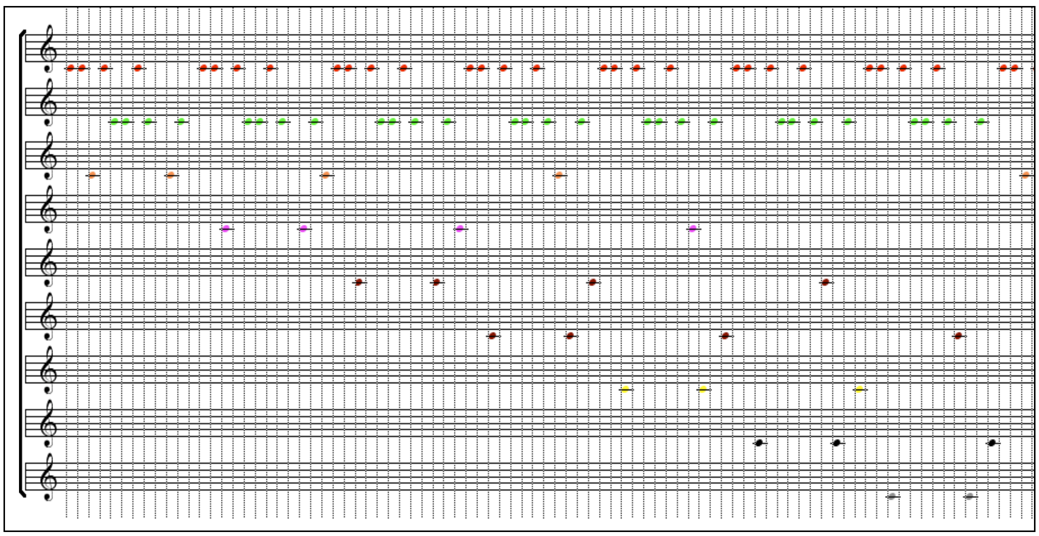

Figure 7: *An augmented tiling canon based on a 3-asymmetric rhythmic pattern*

4. Conclusions

The *OpenMusic* implementation of several families of algebraic-oriented mathematical tools suggests some interesting aspects of the notion of pedagogability in mathematical music theory. We discussed some examples concerning tiling problems in computer-aided composition. Two classes of tiling rhythmic canons are at the moment an active field of research: regular canons (where the voices are a temporal translation of a given rhythmic pattern) and augmented canons (where the voices are augmentations of the first voice). This implementational model is presently taught in many computer-aided composition courses, as well in conservatories (CNSMDP in Paris, Conservatoire of Adria in Italy), schools of music (Escola Superior de Musica de Catalunya,…), and Master programs (ATIAM at Ircam, "Art/Science/Technology" in Grenoble…). It provides a good example of pedagogability of a field of study - Mathematical Music Theory - which naturally aims to transgress the boundaries between teaching and research activities.

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