Journal of Mathematics and Music
Vol. 00, No. 00, Month 200x, 1–35

RESEARCH ARTICLE

Phase retrieval in musical structures

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(*Received 00 Month 200x; final version received 00 Month 200x*)

This paper describes phase retrieval approaches in music by focusing on the particular case of the beltway problem; for several decades, this problem has raised interest in computational musicology and especially set-theoretical methods, and in an independent way and with different vocabulary in crystallography and other scientific areas. The link between these two approachs was only made recently, raising new interesting musical applications and theoretical open problems. We present some old and new results on phase retrieval, and give perspective on future research assisted by computational methods. Extended phase retrieval for generalized musical Z-relation is then introduced with mathematical definitions and motivation from computer-aided composition. We assume from the reader basic knowledge of groups, topological groups, group algebras, group actions, Lebesgue integration, convolution products, and Fourier transform.

1. Introduction

One class of combinatorial problems deals with the problems of reconstruction. Especially, a problem that arises in very different contexts is the reconstruction of a set from the collection of its k-subsets up to isomorphism. The same thing may be done with the reconstruction of graphs from a collection of subgraphs (see [9], [8]). One can come across this type of problems in computer graphic, in physics, in genetics, in crystallography and also in musical composition.

Indeed, the phase retrieval in music extends the concept of Z-relation, a concept introduced by Forte in [13] but already present in Hanson's work [16]. In the classical framework of musical set theory, the n-tone equal temperament is modeled via the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, and each class of $\mathbb{Z}_n$ is said to be a *pitch-class*. Any pitch-class set is simply called *set*.[1] For any set $A \subseteq \mathbb{Z}_n$ one can define the interval vector ($\mathbf{iv}$) as for every $k \in \mathbb{Z}_n$, $\mathbf{iv}(A)_k = \mathbf{ifunc}(A, A)_k = \#\{(s,t) \in \mathbb{Z}_n^2, t - s = k\}$. One might notice that we

[1]We denote any set $\{[a_1]_n, \ldots, [a_s]_n\}$ as $\{a_1, \ldots, a_s\}_n$.

ISSN: 1387-3954 print/ISSN 1745-9745 online

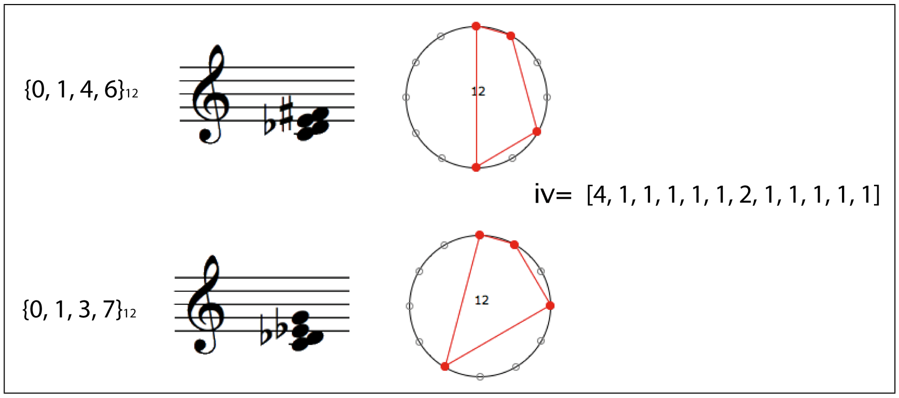

Figure 1. A well-known exemple of Z-related sets, in $\mathbb{Z}_{12}$.

define the **iv** function via the **ifunc** function borrowed from Lewin [22], which is slightly different from (but completely equivalent to) Forte's original **iv** function.

Two sets A and B are said to be *Z-related* if $\mathbf{iv}(A) \equiv \mathbf{iv}(B)$, i.e. if the same number of intervals of each type is showing up in both sets. In other words, A and B share the same interval content. Clearly, transposing or inverting a set does not change his interval content, and thus we have a lot of trivially Z-related sets. To get rid of this trivial case, we may consider the sets up to transposition and inversion, and we notice that there still exists Z-related sets (thus completely unrelated by transposition and inversion). A well-known example is sets $\{0,1,4,6\}_{12}$ and $\{0,1,3,7\}_{12}$ in $\mathbb{Z}_{12}$, which share the same interval vector $[4,1,1,1,1,1,2,1,1,1,1,1]$ – see Figure 1. Some composers have (implicitly or explicitly) dealt with the Z-relation; for example this couple of Z-related sets is exactly the one used by Elliot Carter in his second quartet.

To improve upon the classical model, one can substitute pitch-class sets with multisets, i.e. integer-valued distributions, which might be useful to represent a chord where notes might be repeated (Fig. 2, center); one can even consider rational- or real-valued distributions, which include in the representation the dynamics of each note (Fig. 2, right).[1] In this case, the interval vector is no more sufficient, and must be replaced (as we will see) by the *Patterson function*, which will extend the concept of interval content, as it represents (as suggested by Lewin) the *probability* of hearing a given interval, if the notes of a given set are played randomly.

The name *Patterson function* comes from X-ray crystallography. Let G be an abelian group (with additive notation). Given a distribution $E = \sum_{g\in G} e_g\delta_g$, we call *inversion*[2] of E the distribution $I(E) = \sum_{g\in G} \overline{e}_g\delta_{-g}$, and the k-transposition of E is the distribution $T_k(E) = \sum_{g\in G} e_g\delta_{g+k}$ $(k \in G)$. Then, the Patterson function of any distribution E is the convolution product $E * I(E)$. Now, for any $X \subseteq G$, let $\mathbf{1}_X$ be the distribution $\sum_{g\in X}\delta_g$. By reading [21], we know that $\mathbf{iv}(A) = \mathbf{1}_A * \mathbf{1}_{-A}$, and since $\mathbf{1}_{-A} = I(\mathbf{1}_A)$, we see that

[1] Let K be a field and let G be an abelian group (with additive notation). A distribution on G with coefficients in K has the form $E = \sum_{g\in G} a_g\delta_g$, where $a_g \in K$ and δ_g is the Dirac mass related to the element g. If $a_g \neq 0$ only finitely often, we say that the distribution is finite. Recall that the algebra of such distributions under the convolution product is isomorphic with the group ring K^G, and thus we will sometimes write $E \in K^G$.

[2] The inversion of E, namely $I(E)$, is sometimes found as E' or E^* and referred to as *reflection*.

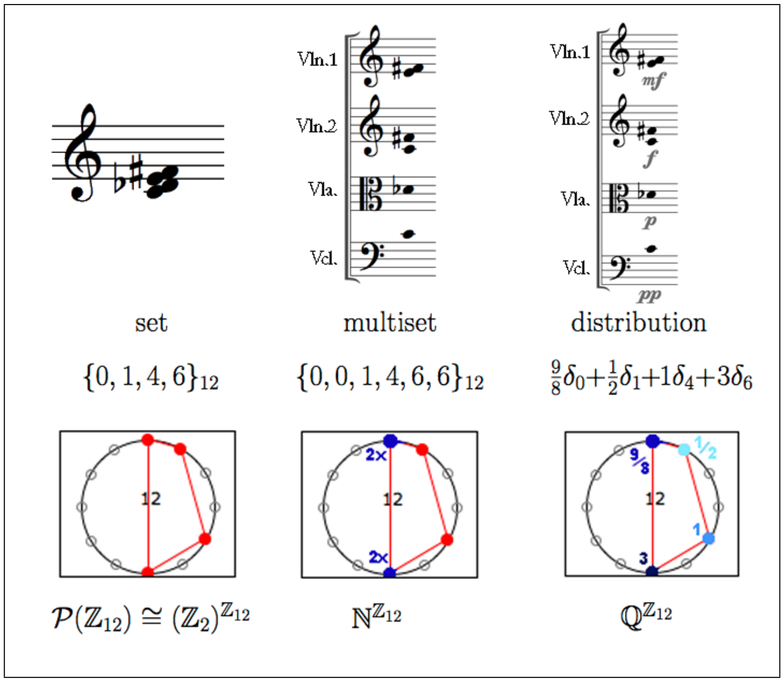

Figure 2. An example showing the usefulness of improving the classical model. A standard set is an element of $\mathcal{P}(\mathbb{Z}_n)$, i.e. a 0-1 distribution on $\mathbb{Z}_n$. If we allow some notes to be repeated, we have a multiset, as in the middle example (the same chord given to a string quartet), i.e. a distribution of $\mathbb{N}^{\mathbb{Z}_n}$. Finally, if we add a dynamic mapping (right example), we can see the chord as a real distribution, i.e. a distribution of $\mathbb{Q}^{\mathbb{Z}_n}$. In this example we have arbitrarily chosen $mf = 1$, $f = 2$, $p = 1/2$, $pp = 1/4$.

the Patterson function is nothing more than a generalization of the interval vector to a generic distribution. In crystallography, the Patterson function is the starting point for solving the phase retrieval problem, i.e. to determine the arrangement of atoms within a crystal, given the module of the Fourier transform[3] of the atoms distribution. Indeed, if we know $D*I(D)$, we know its Fourier transform $\widehat{D}\overline{\widehat{D}}(\omega) = \|\widehat{D}(\omega)\|^2$ for all $\omega \in \mathbb{Z}_n$. Thus, to reconstruct $\widehat{D}(\omega) = \|\widehat{D}(\omega)\|e^{i\phi(\omega)}$ (and D from there by inverse Fourier transform), since we know its module, we just need to retrieve the phase $\phi(\omega)$. This is the central problem that we address in this paper.

In section 2, we introduce topological and measure and integration theory tools that we use on Lewin's Generalized Interval Systems (GIS), then we introduce the interval content, the Patterson function, Z-relation and homometry, and their properties, including two examples of a Z-relation in a non-commutative GIS. In section 3, we define the phase retrieval problem, introduce alternative formulations of it, stressing the role of spectral units in the case of discrete abelian groups, trying to characterize homometric sets in a constructive way. In section 4 we extend the definitions of section 2, by introducing the

[3]Recall that, for $G = \mathbb{Z}_n$, the Fourier transform of a distribution $E = \sum_{g\in\mathbb{Z}_n} e_g\delta_g$ is the map

$$\omega \in \mathbb{Z}_n \mapsto \widehat{E}(\omega) = \sum_{g\in\mathbb{Z}_n} e_g \exp(-2i\pi g\,\omega/n)$$

.

k-deck, the k-Deck, the k-vector, and higher-order generalizations of Z-relation and homometry associated to them. Finally, in section 5, we define the extended phase retrieval problem and the reconstruction index of a cyclic group. The study of k-decks has been widely developed lately, while the k-Deck has been pretty much left aside. After providing some properties of the Z^k-relation, we end up with the first example of 4-Homometric sets.

2. Z-relation and homometry

In this section, we will link vocabulary from musical set theory — Generalized Interval System, interval vector, Z-relation — with vocabulary from crystallography — implicit usage of group structure, Patterson function, homometry. These objects and their elementary properties are presented in a theoretical framework large enough to cover most of the areas wherein homometry and Z-relation have been previously studied.

2.1. Using Generalized Interval Systems (GIS)

2.1.1. Mathematical definition of a GIS

The notion of Generalized Interval System, introduced in [22], formalizes the notion of interval between two points in a set of values of an abstract musical parameter.

Definition 2.1 (Lewin) *A **Generalized Interval System (GIS)** is a triple* $(S, G, \mathbf{int})$*, where* S *is a set called* space *of the GIS,* G *a group called* intervals group *of the GIS, and* $\mathbf{int} : S \times S \to G$ *a map such that*

(A) For every r, s, t *in* S*,* $\mathbf{int}(r, s)\,\mathbf{int}(s, t) = \mathbf{int}(r, t)$*.*

(B) For every s *in* S*,* i *in* G*, there is a unique* t *in* S *such that* $\mathbf{int}(s, t) = i$*.*

It is noted in [32] that

- (A) and (B) in the definition above are equivalent to defining a simply transitive right action of group G on S, such that for every s, t in S, $s\ \mathbf{int}(s, t) = t$;
- the definition of a GIS is analogous with the definition of an affine space, the difference being that the underlying algebraic structure of an affine space is not a group, but a vector space.

In every GIS, the musical parameter space S and the interval group G have the same cardinality; more precisely, condition (B) implies that for every s in S, the **label** map[1] is bijective:

$$\begin{aligned} \mathbf{label} : S &\to G \\ t &\mapsto \mathbf{int}(s, t) \end{aligned}$$

We explicit now two usages of **label** bijections, which are also common with the couple "affine space–vector space".

[1] The denomination **label** comes from [22, beginning of Chapter 3].

The first possibility is using the interval group G itself as the space S: in this case, the group action that defines the GIS is right translation, i.e. for every s, t in G, $\mathbf{int}(s,t) = s^{-1}t$. As a consequence, every group defines a canonical GIS associated with it via this group action. To avoid confusion that may arise from this identification of the group interval G and the GIS space, elements of the space will be called points, elements of the interval group will be called intervals, and unless explicitly mentioned otherwise, subsets of G mean subsets of the GIS space.

The second possibility is using **label** bijections for transferring some additional structure of the interval group G — e.g. a topology, a distance or a measure — onto S. Moreover, if this structure is translation invariant, the resulting structure on S does not depend on a particular $s \in S$ that defines **label** map. This principle of translation-invariant structure transfer for GIS is detailed in [19], and we will use it below.

When G is abelian, we will denote the group operation with a plus sign $+$ instead of a multiplicative notation. Although most of our examples will happen in the commutative case, the definition and several basic properties of the objects that we will define also hold in the non-abelian case. A musically significant example of a non-commutative GIS is the GIS of time spans[22, 4.1.3.1], which is defined as the positive affine group of $\mathbb{R}$, that is the semi-direct product $\mathbb{R} \rtimes_m \mathbb{R}_+^*$ where the group morphism $m : (\mathbb{R}_+^*, .) \to (\mathrm{Aut}(\mathbb{R}), +)$ maps r to the multiplication by r.

2.1.2. Transferring translation-invariant topologies and measures onto a GIS

We are interested in measuring subsets of the space of a GIS. The most straightforward measure of a set is its cardinality; however, many definitions and tools we will present are, under some conditions, still valid with using certain measures — e.g. the Lebesgue measure — on a GIS. More precisely, we need a measure on both the space of a GIS and its interval group, and we require that the measure on the interval group be translation-invariant, so that the measure on the space naturally comes from transferring the measure of the group; we will implicitly assume from now on that defining a translation-stable σ-algebra $\mathcal{A}$ (the borelian subsets, see notations below) on a group G and a measure on $\mathcal{A}$ also defines, through the transfer principle, the same structures on the space of a GIS with G as its interval group. We will exclude non-translation-invariant structures, because giving different weights to a subset and its translations would break the concept of an isotropic GIS with its transfer principle. This generalization of measuring the cardinal of sets in GIS has already been proposed by Lewin in [22, section 6.10], but has never been further elaborated as far as we know. We believe that such a generalization is not gratuitous, from a mathematical point of view. In fact, there are fortunately many groups which may be fitted with a right-translation-invariant measure, thanks to the following result.

Definition 2.2 *Let $(G, \mathcal{A}, \mu)$ be a measured space where G is a group. μ is called* ***right-translation-invariant*** *if $\mathcal{A}$ is right-translation-stable and for every $A \in \mathcal{A}$, $g \in G$, $\mu(A\,g) = \mu(A)$. If, in addition, G is a topological group, and $\mathcal{A}$ is the Borel σ-algebra on G, then μ is called a* ***right Haar measure*** *on G.*

Theorem 2.3 *Any locally-compact Hausdorff topological group G has a right Haar measure μ; moreover, this measure is uniquely defined, up to a multiplicative constant.*

The previous theorem, which is a classical theorem in topology, allows us to define the notion of interval content in any locally-compact topologic group, including every group with the discrete topology — the associated right Haar measure is simply the cardinal function — $\mathbb{R}$, and all products and quotients of such groups.

Since the topology of a topologic group G is translation-invariant, it can be naturally transferred onto the space of a GIS that has G as its interval group. We recall the idea from [19], that using topologies in GIS could help expressing notions of continuity of musical patterns; this would make sense for instance with $\mathbb{R}$, the continuous circle $\mathbb{R}/\mathbb{Z}$, or any product of these groups fitted with their respective usual topology, as an interval group of a GIS.

As we want to be able to compare measures of certain sets and to do some computations on measure values (multiplications, additions, substractions...), we will restrict our study to measurable sets with finite measure, as suggested in [22].

We end this introduction of topological GIS with a (right) Haar measure with some notations, which we will assume throughout the rest of the article. Let G be a locally-compact group, K a field with an involutive automorphism noted $\;\overline{\ }: x \mapsto \overline{x}$; we denote

- $S(X)$ the permutation group of a set X,
- $\mathcal{A}$ the σ-algebra of Borel sets of G,
- μ a right Haar measure on G,
- $\tilde{\mathcal{A}}$ the set of measurable subsets of G with finite measure,
- K^G the K-algebra of maps from G to K, which are also called (K-valued) distributions on G,
- for every $g \in G$, $T_g : K^G \to K^G$
  $$E \mapsto \big(T_g(E) : x \mapsto E(g^{-1}x)\big)$$
  the left translation of distributions by g; we may also write $T_g(A) = gA$ for $A \subset G$ when there is no ambiguity;
- $T(G) = \{x \mapsto gx, g \in G\}$, or simply T, the group of left translations on G,
- $I : K^G \to K^G$, $E \mapsto \big(I(E) : x \mapsto \overline{E(x^{-1})}\big)$ the inversion on distributions; we also overload I by defining for every $A \subset G$ $I(A) = A^{-1}$,
- $D(G)$ (or D) the generalized dihedral group of G, which is the subgroup of $S(G)$ generated by the left translations of G and the inversion $x \mapsto x^{-1}$,
- $\mathcal{D}(G)$ or $\mathcal{D}$ the subgroup of the linear group of K^G generated by $\{T_g, g \in G\} \cup \{I\}$, which is an isomorphic representation of $D(G)$,
- when $K \in \{\mathbb{R}, \mathbb{C}\}$, $\Sigma_{\mathsf{C}}(G, S)$ the algebra of almost everywhere bounded functions with compact support from G to a subset S of K, up to equality almost everywhere; this is the set of functions of which we will define the Patterson function;
- $[x]_H = \{h(x), h \in H\}$ where X is a set, H a subgroup of $S(X)$ and $x \in X$; $[x]_H$ is the orbit of x under the natural group action of H on X, elements of $[x]_H$ are said congruent to x modulo H; the same notation is used with H a subgroup of a group G and for every $g \in G$ $[g]_H = Hg$;
- for every a, b in $\mathbb{Z}$, $[\![a, b]\!] = \{x \in \mathbb{Z}, a \leqslant x \leqslant b\}$.

It should be noticed that, in defining $\mathbf{int}(a, b)$ as $a^{-1}b$, we favor left translations over right translations: for any $a, b, c \in G$, one has $\mathbf{int}(ca, cb) = a^{-1}c^{-1}cb = a^{-1}b = \mathbf{int}(ab)$, but $\mathbf{int}(ac, bc) = c^{-1}a^{-1}bc = c^{-1}\mathbf{int}(a, b)^{-1} \neq \mathbf{int}(a, b)$ in general. Thus this notion of interval is invariant by left translations only.[1] There is, of course, an alternative definition of the interval from a to b, namely $\widetilde{\mathbf{int}}(a, b) = ba^{-1}$, which is invariant under right translation. This explains why we have found not one, but two generalizations of the

[1]This fact is well commented in [22, section 3.4].

hexachord theorem (see subsection 2.7 below). Of course, the abelian case is much simpler, with only one possible notion of interval, and one kind of translation. In the sequel, unless otherwise indicated, we stick to $\mathbf{int}(a,b) = a^{-1}b$.

In general, in a non-abelian locally compact group, the left- and right-invariant Haar measure do not coincide; for instance, in the affine group of maps $x \mapsto ax+b$ on the real line, the left- and right- invariant measures are respectively $da\,db/a^2$ and $da\,db/a$. This motivates the following definition.

Definition 2.4 *A locally compact group is unimodular if it admits a Haar measure that is both right- and left-invariant.*

The unimodularity is a reasonable hypothesis in many cases; in particular, it is satisfied whenever G is compact – see [29, Chap.3, 1(iv)] – and even more easily when G is discrete – since cardinality is both right- and left-translation-invariant.

2.2. Interval vector and Patterson function

Definition 2.5 *Let A, B in $\mathcal{A}$. The* ***interval function*** *between A and B is the function*

$$\begin{aligned}\mathbf{ifunc}(A,B) : \ & G \to \mathbb{R}_+ \\ & g \mapsto \mu(B \cap Ag)\end{aligned}$$

Since $B \cap Ag = \{a \in A, \exists b \in B,\ \mathbf{int}(a,b) = g\}$, this definition is a straightforward generalization of [22, 5.1.3], where **ifunc** is defined for discrete G.

Definition 2.6 *Let $A \in \tilde{\mathcal{A}}$. The* ***interval content*** *of A is the function*

$$\begin{aligned}\mathbf{iv}(A) : \ & G \to \mathbb{R}_+ \\ & g \mapsto \mu(A \cap Ag)\end{aligned}$$

If G is discrete, the interval content is also called interval vector, hence the notation **iv**.

It is clear, from the right translation invariance of μ and the fact that it is real-valued, that for every $A \in \tilde{\mathcal{A}}$ and $g \in G$, $\mathbf{iv}(A)(g) = \mu(Ag^{-1} \cap A) = \overline{\mu(Ag^{-1} \cap A)}$, i.e. $I(\mathbf{iv}(A)) = \mathbf{iv}(A)$. In [21], the interval vector is expressed as a convolution product through the natural bijection between $\tilde{\mathcal{A}}$ and $\Sigma_{\mathsf{C}}(G, \{0,1\})$, i.e. $\mathbf{iv}(A) = \mathbf{1}_A * \mathbf{1}_{A^{-1}}$. However, to include the case of a non-commutative group, the interval content shall be expressed as $\mathbf{iv}(A)(g) = \int \mathbf{1}_A(hg^{-1})\mathbf{1}_A(h)d\mu(h) = I(\mathbf{1}_A) * \mathbf{1}_A(g)$, where $*$ is the convolution product for the right Haar measure – see [29, Chap.3, 3.5 and 5.1]. Then, this definition can be extended to every almost everywhere bounded function on G with compact support, which is customary in crystallography; for example, see the introduction of [31].

In a non abelian group, we can introduce two distinct definitions of the interval content, because there are two different definitions of the interval from a to b.

Definition 2.7 *We note ${}^r\mathbf{ifunc} = \mathbf{ifunc}$, ${}^r\mathbf{iv} = \mathbf{iv}$ the right interval function and interval content already defined above. Let ${}^l\mathbf{ifunc}(A,B)$ be the left interval function:*

$$g \in G \mapsto {}^l\mathbf{ifunc}(A,B) = \mu(B \cap gA) = \int \mathbf{1}_B(h)\mathbf{1}_A(g^{-1}h)d\mu(h)$$

Similarly the left interval content is defined as

$$^{l}\mathbf{iv}(A) : g \in G \mapsto \mu(A \cap gA) = \int \mathbf{1}_A(h)\mathbf{1}_A(g^{-1}h)d\mu(h)$$

Unless otherwise indicated, we will use the rightwise definitions of the interval function and interval content.

DEFINITION 2.8 *For every function $E \in \Sigma_{\mathsf{C}}(G, K)$,*[1] *the **Patterson function** of E is defined by*

$$d^2(E) := I(E) * E : g \in G \mapsto \int \overline{E(hg^{-1})}E(h)\, d\mu(h)$$

As the interval content of a finitely measured subset of G is the Patterson function of its characteristic function, that is $\mathbf{iv}(A) = d^2(\mathbf{1}_A)$, all features of interval contents can and will be expressed in terms of Patterson functions. We introduce below the most basic properties of d^2, which will motivate the ensuing definitions for finitely measured subsets of G that share the same interval contents, and more generally functions in $\Sigma_{\mathsf{C}}(G, K)$ that share the same Patterson function.

PROPOSITION 2.9 [Invariance under transposition and inversion] *If G is unimodular, then for every $E \in \Sigma_{\mathsf{C}}(G, K)$, for every $g \in G$, $d^2(T_g(E)) = d^2(E)$; furthermore, if G is abelian, then $d^2(I(E)) = d^2(E)$.*

Proof: The transposition invariance is implied by the left translation invariance of the Haar measure on G: for every $x \in G$, $d^2(T_g(E))(x) = \int \overline{E(g^{-1}yx^{-1})}E(g^{-1}y)\, d\mu(y) = \int \overline{E(zx^{-1})}E(z)\, d\mu(gz) = \int \overline{E(zx^{-1})}E(z)\, d\mu(z)$, where the variable substitution $y = gz$ is made in the second equality.

If G is abelian, the inversion invariance is a consequence of the commutativity of the convolution product and the involutive property of the inversion: $d^2(I(E)) = I(I(E)) * I(E) = E * I(E) = I(E) * E = d^2(E)$. ■

The invariance under translation may also hold without the hypothesis that G is unimodular, for instance for T_g with g central in G, that is for every $h \in G$, $gh = hg$.

Example 2.10 As a counterexample of the invariance , consider the GIS of major and minor triads with the dihedral group of transpositions and inversions as the interval group, with 24 elements and let for instance $A = \{\{0, 4, 7\}, \{2, 7, 11\}, \{2, 5, 9\}, \{4, 7, 11\}\}$ and $B = I_4(A)$ be its “translate” by the inversion $I_4 : x \mapsto 4 - x$, i.e. $B = \{\{0, 4, 9\}, \{2, 5, 9\}, \{2, 7, 11\}, \{0, 5, 9\}\}$. We can see in Figure 3 that the inversion $I_2 : x \mapsto 2 - x$ occurs twice in B but never in A, i.e. $\mathbf{iv}(B)(I_2) = 2$ while $\mathbf{iv}(A)(I_2) = 0$. Since every transposition T_i is central in G, one can check that $\mathbf{iv}(T_i(A))(g) = \mathbf{iv}(A)(g)$ for all $g \in G$.

[1]It could be defined for a larger set of functions, e.g. the algebra $L^1(\mu)$ of μ-integrable maps from G to $\mathbb{C}$ or the algebra $L^2(\mu)$ of maps from G to $\mathbb{C}$ whose square is μ-integrable, but $\Sigma_{\mathsf{C}}(G, S)$ is sufficient for musical applications.

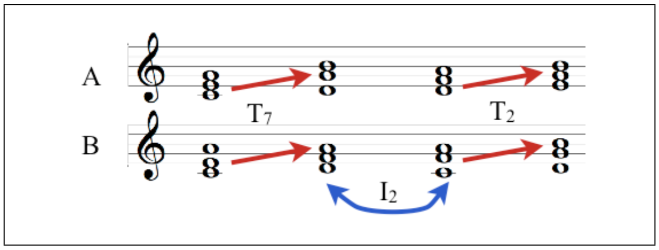

Figure 3. The interval vector changes when A is transformed by I_4.

2.3. Definitions of Z-relation and homometry

Definition 2.11 *The elements of a family $(A_j)_{j\in J}$ valued in $\tilde{\mathcal{A}}$ are said to be **Z-related** if they have the same interval content almost everywhere. If, in addition, for every distinct j,k in J, $[A_j]_D \neq [A_k]_D$, then the elements of $(A_j)_{j\in J}$ are said to be **non-trivially Z-related**.*

Example 2.12 In $\mathbb{Z}_8$, $\{1,2,3,6\}_8$ and $\{0,1,3,4\}_8$ are non-trivially Z-related. It is the simplest example.

Definition 2.13 *Let $(E_j)_{j\in J}$ a family of elements of $\Sigma_{\mathsf{C}}(G,K)$. Elements of $(E_j)_{j\in J}$ are said to be **homometric** if they have the same Patterson function almost everywhere. If, in addition, for every distinct j,k in J, $[E_j]_{\mathcal{D}} \neq [E_k]_{\mathcal{D}}$, the A_j are said to be **non-trivially homometric**.*

It should be noted that the Z-relation as defined by Allen Forte in [13, section 1.9] is what we call non-trivial Z-relation, and that our definition of homometry follows Rosenblatt [31]. We choose these definitions so that Z-relation and homometry are equivalence relations on $\tilde{\mathcal{A}}$ and $\Sigma_{\mathsf{C}}(G,K)$, respectively.

Obviously, subsets of $\mathcal{A}$ are Z-related if and only if their characteristic functions are homometric.

2.4. Elementary properties

We will give now properties of the Patterson function related to monotonicity, periodicity and commutation with quotients.

In order to give a monotonicity property of the Patterson function, we introduce a pointwise order on $\Sigma_{\mathsf{C}}(G,\mathbb{R})$: we note $E \leqslant F$ if, for every x in G, $E(x) \leqslant F(x)$. This order is compatible with the inclusion order on $\tilde{\mathcal{A}}$, i.e. the natural bijection between $\tilde{\mathcal{A}}$ onto $\Sigma_{\mathsf{C}}(G,\{0,1\})$ is an increasing map.

Lemma 2.14 *For all distributions E,F in $\Sigma_{\mathsf{C}}(G,\mathbb{R}_+)$, if $E \leqslant F$, then $d^2(E) \leqslant d^2(F)$, i.e. $d^2 : \Sigma_{\mathsf{C}}(G,\mathbb{R}_+) \mapsto \Sigma_{\mathsf{C}}(G,\mathbb{R}_+)$ is an increasing map. In particular, for every A,B in $\tilde{\mathcal{A}}$, if $A \subset B$ then $\mathbf{iv}(A) \leqslant \mathbf{iv}(B)$.*

Proof: For every x,y in G, $0 \leqslant E(y) \leqslant F(y)$ and $0 \leqslant E(yx^{-1}) \leqslant F(yx^{-1})$, therefore taking the product term by term, $E(yx^{-1})E(y) \leqslant F(yx^{-1})F(y)$; moreover the Lebesgue integral with measure μ is positive, so finally $d^2(E) \leqslant d^2(F)$. ■

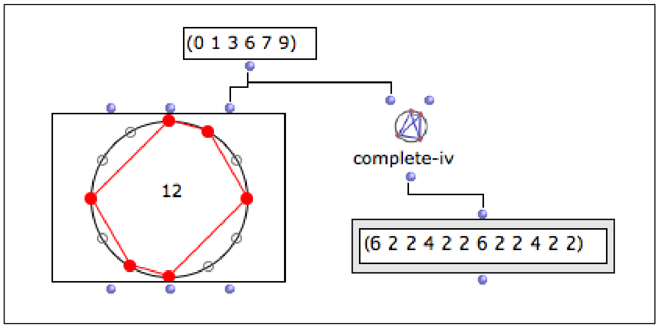

Figure 4. The interval vector of a periodic set is periodic.

PROPOSITION 2.15 *For every distribution E in $\Sigma_{\mathbb{C}}(G)$, for $k \in \mathbb{C}$, $d^2(kE) = |k|^2 d^2(E)$. Moreover, if G is commutative, then for all distributions E, F in $\Sigma_{\mathbb{C}}(G)$ $d^2(E * F) = d^2(E) * d^2(F)$.*

Proof: The first part of the proposition is obvious. To prove the second part, we assume that G is commutative. Let $E, F \in \Sigma_{\mathbb{C}}(G)$. It is straightforward to see that $I(E * F) = I(E) * I(F)$, so we have $d^2(E * F) = I(E * F) * E * F = I(E) * I(F) * E * F$, then the result follows by commutativity of the convolution product. ■

PROPOSITION 2.16 [Periodicity invariance] *Let $E \in \Sigma_{\mathbb{C}}(G)$. If for some $r \in G$, for every $g \in G$, $E(gr^{-1}) = E(g)$, then for every $g \in G$, $d^2(E)(gr^{-1}) = d^2(E)(g)$.*

There is a partial and fuzzy converse result for $\{0, 1\}$-valued distributions: if $A \in \tilde{\mathcal{A}}$ has a finite measure and there is $r \in G$ such that $\mathbf{iv}(A)(r) = \mathbf{iv}(A)(e)$, then there are N, N' μ-negligible subsets of G such that $A \sqcup N = Ar \sqcup N' = A \cup Ar$, that is, A is "almost periodic".

Proof: $d^2(E)(gr^{-1}) = \int \overline{E(h(gr^{-1})^{-1})}E(h)d\mu(h) = \int \overline{E(hrg^{-1})}E(h)d\mu(h)$, so by right translation invariance of μ, $d^2(E)(gr^{-1}) = \int \overline{E(h'g^{-1})}E(h'r^{-1})d\mu(h') = \int \overline{E(h'g^{-1})}E(h')d\mu(h') = d^2(E)(g)$.

As for the second part of the proposition, we have

$$Ar = (A \cap Ar) \sqcup (A^C \cap Ar) \tag{1}$$

$$A = (Ar \cap A) \sqcup (Ar^C \cap A) \tag{2}$$

$$A \sqcup (A^c \cap Ar) = A \cup Ar = Ar \sqcup (Ar^C \cap A) \tag{3}$$

By right translation invariance of μ, $\mu(Ar) = \mu(A)$, so by (1), $\mu(A) = \mu(A \cap Ar) + \mu(A^C \cap Ar)$; moreover, $\mu(A) = \mathbf{iv}(A)(e) = \mathbf{iv}(A)(r^{-1}) = \mathbf{iv}(A)(r) = \mu(A \cap Ar)$ is finite, so $\mu(A^C \cap Ar) = 0$, so $N := A^c \cap Ar$ is negligible. In a similar way, we get from (2) that $N' := Ar^C \cap A$ is negligible. We finally get the result by (3). ■

Example 2.17 The Proposition 2.16 tells us that any periodic distribution has a periodic interval content. Hence the interval content of any of Messiaen's modes of limited transposition will be periodic. For example – see Figure 4 – the interval vector of $A = \{0, 1, 3, 6, 7, 9\}_{12}$ is $\mathbf{iv}(A) = [6, 2, 2, 4, 2, 2, 6, 2, 2, 4, 2, 2]$. Since $T_6(A) = A$, we have $T_6(\mathbf{iv}(A)) = \mathbf{iv}(A)$.

We will use the following simple necessary condition on measure equality for Z-relation.

Lemma 2.18 *If $(A_j)_{j\in J}$ is a family of Z-related subsets of G, then all the A_j have the same measure.*

Proof: For every $j \in J, \mu(A_j) = \mathbf{iv}(A_j)(e)$, where e is the neutral element of G. ■

In particular, if the topology on G is discrete, then any two Z-related subsets of G have the same cardinality.

2.5. Interval structure and interval content

We will now build a link between interval content and interval structure, expressing the former using the latter. We will focus our attention to a restricted class of discrete groups, namely discrete groups with a total order compatible with left translation.

Definition 2.19 *An* ***left-(totally-)ordered group*** *is a couple $(G, \leqslant)$ where G is a discrete group and $\leqslant$ is a total order on G which is compatible with left translation, that is for every f, g, h in G, if $f \leqslant g$ then $hf \leqslant hg$.*

Examples of left-ordered groups are all abelian ordered groups, e.g. $\mathbb{Z}$, $\mathbb{R}$, and the tiemspans group $\mathbb{R} \rtimes_m \mathbb{R}_+^*$ fitted with Lewin's attack order, which is simply the lexicographic order associated with the usual order on $\mathbb{R}$ and $\mathbb{R}_+^*$. Every direct product of left-ordered groups fitted with the lexicographic order associated to the orders of these groups is a left-ordered group too.

Definition 2.20 *Let G be a left-ordered group. For every finite subset A of G, there is a unique strictly increasing family $(a_i)_{i\in[\![1,n]\!]}$ where $n = |A|$, such that $A = \{a_i\}_{i\in[\![1,n]\!]}$. The* ***interval structure*** *of A is the family $\mathbf{is}(A) = (\mathbf{int}(a_i, a_{i+1}))_{i\in[\![1,n-1]\!]}$.*

Example 2.21 Let $A = \{-3, -1, 1, 5, 6\}$ in $\mathbb{Z}$; $\mathbf{is}(A) = (2, 2, 4, 1)$. Let $B = \{(2, 1), (3, 1), (5, 2), (7, \frac{1}{2}), (7 + \frac{1}{2}, \frac{1}{2}), (9, 3)\}$ in the time spans group $\mathbb{R} \rtimes_m \mathbb{R}_+^*$; $\mathbf{is}(B) = ((1, 1), (2, 2), (1, \frac{1}{4}), (1, 1), (3, 6))$.

Proposition 2.22 *Let G be a left-ordered group. The interval structure of every finite subset of G is invariant by left translation, that is for every A finite subset of G, for every g in G, $\mathbf{is}(gA) = \mathbf{is}(A)$. Conversely, if A, B are finite subsets of G such that $\mathbf{is}(A) = \mathbf{is}(B)$, then there is $g \in G$ such that $B = gA$.*

Proof: The invariance of interval structure by left translation directly follows from the preservation of intervals by left translation. As for the second part of the proposition, it is obvious that by defining $g = \min(B)\min(A)^{-1}$ we get by finite induction on the lists defined by ordering A and B that $B = gA$. ■

We shall now define a partition of a non-negative element of a left-ordered group, which naturally generalizes the notion of partition of a positive integer, and a consecutive subfamily of a sequence valued in a left-ordered group.

Definition 2.23 *Let G be a left-ordered group, let e be the neutral element of G, let $p \in G$ such that $p \geqslant e$. An* ***ordered partition*** *of p is a family of elements of G $(d_j)_{j\in[\![1,k]\!]}$ such that $k \in \mathbb{N}$, for all j in $[\![1, k]\!]$ $d_j > e$ and $\prod_{j=1}^{k} d_j = p$.*

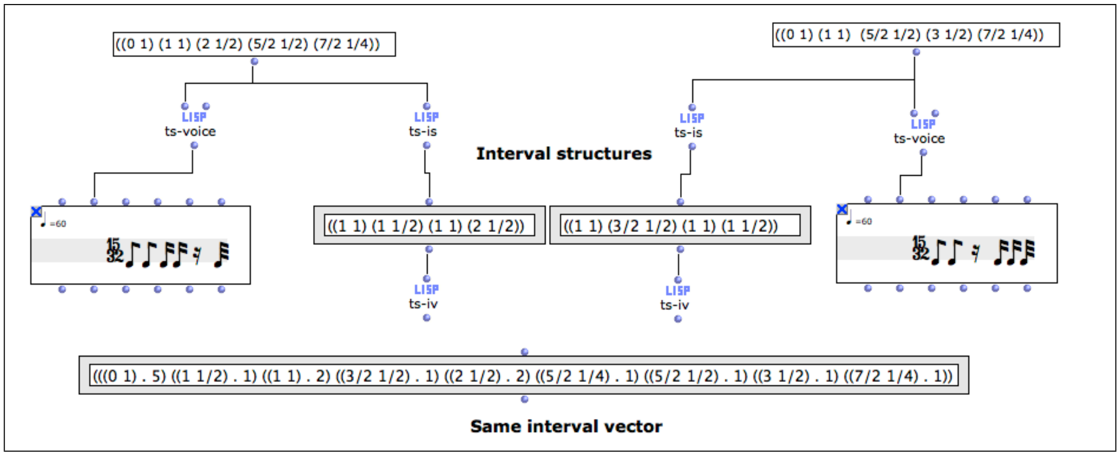

Figure 5. Example of Z-relation between two time spans (non-commutative case).

DEFINITION 2.24 *Let G be a left-ordered group, let $A = (a_j)_{j\in[\![1,k]\!]}$ be a family of elements of G. A **consecutively-indexed subfamily** of A is any subfamily $(a_j)_{j\in J}$ of A such that $J = [\![l,m]\!]$ with $1 \leqslant l \leqslant m \leqslant k$.*

THEOREM 2.25 *Let $A = \{a_i\}_{i\in[\![1,k]\!]}$ be a finite subset of a left-ordered group G, such that $(a_i)_i$ is strictly increasing. We note $(d_i)_{i\in[\![1,k-1]\!]}$ the interval structure of A. For every $p \in G$, we note $\mathcal{I}_p(A) = \{(j,j') \in [\![1,k-1]\!]^2, j+1 \leqslant j' \text{and} \prod_{i=j}^{j'} d_i = |p|\}$, where $|p| = \max(p,p^{-1})$; then $\mathbf{iv}(A)(p) = \#(\mathcal{I}_p(A))$, that is, $\mathbf{iv}(A)(p)$ is equal to the number of consecutively-indexed subfamilies of $\mathbf{is}(A)$ which are partitions of $|p|$.*

Proof: For every $p \in G \setminus \{e\}$, $\mathbf{iv}(A)(p) = \mathbf{iv}(A)(|p|)$, so we can suppose that $p \geqslant e$. The map

$$\begin{aligned} \mathcal{I}_p(A) &\to A \cap Ap \\ (j,j') &\mapsto a_{j'} = a_j p \end{aligned}$$

is well-defined and bijective, and $\#(A \cap Ap) = \mathbf{iv}(A)(p)$. ∎

This theorem may be used to compute the interval content from an interval structure. For instance, the time spans group G is non-commutative and has no central element besides the neutral $(0,1)$, so interval structure and the interval content have exactly the same invariance properties on this group, including invariance by left translation. Thus, an approach for finding Z-related subsets of the time spans is generating interval structures and sorting them by their interval content. For example, by taking $\mathcal{E} = \prod_{j=1}^{4}\{(1+\frac{k}{2}, 2^l)\}_{k=0,\dots,6,l=-1,0,1}$, we find with computer search two and only two interval structures in $\mathcal{E}$ that have the same interval content, and by "integrating them", we obtain that the time spans sets $\{(0,1),(1,1),(2,\frac{1}{2}),(\frac{5}{2},\frac{1}{2}),(\frac{7}{2},\frac{1}{4})\}, \{(0,1),(1,1),(\frac{5}{2},\frac{1}{2}),(3,\frac{1}{2}),(\frac{7}{2},\frac{1}{4})\}$ are Z-related, as shown in Figure 5.

2.6. *Patterson function transfer through quotients*

We keep the same notations as in the previous section. Let H be a closed and normal subgroup of G; then G/H is a locally compact group. Details and proofs for the measure theory results below can be found in [29, Chap.3, 3.3(i) and 4.5].

Let μ be a right Haar measure on G, ν a right Haar measure on H with the topology induced by G, and λ the unique right Haar measure on G/H such that for every E in $\Sigma_{\mathsf{C}}(G)$

$$\int_{G/H}\int_H E(hx)\,d\nu(h)\,d\lambda([x]_H) = \int_G E\,d\mu \tag{4}$$

By defining

$$\begin{aligned}\widetilde{\ }: \Sigma_{\mathsf{C}}(G) &\to \Sigma_{\mathsf{C}}(G/H)\\ E &\mapsto \widetilde{E} : [x]_H \mapsto \textstyle\int_H E(hx)\,d\nu(h)\end{aligned}$$

the equality above is rewritten $\int \widetilde{E}\,d\lambda = \int E\,d\mu$.

In the particular case of $G = \mathbb{Z}$ with the discrete topology, let H be a non-trivial subgroup of $\mathbb{Z}$: $H = n\mathbb{Z}$ for some integer $n > 1$; for all $E \in \Sigma_{\mathsf{C}}(\mathbb{Z})$, $k \in \mathbb{Z}$, $\widetilde{E}([k]) = \sum_{j\in\mathbb{Z}} E(j\,n+k)$.

Theorem 2.26 *With the previous hypotheses and notations, the* $\widetilde{\ }$ *operator defined above and the Patterson function operator "commute", that is, for every* $E \in \Sigma_{\mathsf{C}}(G)$*,* $d^2(\widetilde{E}) = \widetilde{d^2(E)}$*:*

$$\begin{array}{ccc}\Sigma_{\mathsf{C}}(G) & \xrightarrow{d^2} & \Sigma_{\mathsf{C}}(G)\\ \big\downarrow{\scriptstyle \sim} & & \big\downarrow{\scriptstyle \sim}\\ L^1(G/H) & \xrightarrow{d^2} & L^1(G/H)\end{array}$$

Proof: We reuse two results of [29, Chap.3, 5.3], namely that $\widetilde{\ }: \Sigma_{\mathsf{C}}(G) \to \Sigma_{\mathsf{C}}(G/H)$ is a morphism of algebras with the convolution product, and that I and d^2 commute. Thus, for every $E \in \Sigma_{\mathsf{C}}(G)$, $d^2(\widetilde{E}) = I(\widetilde{E}) * \widetilde{E} = \widetilde{I(E)} * \widetilde{E} = \widetilde{I(E) * E} = \widetilde{d^2(E)}$. ■

Corollary 2.27 *Under the same notations and hypotheses as the previous theorem, if* $E_1, \ldots, E_s$ *in* $\Sigma_{\mathsf{C}}(G)$ *are homometric, then* $\widetilde{E_1}, \ldots, \widetilde{E_s}$ *are homometric in* $\Sigma_{\mathsf{C}}(G/H)$*.*

Example 2.28 $A = \{0,1,2,6,8,11\}$ and $B = \{0,1,6,7,9,11\}$ are Z-related in $\mathbb{Z}$, so their projections $\pi(A) = \{0,1,2,6,8,11\}_{12}$ and $\pi(B) = \{0,1,6,7,9,11\}_{12}$ are Z-related in $\mathbb{Z}_{12}$. Actually, the projections $\{0,1,2,6,8,11\}_n$ and $\{0,1,6,7,9,11\}_n$ are homometric for every $n \in \mathbb{N}, n \geqslant 2$, which we will use in 3.4; and they collapse into multisets for $n \leqslant 11$.

Example 2.29 In general, non-triviality is not preserved through quotients. The sets $A = \{0,1,2,3,4,6,7,8,11\}$ and $B = \{0,1,4,5,6,7,8,9,11\}$ are Z-related in $\mathbb{Z}$, and so are their projections on $\mathbb{Z}_{12}$; however, these projections are related by transposition, namely $\pi(B) = T_5(\pi(A))$. It is easy to see that for any Z-relation of subsets of $\mathbb{Z}$ one can always find a n' such that for every $n \geqslant n'$ the non-triviality of a Z-relation is preserved mod

n. In this case, $n' = 12$ is enough: this follows from the fact that in A there are three consecutive integers, a feature invariant under transposition and inversion, while there is no such configuration in B.

A loose but always valid choice for n' is $n' = 2(\max(A) - \min(A)) = 2(\max(B) - \min(B))$.

Note that the converse of Corollary 2.27 is not true: $A = \{0,1,2,5\}_8$ and $B = \{3,4,6,7\}_8$ are Z-related in $\mathbb{Z}_8$, but for every A', B' subsets of $\mathbb{Z}$ such that $\pi(A') = A$ and $\pi(B') = B$, it is easy to see that $\mathrm{diam}(A') \neq \mathrm{diam}(B')$, where diam denotes the diameter, hence A' and B' are not Z-related.

2.7. *The hexachord theorem*

2.7.1. *Patterson functions of generalized hexachords*

The hexachord theorem has been significantly popular in the litterature – see [28, Chapter V, 5.16], [22, section 6.6], [7]. Since it is actually a feature of Patterson functions, we propose here a restatement in the framework of locally compact (not necessarily commutative) GIS, and add a few geometric remarks.

$G, \mathcal{A}, \mu$ are defined as above. We will additionally assume in this subsection that $\mu(G)$ is finite, which is equivalent to the compactness of G.

The initial form of the hexachord theorem by Milton Babbitt is an invariance property of the interval vector by complementation. Wherever there is no ambiguity, $\mathbf{1}_G$ will be written[1] 1, and for every $a \in \mathbb{C}$, $a\mathbf{1}_G$ will be written a. For every measurable subset $A \subset G$, $\mathbf{1}_{A^C} = 1 - \mathbf{1}_A$, where $A^C = G \setminus A$, hence we can naturally extend the complement function to $\Sigma_{\mathsf{C}}(G)$, which we define as $C : E \mapsto 1 - E$. This extension allows us to express a generalization of the hexachord theorem, which results immediately from the following lemma.

LEMMA 2.30 *For every E in $\Sigma_{\mathsf{C}}(G)$, for every $a \in \mathbb{R}$, $d^2(a - E) = a^2\mu(G) - 2a\mathrm{Re}(\int E d\mu) + d^2(E)$. In particular, for $a = 1$, $d^2(C(E)) = \mu(G) - 2\mathrm{Re}(\int E d\mu) + d^2(E)$.*

Proof: The inversion I is linear and $I(a) = \bar{a} = a$, so $d^2(a - E) = I(a - E) * (a - E) = a * a - a * E - I(E) * a + I(E) * E = a^2\mu(G) - a\int E d\mu - a\int \overline{E} d\mu + d^2(E) = a^2\mu(G) - 2a\mathrm{Re}(\int E d\mu) + d^2(E)$. ■

THEOREM 2.31 [Generalized hexachord theorem] *For every E in $\Sigma_{\mathsf{C}}(G)$, $d^2(C(E)) = d^2(E)$ if and only if* $\mathrm{Re}(\int E d\mu) = \mu(G)/2$.

In the non-commutative case, this theorem admits two versions, i.e. it holds with either left or right interval content.

From a geometric point of view, C is the central symmetry relative to constant map 1/2; this means that the hexachord theorem is a condition of invariance of the Patterson function under this kind of symmetry — see Figure 6 — just like its invariance under I, but that is valid only under some normalization condition. If E is a $\{0,1\}$-valued map, i.e. E is the characteristic map of a measurable set $A \subset G$, this normalization condition requires that $\mu(A) = \mu(G)/2$, which in the case where G is discrete means that the cardinality of A is half the cardinality of G, which is already the original result.

[1] All the more so since without loss of generality, one can assume $\mu(G) = 1$.

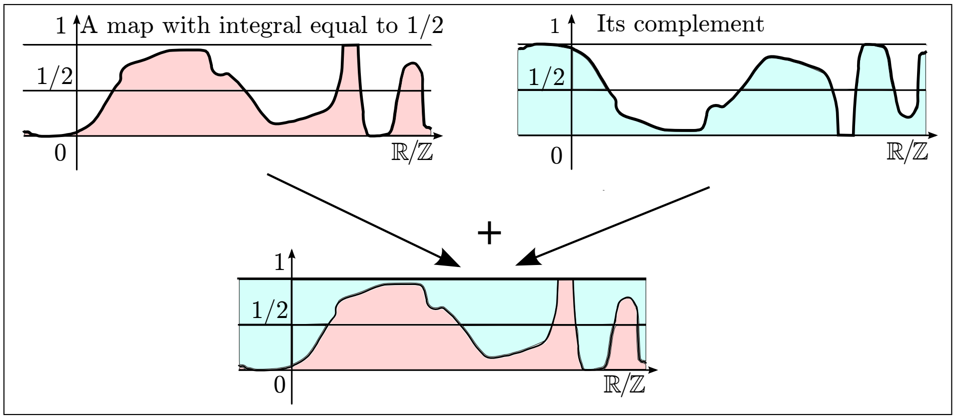

Figure 6. An illustration of the generalized hexachord theorem in the case of $G = \mathbb{R}/\mathbb{Z}$.

A more general formulation of the hexachord theorem – see [28], is computing the difference between the interval contents of a function and of its complement. It entails immediately that homometry is preserved by the complement operator C.

Corollary 2.32 *For every E in $\Sigma_\mathsf{C}(G)$, $d^2(E) - d^2(C(E))$ is a constant map.*

Proof: This results immediately from Lemma 2.30. ∎

Corollary 2.33 *For every $E_1, \ldots, E_s$ in $\Sigma_\mathsf{C}(G)$, $E_1, \ldots, E_s$ are homometric if and only if $C(E_1), \ldots, C(E_s)$ are homometric.*

A previous generalization of Babbitt's hexachord theorem to the unit circle is the subject of [7], but it cannot be further generalized for lack of reference to an integration theory and generalized notion of interval. Nevertheless, the paper mentions the problem of an hexachord theorem on the sphere S^2; unfortunately, since there is no topological group structure on the sphere S^2 (with its usual topology), the notions of interval and interval content in a Generalized Interval System are meaningless.[1,2]

2.7.2. Some examples of the generalized hexachord theorem

- Musical scales can be modelized as elements of a torus, which is the space of a GIS under transposition. Say we define the set of 'in tune' scales as major scales whose maximal deviation from a well-tempered major scale does not exceed 10 cents, e.g. the 'in tune' D major scales would be in $[190, 210] \times [390, 410] \times [590, 610] \times [690, 710] \times [890, 910] \times [1090, 1110] \times [90, 110]$ where each pc is given in cents. So the reunion ITS of all 12 'in tune' major scales is a subset of the torus $\mathbb{T}^7 = (\mathbb{R}/1200\,\mathbb{Z})^7$, with measure $1/60^7$ of the whole torus. Now the complement OTS (out of tune scales) has the same interval content, up to a constant.
- We have explained why, for lack of a group structure, we cannot hope to give a hexachord theorem in the sphere S^2. But in 4 dimensions, the sphere S^3 is a compact Lie group, for instance one can set $G = SU(2) = S^3$:

[1]Only the spheres S^1 (the circle), S^3 (in dimension 4), and in some measure S^7 may be provided with a group structure and a Haar measure compatible with their natural topology.
[2]It is conceivable that a more general notion of interval could be defined as geodesics on manifolds.

Definition 2.34 *The group* $SU(2)$ *is the set of complex matrices* $\begin{pmatrix} z_1 & -z_2 \\ \overline{z}_2 & \overline{z}_1 \end{pmatrix}$ *with determinant 1. As a set it coincides with the sphere in* $\mathbb{C}^2 : \{|z_1|^2 + |z_2|^2 = 1\}$*, e.g. the sphere* S^3 *in* $\mathbb{R}^4$*.*

The group operation is then simply matrix multiplication. It can be shown that, parametrizing S^3 with $z_1 = \cos\theta\, e^{i\phi}, z_2 = \sin\theta\, e^{i\psi}$ with $\theta \in [0, \pi/2], 0 \leq \phi, \psi \leq 2\pi$, the Haar measure is (up to a constant) $\mu_1 = \sin 2\theta\, d\theta\, d\phi\, d\psi$.

With this measure, the hexachord theorem with either right or left interval content hold on S^3.

- We can now turn back to discrete, but non abelian, groups. The Haar measure is the counting measure. For instance, let G be the dihedral group over $\mathbb{Z}_{12}$, which makes a GIS for instance on the space of major and minor triads. For the sake of simplcicity, let G act on itself. A very simple 'hexachord' is the cyclic subgroup T, isomorphic with $\mathbb{Z}_{12}$ (transpositions). The interval vector on T is computed immediately with the following general proposition:

Proposition 2.35 *Let* H *be a subgroup of* G*. Then*

$$^l\mathbf{iv}(H)(g) = {}^r\mathbf{iv}(H)(g) = \begin{cases} \mu(H) & \textit{when } g \in H \\ 0 & \textit{else} \end{cases}.$$

Our generalized hexachord theorem now states that the complement of T (i.e. the inversions) share the same interval vector. The transfer principle from the group to the space of the GIS means that there as many transformations (intervals) between a given triad and the major triads, as there are between this triad and the minor triads.

For a less trivial case, try for instance the diedral subgroup with 12 elements (or equivalently the major triads with root on a whole-tone scales and their inversions), or a random subset of G, for which $^r\mathbf{iv}$ will be different from $^l\mathbf{iv}$ in general.[1]

3. The phase retrieval problem

We have seen so far many properties of the Patterson function of a distribution and the interval contents of a measurable subset of finite measure. In this section, we will focus on the reconstruction problem.

This problem consists in determining whether a given integrable distribution over a locally-compact topological group fitted with its Haar measure, can be uniquely reconstructed – up to translation and inversion, or up to translation – from its Patterson function, and in case it cannot, what are the distributions non-trivially homometric with the given one.

[1]This raises the interesting general concept of those subsets $A \subset G$ for which $^r\mathbf{iv} = {}^l\mathbf{iv}$. For instance, in the diedral group above, there are 2,010 4-elements subsets (out of 10,626) such that $^r\mathbf{iv} = {}^l\mathbf{iv}$. For more examples, see [5].

3.1. Definition of the problem

Let G be a locally compact group with a right-Haar measure μ.

We recall the notation $\mathcal{D}(G)$ for the image in $\mathrm{Aut}(\Sigma_{\mathsf{C}}(G))$ of $D(G)$, the generalized dihedral group on G. Let H be the largest subgroup of $\mathcal{D}(G)$ such that the Patterson function is constant on the orbits of the action of H on $\Sigma_{\mathsf{C}}(G)$, that is for every P in H, every E in $\Sigma_{\mathsf{C}}(G)$, $d^2(P(E)) = d^2(E)$. According to Proposition 2.9, when G is unimodular, H is the group of left transposition operators[1] $\{T_g, g \in G\}$, and when G is abelian, $H = \mathcal{D}(G)$.

Definition 3.1 *The* ***phase retrieval problem*** *consists of*

(1) *determining for every $E \in \Sigma_{\mathsf{C}}(G)$, whether there is some $F \in \Sigma_{\mathsf{C}}(G)$ non-trivially Z-related to E; if there is no such F, one says that E can be* ***uniquely retrieved*** *from its Patterson function up to H;*
(2) *determining, for every $E \in \Sigma_{\mathsf{C}}(G)$ that cannot be uniquely retrieved, a family $F = (F_i)_{i \in I}$ of $\Sigma_{\mathsf{C}}(G)$ such $F \cup (E)$ is a maximal family of non-trivially Z-related distributions.*

In this definition, H is useful in moving out all trivial Z-relatives.

One defines likewise a "restricted" phase retrieval on $\widetilde{\mathcal{A}}$, wherein H is defined as the largest subgroup of $D(G)$ such that for all $A \in \widetilde{\mathcal{A}}, P \in H$, $\mathbf{iv}(P(A)) = \mathbf{iv}(A)$. This restricted phase retrieval problem is identical to the approach used by Forte for classifying pitch class sets in his musical set theory, whereas the definition with distributions comes from crystallography.

3.2. Alternative formulations

We have defined the most general notion of homometry in terms of Patterson functions. But in a number of practical situations, the computations – and indeed the comprehension of the process – are made easier by using the appropriate algebraic tools. A summary of these formulations is shown in Figure 7 at the end of this section.

3.2.1. Polynomials

In the case of distributions on the group $\mathbb{Z}_n$, i.e. maps from $\mathbb{Z}_n$ to some field K, we deal with the algebra $(K^{\mathbb{Z}_n}, +, ., *)$, of which the product law $*$ is essential in defining the Patterson function. It is possible to replace this algebra by the algebra of polynomials.

Definition 3.2 *The characteristic polynomial of a subset $A \subset \mathbb{Z}_n$ is $A(x) = \sum_{k \in A} x^k \in K[x] = K[X]/(X^n - 1)$, where we note $x = X \mod X^n - 1$. More generally, for any distribution $E : \mathbb{Z}_n \to K, E = \sum e_k \delta_k$, we define $E(x) = \sum_{k \in \mathbb{Z}_n} e_k x^k$.*

Proposition 3.3 *The above transformation is an algebra isomorphism between $(K^{\mathbb{Z}_n}, +, ., *)$ and $(K[x], +, ., \times)$, namely $(E * F)(x) = E(x) \times F(x)$.*

Essentially, the translation operator on subsets turns into multiplication by x: $T(A)(x) = x \times A(x)$. This transformation was introduced by Redei *et alii* around 1950

[1] H does not contain I because I does not preserve intervals, that is I does not preserve the interval content of pairs

in the study of tilings by translation. For us, the spotlight is on the Patterson function. Transposing the definitions already given yields the following.

Definition 3.4 *The reciprocal polynomial of* $E(x) = \sum_{k\in\mathbb{Z}_n} a_k x^k$ *is* $I(E)(x) = x^{n-1}\overline{E}(1/x) = \sum_{k\in\mathbb{Z}_n} \overline{a}_k x^{n-k}$. *The Patterson polynomial function associated with the distribution* E *is* $d^2(E)(x) = E(x)I(E)(x) = \sum_{k\in\mathbb{Z}_n} e_k x^k$ *with* $e_k = \sum_{p\in\mathbb{Z}_n} a_p \overline{a}_{p-k}$ *where the indices are computed modulo* n.

Notice that for any root ξ of $E(x)$, both ξ and $1/\xi$ are roots of $d^2(E)(x)$. Also, for $\xi \in S^1$ the unit circle, one gets $d^2(E)(\xi) = E(\xi)\overline{E(\xi)} = |E(\xi)|^2 \in \mathbb{R}_+$.

This approach can be further extended to any finite abelian group, or even any finitely generated abelian group, with polynomials in several variables – one for each element of a generators set of the group. Such constructions are essential in Polya's theory of combinatorics.

Any such polynomial, with degree $d < n$, can be determined uniquely with the values it takes in n different points. A happy choice is to evaluate $E(x)$ in the n^{th} roots of unity, since $E(e^{-2i\,j\pi/n}) = \sum_{k=0}^{n-1} a_k e^{-2i\,j\,k\pi/n}$ is exactly the Fourier transform of the map E.

3.2.2. Fourier transform

Historically, the idea of using the Fourier transform in the theory of intervals goes back to David Lewin's first paper [21]. It was refurbished in recent years, mainly starting from Quinn's PhD [25].

As we have mentioned before, in the case of characteristic functions of subsets of $\mathbb{Z}_n$, the Patterson functions boils down to the much simpler case of discrete Fourier transforms (DFT for short):

$$\widehat{\mathbf{1}_A}(t) = \sum_{k\in A} e^{-2i\pi kt/n}$$

This is indeed closer to the crystallographic origin of the Patterson function: as we mentioned in the introduction, the Fourier transform of the interval content is exactly the module of the DFT of the subset: since $\mathbf{iv}(A) = \mathbf{1}_A * \mathbf{1}_{-A}$, applying the Fourier transform yields $\widehat{\mathbf{iv}(A)} = \widehat{\mathbf{1}_A} \times \widehat{\mathbf{1}_{-A}} = \widehat{\mathbf{1}_A} \times \overline{\widehat{\mathbf{1}_A}} = |\widehat{\mathbf{1}_A}|^2$

It is perhaps interesting to mention the slightly more complicated equation used by Lewin: he aimed to retrieve a pc-set A, knowing pc-set B and the interval function between the two: $\mathbf{ifunc}(A,B)(t) = \#\{(a,b) \in A\times B, a+t=b\} = \mathbf{1}_B * \mathbf{1}_{-A}(t)$.

Since $\widehat{\mathbf{ifunc}(A,B)} = \widehat{\mathbf{1}}_B \times \overline{\widehat{\mathbf{1}_A}}$ the retrieval of A is always possible *provided that* $\widehat{\mathbf{1}}_B$ *does not vanish*. As we will see below, this condition arises in the discussion of k-decks. It is also instrumental in numerous problems, for instance rhythmic tilings. For practical retrieval, see the following section.

This approach can of course be extended to distributions on $\mathbb{Z}_n$, enlarging the codomain from $\{0,1\}$ to $\mathbb{C}$; but also to any commutative group instead of $\mathbb{Z}_n$, with the Fourier transform defined in terms of characters. This will be useful again below: in the study of k-decks we will introduce multi-dimensional Fourier transform.

An interesting alternative, introduced by Thomas Noll, is the case $\mathbb{Z}_n \to \mathbb{T}^n \subset \mathbb{C}^n$, modelizing ordered sequences of n notes, e.g. musical scales. On these topics, see [2, 4].

3.2.3. *Circulating matrices*

Circulating matrices of order n are defined as $\mathcal{C}_n(K)$ or $\mathcal{C}_n$ for short, the (commutative) algebra of matrices of the form $\begin{pmatrix} a_0 & a_{n-1} & \cdots & a_1 \\ a_1 & a_0 & \cdots & a_{n-1} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & \cdots & a_1 & a_0 \end{pmatrix}$, with coefficients in any field K.

This algebra is actually the algebra of polynomials in the matrix $J = \begin{pmatrix} 0 & \cdots & & 0 & 1 \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$.
J can be seen as the matrix for the elementary translation operator, $\mathbb{Z}_n \ni k \mapsto k+1$.

There is a natural mapping from distributions on $\mathbb{Z}_n$ onto $\mathcal{C}_n$, setting for any map $E \in K^{\mathbb{Z}_n}$ $a_k = E(k)$. For instance if $E = \mathbf{1}_A$, one gets $a_k = 1 \iff k \in A$ and $a_k = 0$ if $k \notin A$. What makes this bijection interesting is the following:

Proposition 3.5 *The above mapping is an isomorphism of algebras between* $(K^{\mathbb{Z}_n}, +, ., *)$ *and* $(\mathcal{C}_n, +, ., \times)$.

In other words, this matricial representation turns the cumbersome convolution product into the (slightly less cumbersome) matricial product.[1] This is easily checked by a direct computation, left to the reader. But the deep reason for this apparent miracle is linked to simultaneous diagonalization of these matrices:

Theorem 3.6 *Let* $\Omega = \frac{1}{\sqrt{n}}\left[e^{-2i\pi j\,k/n}\right]_{j,k=0\ldots n-1}$ *be the* ***Fourier matrix***.[2] *Then for any circulating matrix* S *associated with* $E : k \mapsto a_k$,

$$\Omega^{-1} S\, \Omega = \begin{pmatrix} \psi_0 & 0 & \cdots & 0 \\ 0 & \psi_1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \psi_{n-1} \end{pmatrix}.$$

where the $\psi_k = \widehat{E}(k)$ *are the Fourier coefficients of map* E.

Proof: It is straightforward to check that the colums of Ω are eigenvectors of the matrix J, with eigenvalue equal to the first element of the column. Hence $\Omega^{-1} J\, \Omega = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{-2i\pi/n} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & e^{-2i\pi(n-1)/n} \end{pmatrix}$ and for $S = a_0 I + a_1 J + \cdots a_{n-1} J^{n-1}$, $\Omega^{-1} S\, \Omega = \begin{pmatrix} \psi_0 & 0 & \cdots & 0 \\ 0 & \psi_1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \psi_{n-1} \end{pmatrix}$ where $\psi_k = \sum_{j=0}^{n} a_j e^{-2i\pi j\,k/n}$. ∎

So the miracle of the algebra morphism is just the fact that convolution $*$ is turned into ordinary product by the Fourier transform. Here the Fourier transform is read as a diagonal matrix, whose algebra is clearly isomorph to K^n with term-to-term product.

This matrix representation is still close enough to the musical material (the distribution can be read *verbatim* in the first column) and introduces the whole, powerful machinery of linear algebra. For some fascinating applications, see [3]. We will sample here a few results or techniques related to our topic:

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[1]Actually one of the authors first introduced this algebra as the natural representation of $K^{\mathbb{Z}_n}$ acting on itself by way of the adjunction operator $E \mapsto (F \mapsto E * F)$.
[2]Notice that ${}^t\overline{\Omega} = \Omega^{-1}$.

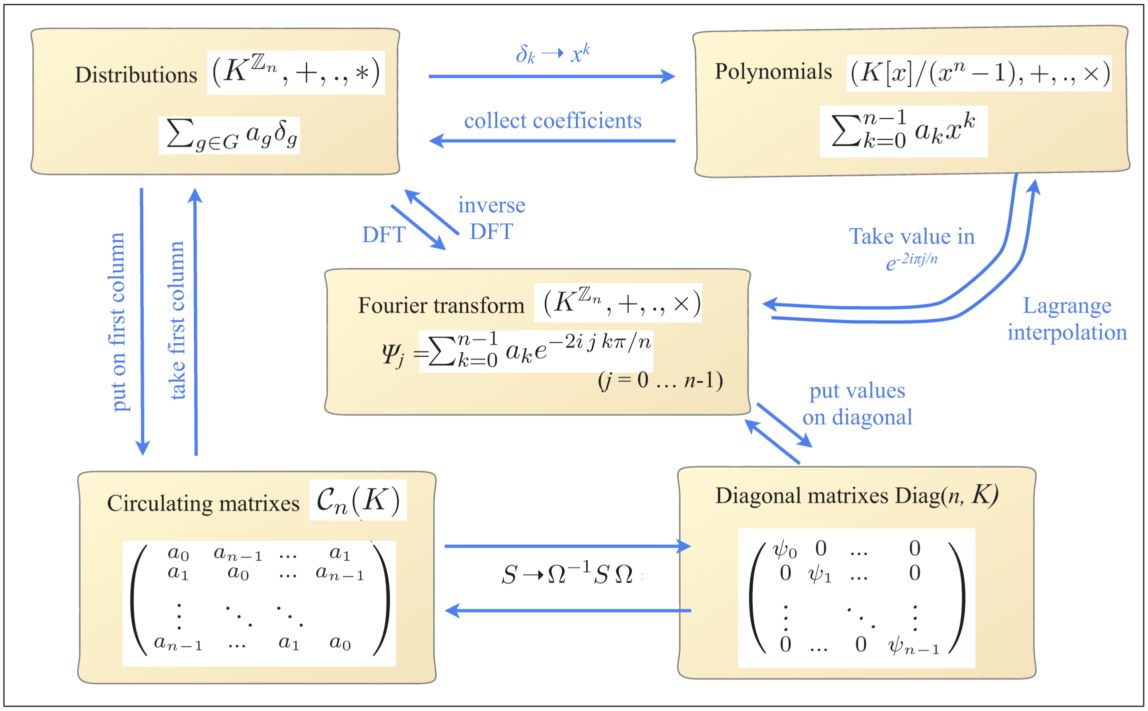

Figure 7. Isomorphisms between different algebras.

- The Fourier transform is non-vanishing iff the determinant of the matrix is different from 0. This is a straightforward criteria for all of Lewin's 'special cases', which was hitherto a messy catalogue of obscure conditions.
- The matrix associated with $\mathbf{ifunc}(A,B)$ (resp. $\mathbf{iv}(A)$) is ${}^tS_AS_B$ (resp. ${}^tS_AS_A$) where S_A, S_B are the matrices associated with $\mathbf{1}_A, \mathbf{1}_B$. For more general distributions (complex valued instead of 0,1) the conjugate must be used, e.g. if S is associated with map E, then its $\mathbf{iv}$ is associated with ${}^t\overline{S}S$.
- Hence "Lewin retrieval" (finding A from $\mathbf{ifunc}(A,B)$) is accomplished by inverting S_B (notice the condition on non-vanishing Fourier coefficients again here).
- S_A and S_B are homometric iff ${}^tS_AS_A = {}^tS_BS_B$. Diagonalizing, this in turn is equivalent to the existence of some matrix U such that
  (1) U is a circulating matrix [it diagonalizes with the same eigenvectors as all others] and
  (2) U is unitary: its eigenvalues lie on the unit circle (or equivalently: ${}^t\overline{U}U = I_n$, the identity matrix).
  (3) $S_B = US_A$.

We will elaborate below on these so-called spectral units, see 3.3. A straightforward example is J, the equation $S_B = JS_A$ expressing that $B = T_1(A)$. It is, however, much less easy to characterize the inversion operator I in terms of spectral units.

In this paragraph, we have restricted ourselves to (distributions on) the cyclic group; nonetheless we look forward to further research making use of group representation theory, of which this is but one of the most elementary examples. It might be the best access to the non-commutative case.

3.3. *Spectral units*

As the Patterson function of a bounded distribution with compact support is defined using a convolution product, it is natural to ask whether there exist distributions U such that the convolution with U does not change the value of the Patterson function, i.e. such that for every $E \in \Sigma_{\mathbb{C}}(G)$ $E * U * I(E * U) = E * I(E)$, which is equivalent – if $d^2(U)$ is well defined and G abelian – to $d^2(E) * d^2(U) = d^2(E)$, i.e. E and $E * U$ are homometric.

When the algebra $(L^1(\mu), +, ., *)$ does not have a unit, which is equivalent to G having a non-discrete topology [29, Chap.3, 5.6], in order to formulate the Z-relation and phase retrieval problem, it is necessary to enlarge the algebra to the space of distributions, which has been done in depth for $G = \mathbb{R}$ in [30].

When G is discrete and abelian, which we will assume henceforth, such distributions U are easily characterized as distributions homometric to the unit of $L^1(\mu)$, which is the Dirac distribution in e, the neutral element of G.

Definition 3.7 *A distribution $U \in \Sigma_{\mathbb{C}}(G)$ is called a spectral unit of G if $I(U) * U = \delta_e$.*

Proposition 3.8 *The set of spectral units of G is a subgroup of the group of invertible elements of the algebra $L^1(G)$.*

Rosenblatt has proven that any pair of homometric distributions is connected by a spectral unit.[1] Hence, in a way, enumerating all spectral units would solve the phase-retrieval problem. In practice it is not so, because we do not want $E * U$ to be just any distribution; for instance for pc-sets, we would want its codomain to be $\{0, 1\}$. As we will see below, even in the simplest case of distributions in $\mathbb{Z}_n$, this is far from obvious.

3.4. *Phase retrieval in the case of cyclic groups: the beltway problem*

3.4.1. *Spectral units of $\mathbb{Z}_n$*

Putting together circular matrices and spectral units, we are looking for unitary circulating matrices: ${}^t\overline{U^{-1}} = U \in \mathcal{C}_n$. Then any pair of circulating matrices S, T such that $S = UT$ provides homometric distributions. For convenience, let us generally denote by shortcase $s \in K^n$ the first column of uppercase $S \in \mathcal{C}_n$.

For instance, let the first column of S be $s = (1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0)$ (the C minor triad) and T defined by the first column $t = (1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0)$ (the C major triad). Then transposition is achieved my multiplication by $j = (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ and its powers, e.g. E flat minor triad is obtained with the matrix product J^3S, or equivalently $j^3 * s = (0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0)$. It is, however, much less straightforward to achieve inversion by way of a spectral unit: from C major to C minor we must have $U = S^{-1}T$, which yields $u = \frac{1}{15}(7, 4, -2, 1, 7, 4, -2, 1, -8, 4, -2, 1)$. Contrary to transposition, the spectral unit achieving inversion depends on the inversed subset (or distribution), and even more strangely, in general, such units are of infinite order in the group of units, like u in the example above.

Still, we managed to completely characterize rational[2] spectral units with finite order:

[1] There are some conditions about the field where the computations are made.

[2] For many musical applications, homometric distributions will be Z-related (multi)sets, and because of the matricial equation between them, a spectral unit connecting them must have rational

THEOREM 3.9 *Any spectral unit with finite order is completely determined by the values of the subset $\{\xi_j, j \mid n\}$ of its eigenvalues. The possibilities are listed* infra*:*

- $\xi_0 = \pm 1$;
- *When n is odd, for all $j \mid n$, ξ_j OR $-\xi_j$ is any power of $e^{2ij\pi/n}$.*
- *When n is even, ξ_j is any power of $e^{2ij\pi/n}$ if n/j is even, or any power of $e^{ij\pi/n}$ if n/j is odd.*

Then for any k coprime with n, $\xi_{kj} = \xi_j^k$ (or $-\xi_j^k$ in a specific case, cf. [6]).

For instance, for $n = 12$ the structure of the group is $\mathbb{Z}_{12} \times (\mathbb{Z}_6)^2 \times \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$, with 6,912 elements. In general, the group of rational spectral units with finite order is isomorphic with $\prod_{d|n} \mathbb{Z}/(\mathrm{lcm}(2, d)\mathbb{Z})$.

Notice the similarity with 5.1.2 below. Proofs and details can be found in [6].

3.4.2. Existence of non-trivially Z-related subsets of $\mathbb{Z}_n$

THEOREM 3.10 *Let $n \in \mathbb{N}$ with $n \geqslant 2$. There exist A, B non-trivially Z-related subsets of $\mathbb{Z}_n$ if and only if $n = 8$ or $n = 10$ or $n \geqslant 12$.*

Proof: If $n = 8$, sets A, B that fit are given in the example 2.12: $\{0, 1, 3, 4\}_8, \{2, 5, 6, 7\}_8$. If $n = 10$, $A = \{0, 1, 3, 4, 8\}_{10}, B = \{2, 5, 6, 7, 9\}_{10}$ fit. It is easily seen that these two cases are instances of the (Generalized) Hexachord Theorem 2.31; the non-triviality of the Z-relation comes from the fact that there is a sequence of three consecutive elements in B, whereas there is no such sequence in A.

Let us suppose now that $n \geqslant 12$; we note $\pi_n : \mathbb{Z} \to \mathbb{Z}_n$ the canonical projection. $A = \{0, 1, 2, 6, 8, 11\}$ and $B = \{0, 1, 6, 7, 9, 11\}$ are Z-related in $\mathbb{Z}$, so using Corollary 2.27, $\pi_n(A)$ and $\pi_n(B)$ are Z-related; moreover, there is a sequence of three (four in the case $n = 12$) consecutive integers in $\pi_n(A)$, whereas there is no such sequence in $\pi_n(B)$, so this Z-relation is not trivial.

Conversely, for $n \leqslant 7$, $n = 9$ and $n = 11$, it is easy to check by computer search that there are no non-trivially Z-related subsets in $\mathbb{Z}_n$. ■

3.5. Is there a group action representing the Z-relation?

An appealing formulation of the phase retrieval problem is asking whether there is a non-trivial group action on $\Sigma_{\mathbb{C}}(G)$ wherein the orbits are the equivalence classes of the homometry, and whether there is a non-trivial group action on the set of elements of $\mathcal{A}$ of finite measure wherein the orbits are the equivalence classes of the Z-relation.

A "trivial group action" can always be achieved with the direct sum of the permutation groups of the equivalence classes, which is both a huge and uninteresting group. Precluding this is essential in practice if one is to use properties of group actions of which both

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coefficients. Conversely of course, a rational spectral unit will not necessarily yield integer coefficients when multiplied with the characteristic function of a (multi)set. Finally, the whole group of rational (or real) spectral unit matrices can be described implicitely by the equations $(E_k) : \sum_{j=0}^{n-1} a_j a_{j+k} = 0, k = 1 \dots \lfloor \frac{n-1}{2} \rfloor$, and the condition $\sum a_j^2 = 1$, where indices are taken modulo n. For instance, for $n = 3$ the group of real spectral units is the pair of circles made of the matrices $\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$ with $a^2 + b^2 + c^2 = 1$ and $a + b + c = \pm 1$.

the group and the set are finite, for instance computing effectively the number of orbits using the equation of Burnside-Frobenius. We prove below that, in essence, there is no reasonable group action whose orbits are the homometric classes.

THEOREM 3.11 *Let $n \in \mathbb{N}$ with $n \geqslant 2$. If $n = 8$, $n = 10$ or $n \geqslant 12$, then for every field K and for every subgroup H of the linear group $\mathrm{GL}_n(K)$ such that the natural group action of H on $\mathcal{P}(\mathbb{Z}_n)$ identified with $\{0,1\}^n$ is well-defined, the orbits of this group action are not identical with the equivalence classes of the Z-relation.*

Proof: We suppose that $n = 8$, $n = 10$ or $n \geqslant 12$. Let K be a field, let H be a subgroup of $\mathrm{GL}_n(K)$ such that the natural group action of H on K^n can be restricted to a group action of H on $\{0_K, 1_K\}^n$; note that this restriction is well-defined if and only if $\{0_K, 1_K\}^n$ is a union of some orbits of the group action of H on K^n.

We note the natural injective group morphism into permutation matrices

$$\begin{aligned} P\colon\ S(\mathbb{Z}_n) &\to \mathrm{GL}_n(K) \\ \sigma &\mapsto P_\sigma = (\delta_{i,\sigma(j)})_{i,j\in\mathbb{Z}_n} \end{aligned}$$

From theorem 3.10, there exist two non-trivially Z-related subsets A, B of $\mathbb{Z}_n$. If we assume that the orbits of H are the classes of Z-related sets, B is in the orbit $[A]_H$ of A, so there exists $M \in H$ such that $M\mathbf{1}_A = \mathbf{1}_B$, and since the homometry between $\mathbf{1}_A$ and $\mathbf{1}_B$ is not trivial, M is not in $P(D(\mathbb{Z}_n))$.

On the other hand, we get from Lemma 2.18 that any distribution with codomain $\{0,1\}$ homometric to $\mathbf{1}_{\{0\}}$ is a $\mathbf{1}_{\{k\}}$ for some $k \in \mathbb{Z}_n$. In particular, as $M\mathbf{1}_{\{0\}}$ and $\mathbf{1}_{\{0\}}$ are homometric, there is a $k \in \mathbb{Z}_n$ such that $M\mathbf{1}_{\{0\}} = \mathbf{1}_{\{k\}}$. Identifying the distributions with circulating matrices (cf. 3.2.3), we find $MI_n = J^k$, so $M \in P(D(\mathbb{Z}_n))$, which leads to a contradiction. ∎

4. Homometry and Z-relation of higher order

4.1. k-vector, k-deck and k-Deck

We have seen that the Patterson function does not, in general, provide enough information for the reconstruction of a distribution. So we need to extend these concepts far enough to describe exactly the content of our distribution.

DEFINITION 4.1 *Let H be a subgroup of $S(\mathbb{Z}_n)$. Let us define a H-**copy** of a set $S \subset \mathbb{Z}_n$ as any set of the form $h(S)$, with $h \in H$.*

Two interesting cases are $H = T$, the cyclic group of transpositions; and $H = D$, the dihedral group of transpositions and inversions.

We begin by noticing that the interval vector of a set A is simply counting, in its i-th place, how many D-copies of the set $\{0, i\}$ are embedded in A, and this correspond exactly to $\#(A\cap(A-i))$. Analogously, the coefficient of δ_i (or x^i in polynomial representation) of the Patterson function of a distribution $\mathbf{1}_A$ tells us how many D-copies of the distributions $\delta_0 + \delta_i$ are embedded in $\mathbf{1}_A$, which still correspond exactly to $\#(A \cap (A - i))$. We may now ask, more generally, how many D-copies of some general k-subsets are contained in A. This has been done on the musical side in [22] and then [11] and [14].

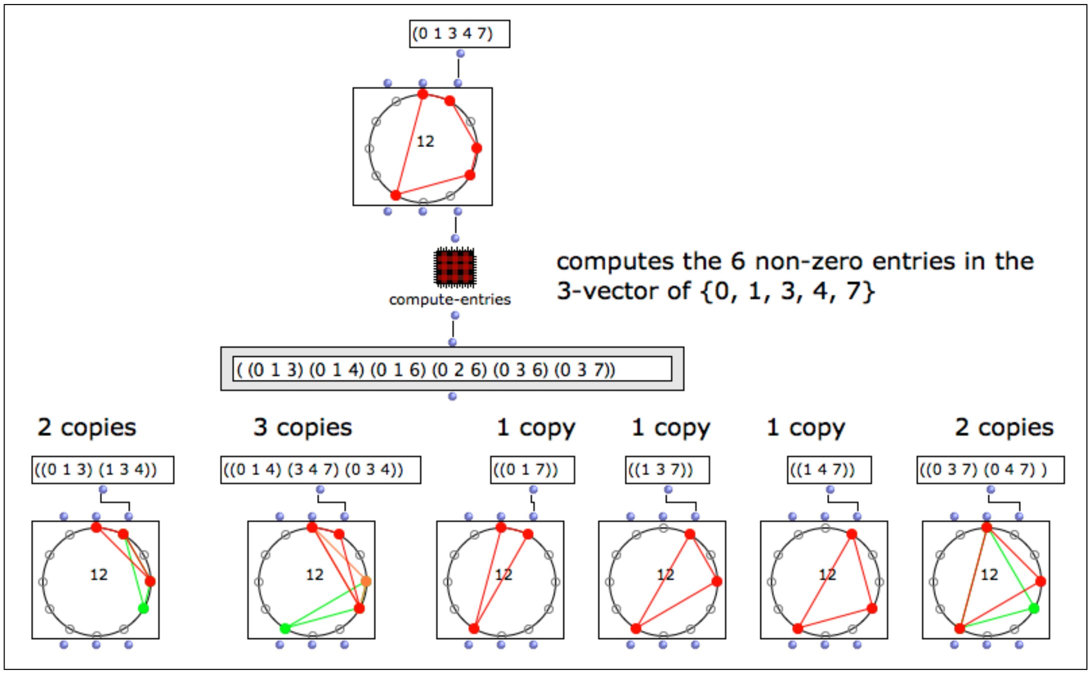

Figure 8. Computation of a 3-vector in $\mathbb{Z}_{12}$.

Following these works, we define the concept of k-**vector**[1]:

DEFINITION 4.2 *Given a set $A \subset \mathbb{Z}_n$, if S is a k-set, we call k-**vector** of A, and we denote by $\mathbf{mv}^k(A)_S$, the number of D-copies of S embedded in A.*

Example 4.3 The set $A = \{0, 1, 3, 4, 7\}_{12}$ has essentially only 6 non-zero entries in its 3-vector, as shown in Figure 8.

$$\begin{array}{ll} \mathbf{mv}^3(A)_{\{0,1,3\}_{12}} = 2 & \mathbf{mv}^3(A)_{\{0,1,4\}_{12}} = 3 \\ \mathbf{mv}^3(A)_{\{0,1,6\}_{12}} = 1 & \mathbf{mv}^3(A)_{\{0,2,6\}_{12}} = 1 \\ \mathbf{mv}^3(A)_{\{0,3,6\}_{12}} = 1 & \mathbf{mv}^3(A)_{\{0,3,7\}_{12}} = 2 \end{array}$$

Indeed, $\mathbf{mv}^3(A)_{\{0,1,3\}} = 2$ since there are two D-copies of $\{0, 1, 3\}_{12}$ embedded in A (they are $\{0, 1, 3\}_{12}$ and $\{1, 3, 4\}_{12}$); $\mathbf{mv}^3(A)_{\{0,1,4\}} = 3$ since there are three D-copies of $\{0, 1, 4\}_{12}$ embedded in A (they are $\{0, 1, 4\}_{12}$, $\{0, 3, 4\}_{12}$ and $\{3, 4, 7\}_{12}$); and so on.

Since $\mathbf{iv}(A)_h = \mathbf{mv}^2(A)_{\{0,h\}}$, this definitions extends the concept of interval vector. Analogously we define the concept of k-**deck**:

DEFINITION 4.4 *Let G be a locally compact group, and let $E = \sum_{g \in G} e_g \delta_g$ be a real distribution on G. We call*

[1]Notice that this definition does not depend on the set S, but on the set-class $[S]_D$. So there are as many essentially significant entries in the k-vector, as the number of set classes of cardinality k in $\mathbb{Z}_n$.

DEFINITION 4.5 *k-deck of E the function* $d^k(E)\colon G^{k-1} \to \mathbb{Q}$ *defined by*

$$d^k(E)(s_1, \ldots, s_{k-1}) = \sum_{g \in G} e_g e_{g+s_1} e_{g+s_2} \cdots e_{g+s_{k-1}}. \tag{5}$$

Notice that, since $E * E' = \sum_{g\in G}\sum_{h\in G} e_g e_{-h}\delta_{g-h} = \sum_{t\in G}\left(\sum_{s\in G} e_s e_{t+s}\right)\delta_t$, when $k = 2$, $d^k(E)(s) = \sum_{g\in G} e_g e_{g+s}$ is exactly the s-th term of the Patterson function of E, and thus the k-deck extends the Patterson function.

Now, let $G = \mathbb{Z}_n$ and $E = \mathbf{1}_A$; then all the e_g's are either 0 (if $g \in A$) or 1 (otherwise), and thus $d^k(\mathbf{1}_A)(s_1, \ldots, s_{k-1}) = \#(A \cap (A - s_1) \cap \ldots \cap (A - s_{k-1}))$, which is non zero if and only if there's a T-copy of $\{0, s_1, \ldots, s_{k-1}\}$ in A. In other words, the k-deck of $\mathbf{1}_A$ tells us how many T-copies of $\{0, s_1, \ldots, s_{k-1}\}$ are contained in A.

These two definitions extend (respectively) the concept of interval vector and the concept of Patterson function. Indeed $\mathbf{iv}(A)_h = \mathbf{mv}^2(A)_{\{0,h\}}$, and $d^2(A)(s) = \#(A \cap (A - s))$ is the coefficient of δ_s in the Patterson function of A.

We see that the k-vector and the k-deck are quite similar objects, with the difference that the first one counts the D-copies, while the last one counts the T-copies. We may solve this discrepancy by introducing the k-Deck (following [26]):

DEFINITION 4.6 *Let* $E = \sum_{g\in\mathbb{Z}_n} e_g\delta_g$ *be a real distribution on* $\mathbb{Z}_n$. *We call* k***-Deck*** *of* E *the function* $d^k(E)\colon (\mathbb{Z}_n)^{k-1} \to \mathbb{Q}$ *defined by* $D^k(E) = d^k(E) + d^k(I(E))$.

In this way, we get back the invariance by inversion, and since $d^k(\mathbf{1}_A)$ is the number of D-copies of $\{0, s_1, \ldots, s_{k-1}\}$ in A, the k-Deck is nothing more than the extension of the k-vector to a generic distribution.[1]

4.2. Z^k-relation, k-homometry, k-Homometry

As we have extended the interval vector and the Patterson function, we are now able to extend also the Z-relation and the homometry.

DEFINITION 4.7 *Sets* $A_1, \ldots, A_s$ *are* ***Z^M-related*** *if* $\mathbf{mv}^M(A_1)_S = \mathbf{mv}^M(A_2)_S = \ldots = \mathbf{mv}^M(A_s)_S$ *for all* $S \subseteq \mathbb{Z}_n$, $\#S = M$.

DEFINITION 4.8 *Distributions* $E_1, \ldots, E_s$ *are* k-***homometric*** *if* $d^k(E_1) = d^k(E_2) = \ldots = d^k(E_s)$.

DEFINITION 4.9 *Distributions* $E_1, \ldots, E_s$ *are* k-***Homometric*** *if* $D^k(E_1) = D^k(E_2) = \ldots = D^k(E_s)$.

Clearly, the Z^2-relation is the Z-relation and the 2-homometry (which is equivalent to 2-Homometry) is plain homometry. Again, for all these definitions, we will add the "non-trivially" prefix if the sets (or distributions) belong to different classes under the action of D (for Z^M-relation and k-Homometry) or T (for k-homometry). This vocabulary makes sense because of the following straightforward result.

[1]More precisely, if $I(\{0, s_1, \ldots, s_{k-1}\}) = T_h(\{0, s_1, \ldots, s_{k-1}\})$ for some h, each D-copy is counted twice. If we want complete accordance between the two definitions, we must treat separately that case. But this does not scupper the equivalence between $D^k(A)$ and $\mathbf{mv}^k(A)$. By the way, the Definition 4.6 of the k-Deck we in this form will be very useful later.

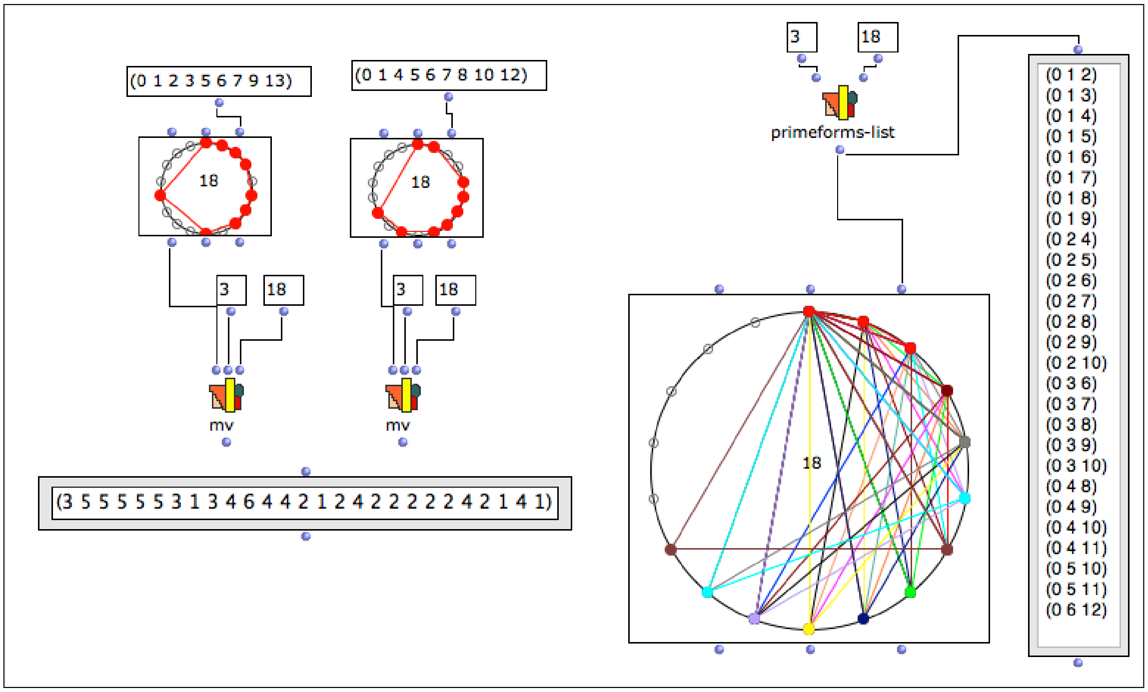

Figure 9. A non-trivial Z^3-relation in $\mathbb{Z}_{18}$. The two sets share the same 3-vector, whose entries specify the number of copies of the corresponding elements in the prime forms list (given in the right part of the figure). For instance, in both sets there are exactly 3 copies of $\{0,1,2\}_{18}$, 5 copies of $\{0,1,3\}_{18}$, and so on.

LEMMA 4.10

(i) *If* $A \subset \mathbb{Z}_n$ *and* $B = I(T_h(A))$ *or* $B = I^s(T_h(A))$, $s \in \{0,1\}$, $h \in \mathbb{Z}_n$, *then* $\mathbf{mv}^M(A)_S = \mathbf{mv}^M(B)_S$, *for all* $M \geqslant 2$, $S \subset \mathbb{Z}_n$, *such that* $\#S = M$.
(ii) *If* $E \in \mathbb{R}^{\mathbb{Z}_n}$ *and* $F = T_h(E)$, $h \in \mathbb{Z}_n$, *then* $d^k(E) \equiv d^k(F)$.
(iii) *If* $E \in \mathbb{R}^{\mathbb{Z}_n}$ *and* $F = I^s(T_h(E))$, $s \in \{0,1\}$, $h \in \mathbb{Z}_n$, *then* $D^k(E) \equiv D^k(F)$.

Proof: (i) is straightforward, since there is an obvious 1-to-1 correspondance between the 3-sets embedded in A and the 3-sets embedded in $I^s(T_h(A))$; (ii) and (iii) come from an easy direct computation. ■

Non-trivial Z^3-related sets exist, as first shown by Collins [11].

Example 4.11 Let us consider, in $\mathbb{Z}_{18}$, the two sets $A = \{0,1,2,3,5,6,7,9,13\}_{18}$ and $B = \{0,1,4,5,6,7,8,10,12\}_{18}$. They are not related by translation/inversion, but $\mathbf{mv}^3(A)_S = \mathbf{mv}^3(B)_S$ for all S, as illustrated by Figure 9.

4.3. Nesting

Following Jaming[1] [18], we notice that, if $E = \sum_{g\in\mathbb{Z}_n} e_g\delta_g$ is a non negative real distribution on $\mathbb{Z}_n$ ($e_g \geqslant 0$), then $\sum_{s_1,\ldots,s_{k-1}\in\mathbb{Z}_n} d^k(E)(s_1,\ldots,s_{k-1}) = \left(\sum_{g\in\mathbb{Z}_n} e_g\right)^k$ and so, if two positive distributions $E = \sum_{g\in\mathbb{Z}_n} e_g\delta_g$, $F = \sum_{g\in\mathbb{Z}_n} f_g\delta_g$ have the same k-deck, they surely satisfy $\sum_{g\in\mathbb{Z}_n} e_g = \sum_{g\in\mathbb{Z}_n} f_g$ i.e. they have the same 1-deck. Then we notice also that

$$\sum_{s_{k-1}\in\mathbb{Z}_n} d^k(E)(s_1,\ldots,s_{k-1}) = d^{k-1}(E)(s_1,\ldots,s_{k-2}) \sum_{g\in\mathbb{Z}_n} e_g \tag{6}$$

and thus we immediately have the following lemma:

Lemma 4.12 *Let $E, F \in \mathbb{Q}^{\mathbb{Z}_n}$. If $d^k(E) \equiv d^k(F)$ for some k, then $d^h(E) \equiv d^h(F)$ for all $h \leqslant k$.*

This lemma is crucial, since it states that, as k increases, the information given by the k-deck is more and more precise; in particular, the sets which share the same k-deck, as k increases, are *nested*. By definition of the k-Deck, this result applies equally to the case of D. So, the k-homometric sets and the k-Homometric sets, as k increase, are nested. On the musical side, the k-vector version of the Nesting Lemma has been independently developed by Collins [11], starting from a reconstruction formula given by Lewin [22].

Lemma 4.13 *Let A, B be sets in $\mathbb{Z}_n$. If $\mathbf{mv}^k(A) \equiv \mathbf{mv}^k(B)$ for some $k \leqslant \min(\#A, \#B)$, then $\mathbf{mv}^h(A) \equiv \mathbf{mv}^h(B)$ for all $h \leqslant k$.*

5. The Extended Phase Retrieval Problem

The **extended phase retrieval problem** deals precisely with the question of where this nesting stops. If we know that in $\mathbb{Z}_n$ there exist $(r-1)$-homometric sets but no r-homometric sets, it means that r-decks provide enough information for phase retrieval.

Definition 5.1 *The T-**reconstruction index** $r(n)$ is the minimum integer k for which there exist no k-homometric 0-1 distributions in $\mathbb{Z}_n$. The D-**reconstruction index** $R(n)$ is the minimum integer k for which there exist no k-Homometric 0-1 distributions in $\mathbb{Z}_n$. We define $r_{\mathbb{Q}}(n)$ and $R_{\mathbb{Q}}(n)$ analogously, but for general distributions in $\mathbb{Q}^{\mathbb{Z}_n}$.*

Clearly, $r(n) \leqslant r_{\mathbb{Q}}(n)$ and $R(n) \leqslant R_{\mathbb{Q}}(n)$. By the way, it is interesting to notice how $R(n)$ finds its musical mirror-image in the concept of “uniqueness of pitch class spaces”, independently developed by Collins in [11].

Direct computer search can give the values of $r(n)$, $R(n)$ for small n, but we need some algebra to explore the general cases.

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[1]We correct here two small typos in the paper, concerning the exponent of the norm and a sign of an inequality.

5.1. The k-deck problem

If E, F are k-homometric distributions, i.e. $d^k(E)(s_1, \ldots, s_{k-1}) = d^k(F)(s_1, \ldots, s_{k-1})$ for all $(s_1, \ldots, s_{k-1}) \in \mathbb{Z}_n^{k-1}$, then we can take the discrete Fourier transform of these k-decks, considered as functions in the $k-1$ variables $s_1, \ldots, s_{k-1}$.

It is then easy to check that the homometry condition is equivalent to

$$\widehat{E}(\omega_1)\widehat{E}(\omega_2)\cdots\widehat{E}(\omega_{k-1})\widehat{E}(-\omega_1 - \ldots - \omega_{k-1}) = \\ = \widehat{F}(\omega_1)\widehat{F}(\omega_2)\cdots\widehat{F}(\omega_{k-1})\widehat{F}(-\omega_1 - \ldots - \omega_{k-1}) \quad (7)$$

for every $(\omega_1, \ldots, \omega_{k-1}) \in \mathbb{Z}_n^{k-1}$.

We now assume that $E, F \in \mathbb{R}_+^{\mathbb{Z}_n}$, i.e. they are non negative distributions. In this case, $\widehat{E}(0) = \sum_{g \in \mathbb{Z}_g} e_g > 0$. By choosing $\omega_i = 0$ for all i, we get immediately that $(\widehat{E}(0))^k = (\widehat{F}(0))^k$, and then $\widehat{E}(0) = \widehat{F}(0)$. By choosing $\omega_1 = \omega$ arbitrary and $\omega_2 = \ldots = \omega_{k-1} = 0$ we reach the Patterson equality $\|\widehat{E}(\omega)\|^2 = \|\widehat{F}(\omega)\|^2 \quad \forall\omega$. This is not surprising (we know that the k-deck information is nested as k increase), but it tells us that $\operatorname{supp}\widehat{E} = \operatorname{supp}\widehat{F}$, i.e. either the two Fourier transforms are both nil, or they are both non-zero.[1] Moreover, the Patterson equality allows us to perform the substitution $\widehat{F}(\omega) = e^{i\phi(\omega)}\widehat{E}(\omega)$ and to get equivalentely (after simplifying)

$$\phi(\omega_1 + \omega_2 + \ldots + \omega_{k-1}) = \phi(\omega_1) + \phi(\omega_2) + \ldots + \phi(\omega_{k-1}) \mod 2\pi \quad (8)$$

which must be valid for all $\omega_1, \ldots, \omega_{k-1} \in \operatorname{supp}\widehat{E}$ such that $\omega_1 + \ldots + \omega_{k-1} \in \operatorname{supp}\widehat{E}$.

5.1.1. The Case $\operatorname{supp}\widehat{E} = \mathbb{Z}_n$

We can easily show that if the Fourier transform never vanishes on $\mathbb{Z}_n$, then the 3-deck suffices for the reconstruction. The 3-deck version of the (8) is

$$\phi(\omega_1 + \omega_2) = \phi(\omega_1) + \phi(\omega_2) \mod 2\pi \quad (9)$$

for *all* the $(\omega_1, \omega_2) \in \mathbb{Z}_n^2$, which tells us that the function $\psi\colon \omega \mapsto e^{i\phi(\omega)}$ is a character of $\mathbb{Z}_n$. Since we know the form of the characters of $\mathbb{Z}_n$, necessarily there exist a k_0 such that $\psi(\omega) = e^{2i\pi k_0 \omega/n}$. But this means that

$$\widehat{F}(\omega) = e^{i\phi(\omega)}\widehat{E}(\omega) = \psi(\omega)\widehat{E}(\omega) = e^{2i\pi k_0\omega/n}\widehat{E}(\omega) = \widehat{E * \delta_{k_0}}(\omega) = \widehat{T_{k_0}(E)}(\omega)$$

where we have applied the shift theorem for the DFT. Thus $F = T_{k_0}(E)$, which means that, if the Fourier transforms never vanish, two distributions with the same 3-deck are necessarily related by transposition, and the (extended) phase retrieval is succesful.

5.1.2. The Case $\operatorname{supp}\widehat{E} \neq \mathbb{Z}_n$

If $\widehat{E}(\omega)$ vanishes for some ω, the function $\phi(\omega)$ is not everywhere defined, the (9) is no more valid for all the $(\omega_1, \omega_2) \in \mathbb{Z}_n^2$, and thus ψ is no more a character, being defined

[1] We denote as $\operatorname{supp}\widehat{E}$ the support of $\widehat{E}$ i.e. the set of values on which $\widehat{E}$ does not vanish.

only on supp $\widehat{E}$. However, if we succeed in showing that we can extend $\psi(\omega) = e^{i\phi(\omega)}$ to a character on all $\mathbb{Z}_n$, we can apply the shift theorem again, and thus prove that F and E are related by transposition.

We will follow the lead of Jaming and Kolountzakis in [17], and we start by gathering some information about the position of the zeros of $\widehat{E}$.

Lemma 5.2 *If $\widehat{E}(\omega) = 0$ for some $\omega \neq 0$, then $\widehat{E}(\eta) = 0$ for all η such that* $\gcd(\omega, n) = \gcd(\eta, n)$.

Proof: First recall (see 3.2.1), denoting $\zeta_n = e^{2\pi i/n}$, that $\widehat{E}(\omega) = E(\zeta_n^\omega)$, i.e. computing the Fourier transform is equivalent to the evaluation of the polynomial $E(x)$ in the powers of an n-th primitive root of unity ζ_n.

If $\widehat{E}$ vanishes on ω, then $E(\zeta_n^\omega) = 0$, which means that $(x - \zeta_n^\omega)$ divides $E(x)$. But ζ_n^ω is a primitive $n/\gcd(\omega, n)$-root of the unity, and thus if $(x - \zeta_n^\omega)$ divides $E(x)$, necessarily all the cyclotomic polynomials $\Phi_{n/\gcd(\omega,n)}(x)$, which are irreducible in $\mathbb{Q}[x]$, divide $E(x)$ in $\mathbb{Q}[x]$, and in particular it will vanish also for all other roots of unity with the same order, i.e. $E(x) = 0$ for all $x = \zeta_n^\eta$ such that $\gcd(\omega, n) = \gcd(\eta, n)$. For such η, $\widehat{E}(\eta) = E(\zeta_n^\eta) = 0$. ■

This means that we can partition $\mathbb{Z}_n = \bigsqcup_{i \in \mathbb{Z}, i|n} \mathcal{C}_i$ where each $\mathcal{C}_i = \{a \in \mathbb{Z}_n : \gcd(a, n) = i\}$ and if a transform vanishes on a certain element of $\mathcal{C}_i$, then it must vanish on all the class $\mathcal{C}_i$.

Lemma 5.3 *The class $\mathcal{C}_i$ ($i < n$) of the partition of $\mathbb{Z}_n$ is isomorphic to the multiplicative group $\mathbb{Z}^*_{n/i}$.*

Proof: Consider the subgroup $\mathbb{Z}_{n/i}$ of $\mathbb{Z}_n$, in the sense of the injection $\iota\colon \mathbb{Z}_{n/i} \to \mathbb{Z}_n$ defined by $\iota([a]_{n/i}) := [ia]_n$, $a \in \mathbb{Z}$. The generic element of $\mathcal{C}_i$ is of the type $[ia]_n$, with $\gcd(a, n) = 1$, and thus we can apply ι^{-1} to get to $[a]_{n/i}$. Since a is coprime with n, it is also coprime with n/i, and thus $[a]_{n/i} \in \mathbb{Z}^*_{n/i}$. It is immediate to see that ι^{-1} is an isomorphism between $\mathcal{C}_i$ and $\mathbb{Z}^*_{n/i}$. ■

Example 5.4 We can easily decompose $\mathbb{Z}_{12} = \mathcal{C}_1 \sqcup \mathcal{C}_2 \sqcup \mathcal{C}_3 \sqcup \mathcal{C}_4 \sqcup \mathcal{C}_6 \sqcup \mathcal{C}_{12}$ where

$$\begin{array}{ll} \mathcal{C}_1 = \{[1]_{12}, [5]_{12}, [7]_{12}, [11]_{12}\} = \mathbb{Z}^*_{12} & \mathcal{C}_4 = \{[4]_{12}, [8]_{12}\} \cong \mathbb{Z}^*_3 \\ \mathcal{C}_2 = \{[2]_{12}, [10]_{12}\} \cong \mathbb{Z}^*_6 & \mathcal{C}_6 = \{[6]_{12}\} \cong \mathbb{Z}^*_2 \\ \mathcal{C}_3 = \{[3]_{12}, [9]_{12}\} \cong \mathbb{Z}^*_4 & \mathcal{C}_{12} = \{[0]_{12}\} \end{array}$$

Proposition 5.5 *If p is prime, in $\mathbb{R}^{\mathbb{Z}_p}$ the 3-deck suffice for the reconstruction, i.e. if $d^3_E \equiv d^3_F$ then $F = T_{k_0}E$ for some $k_0 \in \mathbb{Z}_p$.*

Proof: We have just 2 classes $\mathcal{C}_i$: $\mathcal{C}_p = \{[0]_p\}$ and $\mathcal{C}_1 = \mathbb{Z}^*_p$. Since $\widehat{E}(0) > 0$ (we're assuming that E is a positive real distribution), we have 2 cases:

(1) supp $\widehat{E} = \mathbb{Z}_n$, which has already been seen in section 5.1.1.
(2) supp $\widehat{E} = \{[0]_n\}$, which means $\widehat{E} \equiv \widehat{F}$, since $\widehat{E}(0) = \widehat{F}(0)$ (the 3-homometry implies the 1-homometry). ■

With this kind of argumentations Pebody in [23] and [24], and Jaming and Kolountzakis in [17] have shown that the 3-deck suffices also in the cases $n = p^a$, $n = pq$, $n = p^2q$ and $n = pqr$ (p, q, r odd primes, $a > 2$). Pebody in [23] reaches a complete determination of the function $r_{\mathbb{Q}}(n)$ (see Theorem 5.9).

The most general case is addressed by Grünbaum and Moore (see [15] for the details of the proof), who reach the following theorem.

THEOREM 5.6 *Let* $E, F \in \mathbb{Q}^{\mathbb{Z}_n}$, $d_E^6 \equiv d_F^6$. *Then* $F = T_{k_0}(E)$ *for some* $k_0 \in \mathbb{Z}_n$.

Grünbaum and Moore in [15] also give an interesting corollary for 0-1 distributions.

COROLLARY 5.7 *Let* A, B *be two sets in* $\mathbb{Z}_n$. *If* $d_A^4 \equiv d_B^4$ *and* $\mathbb{Z}_n^* \subseteq supp\,\widehat{\mathbf{1}_A}$, *then* $A = T_{k_0}B$ *for some* $k_0 \in \mathbb{Z}_n$.

Unfortunately there is a technical assumption (the transform does not vanish on $\mathbb{Z}_n^*$) which one would like to dispense with. This seems to have been discarded of late, and the case of the k-deck seems to be considered as good as solved. But, actually, that assumption is a strong one.

So, it turns out that, while the behaviour of the functions $r_{\mathbb{Q}}(n)$ is completely known (see [23] for details), the same thing is *almost* true for $r(n)$: Pebody in [24] gives all the boundaries for odd n. Notwithstanding the aforementioned unwanted assumption, we know that $r(n) \leqslant 4$ for n even. Computer calculation shows that $r(2) = 1$, $r(4) = 2$, $r(6) = r(8) = r(10) = 3$; to complete the boundaries, we just have to prove that:

LEMMA 5.8 *If* n *is an even integer,* $n \geqslant 12$, *then* $r(n) \geqslant 4$.

Proof: Let $n = 2m$ and consider $A = \{0, 1, 2, \ldots, m-4, m-1, 2m-3, 2m-2\}_{2m}$ and $B = \{0, 1, 2, \ldots, m-4, m-2, m-1, 2m-3\}_{2m}$

Notice that B is obtained from A by a one-pitch shift of m (which is very similar to what Althuis and Göbel did in [1] to find some Z-related families). More precisely: $C = \{0, 1, 2, \ldots, m-4, m-1, 2m-3\}_{2m}$, $A = C \cup \{2m-2\}_{2m}$, $B = C \cup \{m-2\}_{2m}$. So we just need to show that there's a 1-to-1 correspondance between the 3-subsets of A containing $2m-2$ and the 3-subsets of B containing $m-2$. We shall give it explicitely. Let $a \in \{0, 1, 2, \ldots, m-4\}_{2m}$. Then the correspondance is the following one:

$$\begin{aligned}
\{2m-2, a, a+1\}_{2m} &\mapsto \{m-4-a, m-2, m-1\}_{2m}, \text{ for } a \neq [m-4]_{2m} \\
\{2m-2, a, a+k\}_{2m} &\mapsto \{3m–a–4–k, m–2–k, m–2\}_{2m}, \text{ for } k \in \{[2]_{2m}, \ldots, [m-4-a]_{2m}\} \\
\{a, m-1, 2m-2\}_{2m} &\mapsto \{a-1, m-2, 2m-3\}_{2m}, \text{ for } a \neq [0]_{2m} \\
\{0, m-1, 2m-2\}_{2m} &\mapsto \{m-2, 2m-3, m-4\}_{2m} \\
\{m-1, 2m-3, 2m-2\}_{2m} &\mapsto \{0, m-2, m-1\}_{2m} \\
\{a, 2m-3, 2m-2\}_{2m} &\mapsto \{m-2, m-5-a, m-4-a\}_{2m}, \text{ for } a \neq [m-4]_{2m} \\
\{m-4, 2m-3, 2m-2\}_{2m} &\mapsto \{2m-3, m-2, m-1\}_{2m}
\end{aligned}$$

Notice that, as requested, $2m-2$ is always present in the left 3-subsets and $m-2$ is always present in the right ones. So A and B have the same 3-deck.

To complete the proof, we notice that, if $n \geqslant 12$, the homometry is non-trivial - just look at the intervals between the pitch classes and at the order of the intervals bigger than 1 ($3, m-2, 2$ for A and $2, m-2, 3$ for B), which cannot be related by transposition if $m \geqslant 6$. Instead, for $n = 10$ the sets become $A = \{0, 1, 4, 7, 8\}$ and $B = \{0, 1, 3, 4, 7\}$ which are transpositionally related. The same thing happens for $n = 8$. ■

We are ready to summarize all the results in the following theorem.

THEOREM 5.9 *Let* p, q *be odd primes and let* α, β *be integers* $\alpha >= 1$, $\beta > 1$. *Then*

$$r_{\mathbb{Q}}(n) = \begin{cases} 1 & \textit{if } n = 1 \\ 2 & \textit{if } n = 2 \\ 3 & \textit{if } n = p^\alpha \textit{ or} \\ & \textit{if } n = pq \\ 4 & \textit{if } n \textit{ is any other} \\ & \textit{ odd number or} \\ & \textit{if } n = 2^\beta \textit{ or} \\ & \textit{if } n = 2p^\alpha \\ 5 & \textit{if } n = 2^\beta p^\alpha \\ 6 & \textit{if } n \textit{ is any other} \\ & \textit{ even number} \end{cases} \qquad r(n) = \begin{cases} 1 & \textit{if } n = 1,2,3 \\ 2 & \textit{if } n = 4,5 \\ 3 & \textit{if } n = p^\alpha > 5 \textit{ or} \\ & \textit{if } n \textit{ has less than } 4 \\ & \textit{ not-necessarily distinct} \\ & \textit{ odd prime factors or} \\ & \textit{if } n = 6,8,10 \\ 4 & \textit{if } n \textit{ is any other odd number, or} \\ & \textit{if } n \textit{ if any other even number} \\ & \textit{ (under the hypothesis of non-} \\ & \textit{ vanishing transform on } \mathbb{Z}_n^*\textit{)} \end{cases}$$

5.2. The k-Deck problem

Let us try to do the same thing with the k-Deck problem, which is the case we are most interested in, since there is an exact correspondence between k-Homometry and Z^M-relation. If E, F are k-Homometric distributions, $D^k(E)(s_1, \ldots, s_{k-1}) = D^k(F)(s_1, \ldots, s_{k-1})$ i.e. $d^k(E)(s_1, \ldots, s_{k-1}) + d^k(E)(-s_1, \ldots, -s_{k-1}) = d^k(F)(s_1, \ldots, s_{k-1}) + d^k(F)(-s_1, \ldots, -s_{k-1})$, and taking again the Fourier transform, we get

$$\mathrm{Re}\left(\widehat{E}(\omega_1)\widehat{E}(\omega_2)\cdots\widehat{E}(\omega_{k-1})\widehat{E}(-\omega_1 - \ldots - \omega_{k-1})\right) =$$
$$= \mathrm{Re}\left(\widehat{F}(\omega_1)\widehat{F}(\omega_2)\cdots\widehat{F}(\omega_{k-1})\widehat{F}(-\omega_1 - \ldots - \omega_{k-1})\right) \quad (10)$$

The difference between (7) and (10), the real parts, is the main obstacle in pursuing the analysis. Indeed, (10) leads to either one of the following equations:

$$\widehat{E}(\omega_1)\cdots\widehat{E}(\omega_{k-1})\overline{\widehat{E}(\omega_1 + \ldots + \omega_{k-1})} = \widehat{F}(\omega_1)\cdots\widehat{F}(\omega_{k-1})\overline{\widehat{F}(\omega_1 + \ldots + \omega_{k-1})} \quad (11)$$
$$\widehat{E}(\omega_1)\cdots\widehat{E}(\omega_{k-1})\overline{\widehat{E}(\omega_1 + \ldots + \omega_{k-1})} = \overline{\widehat{F}(\omega_1)\cdots\widehat{F}(\omega_{k-1})}\widehat{F}(\omega_1 + \ldots + \omega_{k-1}) \quad (12)$$

and things are complicated because (11) might be valid for some values of ω_i's while (12) might be valid for others.

By choosing $\omega_i = 0$ for all i, we still get to $\widehat{E}(0) = \widehat{F}(0)$, and by arbitrarily choosing $\omega_1 = \omega$ and $\omega_2 = \ldots = \omega_{k-1} = 0$ we get again the Patterson equality $\|\widehat{E}(\omega)\|^2 = \|\widehat{F}(\omega)\|^2 \quad \forall\omega$ which is little surprising, since the 2-deck and the 2-Deck coincide.

Considering these obstacles, the best we can do is to provide a list of computer-calculated values for $n \leqslant 37$:

Proposition 5.10

$$R(n) = \begin{cases} 1 & \textit{if } n = 1,2,3 \\ 2 & \textit{if } n = 4,5,6,7,9,11 \\ 3 & \textit{if } n = 8,10,12,13,14,15,16,17,19,22,23,25,29,31,37 \\ 4 & \textit{if } n = 18,20,21,24,26,27,28,30,32,33,34,35 \\ 5 & \textit{if } n = 36 \end{cases}$$

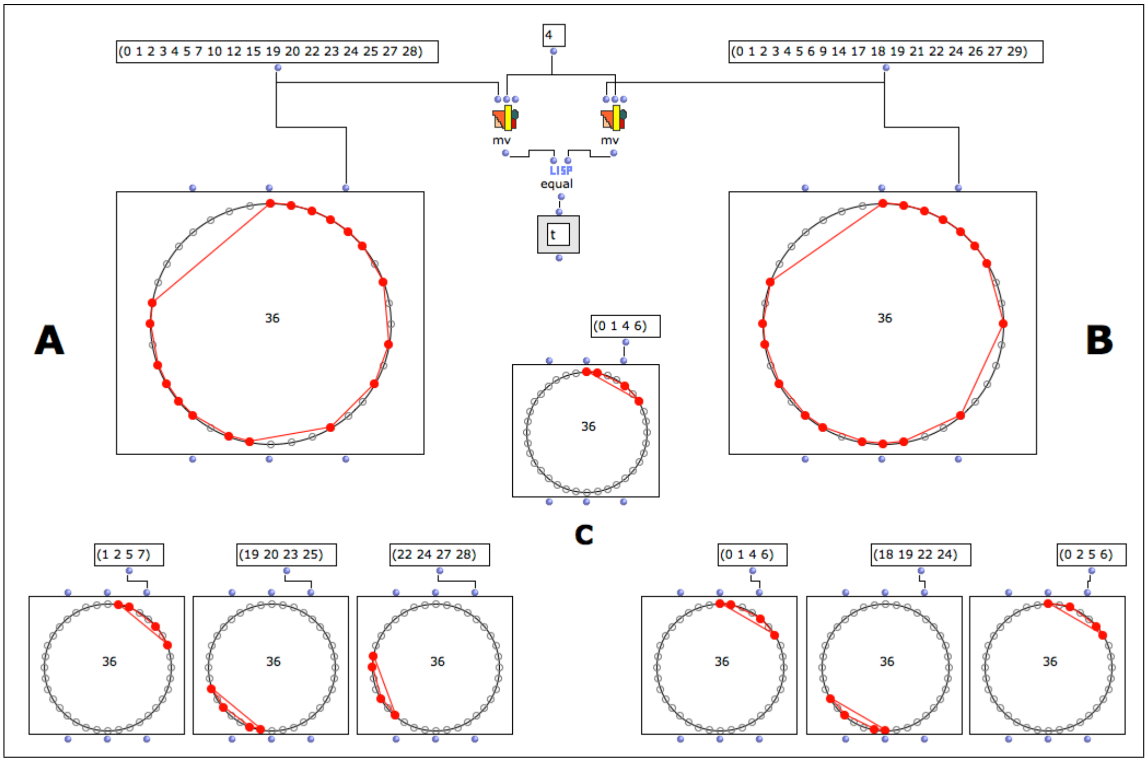

Figure 10. An OpenMusic patch showing two Z^4-related sets A and B in $\mathbb{Z}_{36}$. As an example, we consider the subset $C = \{0, 1, 4, 6\}_{36}$, and we show that the same number of copies, up to transposition and inversion (3, in this case) are included in the two initial sets. The patch shows also that this is true for any other 4-subset, by comparing the two **mv** functions.

5.2.1. *An upper bound*

As a direct consequence of Theorem 4 in [27], by Radcliffe and Scott, one easily gets

$$R(n) \leqslant 2r(n).$$

Thus, under the hypothesis of non vanishing Fourier transform on $\mathbb{Z}_n^*$, $R(n) \leqslant 8$.

5.2.2. *Existence of Z^4-related Sets*

Notice, in particular, that $R(36) = 5$, which means that there are some Z^4-related sets.

To stress the interest of the research in the k-Deck problem, and the intimate difference with the k-deck problem, we finish with an explicit example of Z^4-related sets, obtained by computer search. In $\mathbb{Z}_{36}$ consider the sets

$$A = \{0, 1, 2, 3, 4, 5, 7, 10, 12, 15, 19, 20, 22, 23, 24, 25, 27, 28\}_{36}$$
$$B = \{0, 1, 2, 3, 4, 5, 6, 9, 14, 17, 18, 19, 21, 22, 24, 26, 27, 29\}_{36}$$

They are not related by transposition or inversion, but $\mathbf{mv}^4(A) \equiv \mathbf{mv}^4(B)$, or equivalently $D^4(A) \equiv D^4(B)$, see Figure 10.

6. Conclusions and open problems

We have extended and unified the definition of interval content and Patterson function to a larger framework using common mathematical tools, namely Haar measures and Lebesgue integration theory. This approach has allowed us to obtain the following results on Patterson functions, also valid in the non-commutative case:

- translation and periodicity invariance;
- transfer through quotients;
- a generalization of the hexachord theorem to a large class of GIS;
- first musical examples of Z-relation in a non-commutative GIS.

After providing the general definition of the phase retrieval problem, we have resumed the recent results on the characterization of the homometry and the k-homometry, then started the analysis of the k-Homometry, and finally given the first example of Z^4-related sets.

There is still a certain number of outstanding open problems. In particular:

- there is still no constructive characterization of the homometry, i.e. there is no reasonable way to determine, given a set (a distribution, respectively), whether it is non-trivially Z-related to other sets (homometric to other distributions, respectively) and to reconstruct them;
- the phase retrieval problem in the GIS of time spans (see section 2.5) is still to be solved; because of the non-commutativity of the group, one of the usual approaches based on Fourier transform cannot be applied, which calls for the search of new mathematical constructions, as suggested, for example, by Pebody in [24];
- the full determination of $r(n)$ still depends upon the hypothesis of non-vanishing coefficients of the Fourier transform (Corollary 5.7);
- the behaviour of $R(n)$ as n increase is still unknown. It likely has a finite upper bound, like $r(n)$; given the example in section 5.2.2, we only know that the upper bound of $R(n)$ is greater or equal to 5; to study $R(n)$, probably we will need a way to get around the problem of having two possible relations (11) and (12).

Although we have implemented original algorithms for searching Z and Z^k relations, an extensive review of them will be necessary to a real application in computer-assisted musical composition and analysis.

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