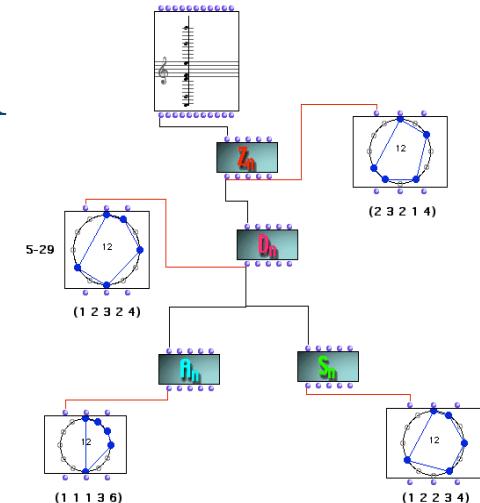
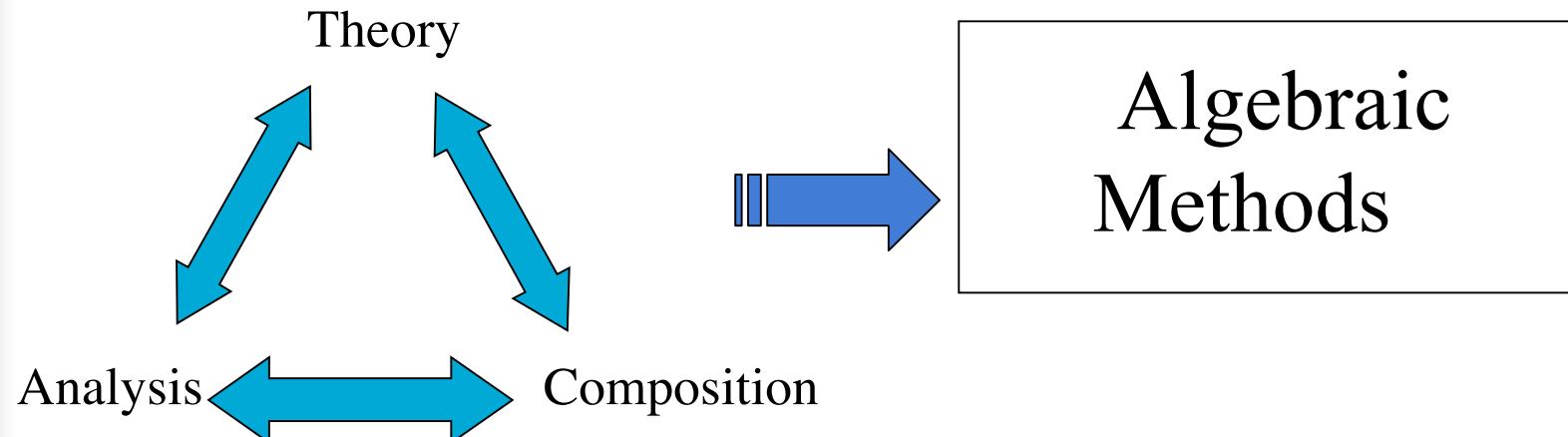


Modern Algebra and the Object/Operation duality in music

Moreno Andreatta
Music Representations Team
IRCAM/CNRS
Moreno.Andreatta@ircam.fr



Mathematics/Music/Philosophy

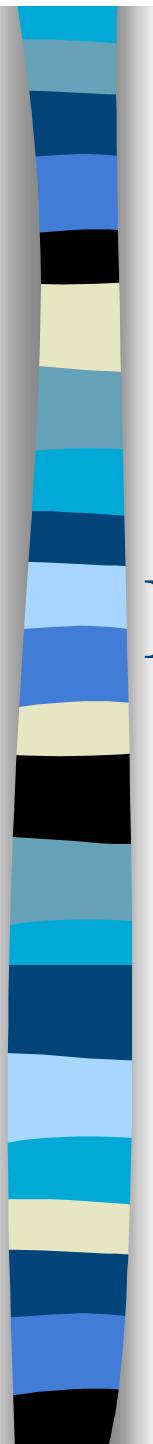


- **Articulation objectal/opératoire (Granger)**
 - Pour la connaissance philosophique (1988)
 - Formes, opérations, objets (1994)
- **Group theory and perception (Cassirer)**
 - The concept of group and the theory of perception (1944)
- **Category theory and (neo)structuralism (Piaget)**
 - Morphismes et Catégories. Comparer et transformer (1990)

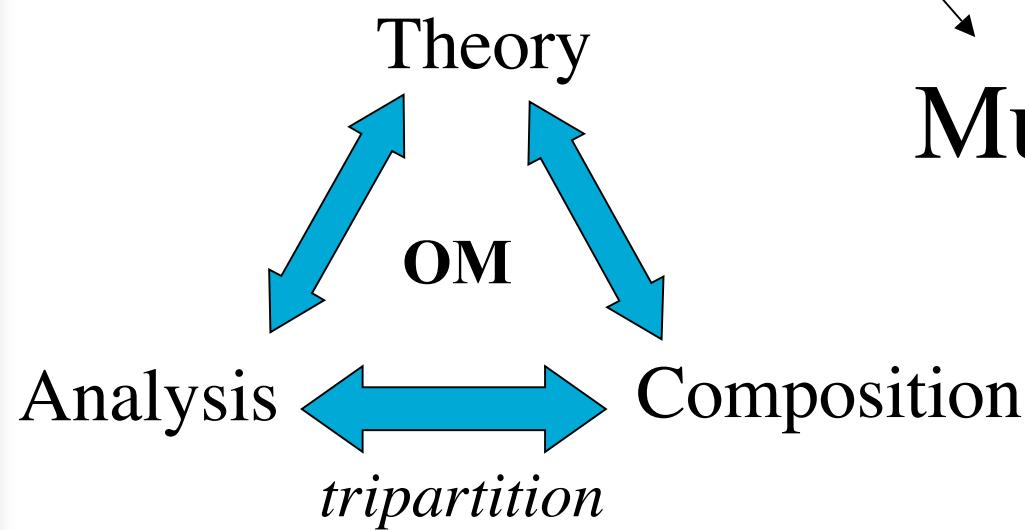


Méthodes algébriques en Musique et Musicologie du XX^e siècle : aspects théoriques, analytiques et compositionnels

www.ircam.fr/equipes/repmus

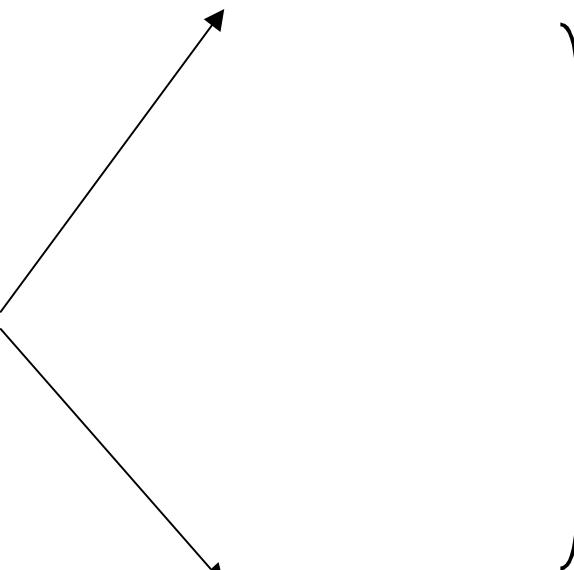


Mathematics (algebra)



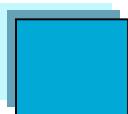
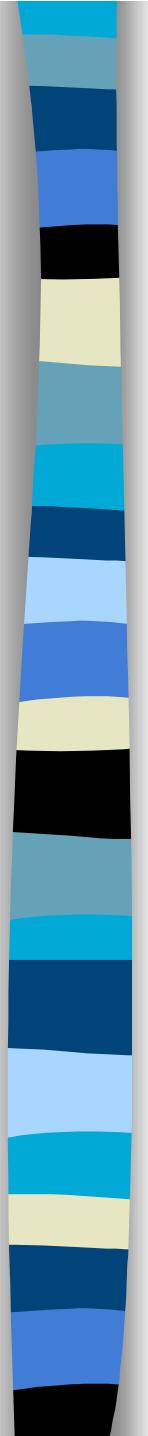
Double perspective

Music

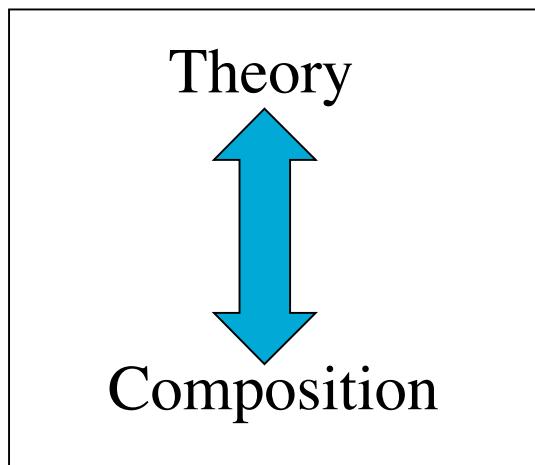


} XXth
century

Musicology



Algebra



Music

Theorists/Composers

- Ernst Krenek
- Milton Babbitt
- Iannis Xenakis
- Anatol Vieru
- Pierre Barbaud
- Michel Philippot
- André Riotte
- Robert Morris
-



Babbitt : *The function of Set Structure*

in the Twelve-Tone System, PhD 1946/1992

Xenakis : *Musiques formelles*, 1963

The *group* structure in music

Vieru : *Eléments d'une théorie générale des modes*, 1967

Theory

Cyclic group $\mathbf{Z}/n\mathbf{Z}$ (Vieru, ...)

Dihedral group \mathbf{D}_n (Forte, ...)

Affine group \mathbf{Aff}_n (Mazzola, Morris, ...)

Composition

Klein group (Twelve-Tone System)
Group of rotations of the cube S_4
Cyclic group (rhythmic canons, ...)

Algebraic Methods in music and musicology

→ *Algebra and the formalization of music theory*

Set Theory

« Musiques formelles »

Modal Theory

Mathematical Music Theory

Babbitt
Lewin

Xenakis
Barbaud

Vieru

Mazzola

Adler
Seeger

Riotte
Mesnage
Assayag

→ *Computational Musicology*

Historical vs Systematic Musicology

CAO, AAO and TAO

« Paradigmatic Analysis » in *OpenMusic*

Models of the compositional process

The place of Mathematics in Systematic Musicology

Guido Adler : « Umfang, Methode und Ziel der Musikwissenschaft » (1885)

II. Systematisch.

Aufstellung der in den einzelnen Zweigen der Tonkunst zu höchst stehenden Gesetze.

A. Erforschung und Begründung derselben in der

1. *Harmo-* 2. *Rhyth-* 3. *Melik*
nik *mik* *(Cohärenz*
(tonal od. (temporär von tonal Relation mit den apper-
tonlich). oder und tem- cipirenden Subjecten
zeitlich). porär). behufs Feststellung der

B. Aesthetik der Tonkunst.

1. Vergleichung und Werthschätzung der Gesetze und deren Relation mit den apper- cipirenden Subjecten behufs Feststellung der *Kriterien des musikalisch Schönen.*
2. Complex unmittelbar und mittelbar damit zusammenhängender Fragen.

C. Musikalische Pädagogik und Didaktik (Zusammenstellung der Gesetze mit Rücksicht auf den Lehrzweck)

1. Tonlehre,
2. Harmonielehre,
3. Kontrapunkt,
4. Compositionslehre,
5. Instrumentationslehre,
6. Methoden des Unter- richtes im Gesang und Instrumentalspiel.

D. Musikologie (Untersuchung und Vergleichung zu ethnographischen Zwecken).

Hilfswissenschaften: Akustik und Mathematik.

Physiologie (Tonempfindungen).

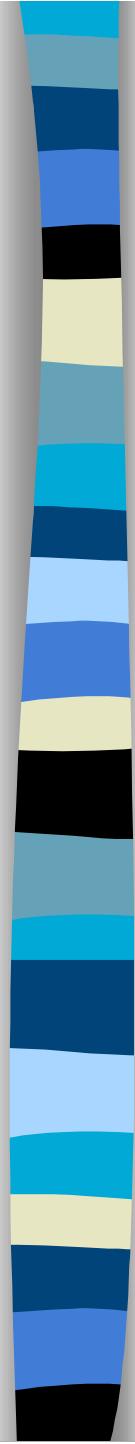
Psychologie (Tonvorstellungen, Tonurtheile und Tongefühle).

Logik (das musikalische Denken).

Grammatik, Metrik und Poetik.

Pädagogik

Ästhetik etc.



The axiomatic approach in mathematics

David Hilbert and the axiomatic foundation of geometry

« In order to be constructed in a right way, geometry [...] only needs few simple principles. These principles are called the **axioms** of the geometry. [...] This study (of the axioms) goes back to the **logical analysis of our spatial intuition** »

David Hilbert : *Grundlage der Geometrie*, 1899



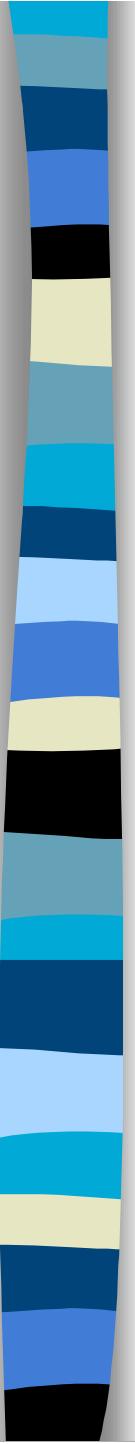
The axiomatic approach and the role of intuition

David Hilbert and the “Anschauliche Geometrie”

« At the moment [...] there are two tendencies in mathematics. From one side, the tendency toward abstraction aims at ‘cristallizing’ the logical relations inside of a study object [...] and at organizing this material in a systematic way.

But there is also a tendency towards the **intuitive understanding** which aims at understanding the **concret meaning of their relations** »

D. Hilbert, S. Cohn-Vossen: *Anschauliche Geometrie*, 1932



Towards the modern concept of music theory

Ernst Krenek and the axiomatic method in music

- *The Relativity of Scientific Systems*
- *The Significance of Axioms*
- *Axioms in music*
- *Musical Theory and Musical Practice*

« *Physicists and mathematicians are far in advance of musicians in realizing that their respective sciences do not serve to establish a concept of the universe conforming to an objectively existent nature. They are fully cognizant of the fact that, conversely, their task is to make an existing conception of the universe conform to the largest number of observations demonstrable by scientific experiments* »

Ernst Krenek : *Über Neue Musik*, 1937 (Engl. Transl. *Music here and now*, 1939, p. 202).



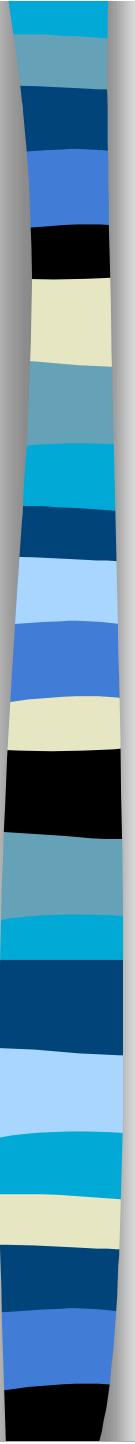
Towards the modern concept of music theory

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- *Axioms in music*
- *Musical Theory and Musical Practice*

« By axiom, one understands a proposition which cannot be reduced to another by logical deductions, or, in other words, which cannot be ‘proved’. [...] Axioms are **free statements of our minds**, enunciated for the purpose of making geometry possible ».

Ernst Krenek : *Über Neue Musik*, 1937 (Engl. Transl. *Music here and now*, 1939, p. 203-204).



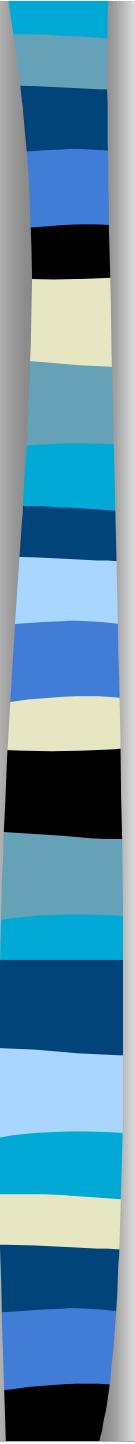
Towards the modern concept of music theory

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- *Musical Theory and Musical Practice*

« *Musical systems [...] have not been created by nature or by some mystical Supreme Being, but have been produced by man to render music possible within a certain sphere. [...] As the study of axioms eliminates the idea that axioms are something absolute, conceiving them instead as **free propositions of the human mind**, just so would this **musical theory** free us from the concept of major/minor tonality [...] as an irrevocable law of nature ».*

Ernst Krenek : *Über Neue Musik*, 1937 (Engl. Tr. *Music here and now*, 1939, p. 206).



Towards the modern concept of music theory

Ernst Krenek and the axiomatic method in music

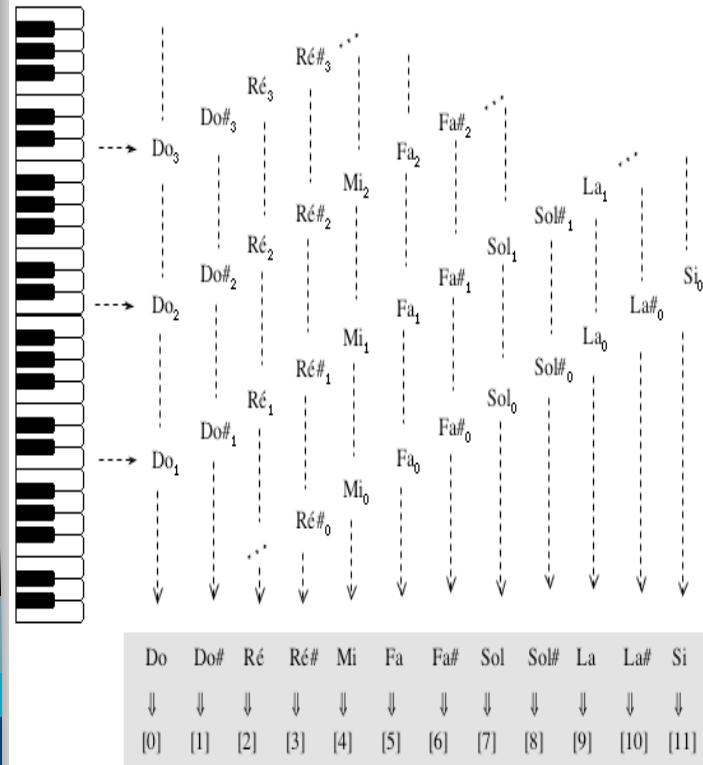
- *The Relativity of Scientific Systems*
- *The Significance of Axioms*
- *Axioms in music*
- *Musical Theory and Musical Practice*

« Whereas geometric axioms are sufficiently justified if their combinations prove them to be both independent of and compatible with each other, the **accuracy of musical axioms** can be proved exclusively by their fitness for musical practical use. [...] A system of musical axioms can never be established **in theory** until it has been demonstrated **in practice** ».

Ernst Krenek : *Über Neue Musik*, 1937 (Engl. Ed. *Music here and now*, 1939, p. 207).

Towards an algebraic approach in music

The congruence relation mod 12



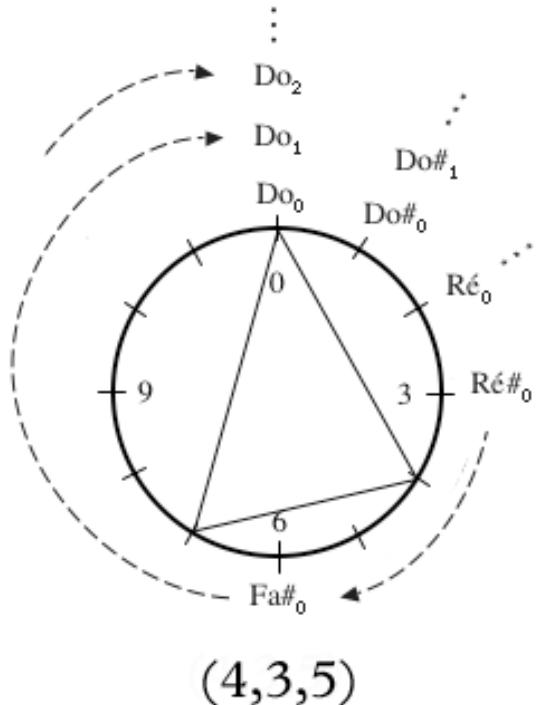
Camille Durutte:

- *Technie, ou lois générales du système harmonique* (1855)
- *Résumé élémentaire de la Technie harmonique, et complément de cette Technie* (1876)

« Two elements are congruent modulo 12 if their difference is equal to a multiple of 12 »

(M. Babbitt: *The function of Set Structure in the Twelve-Tone System*, 1946)

The emergence of group structure in music



↓
*Cyclic group
 $\mathbb{Z}/12\mathbb{Z}$*

The congruence modulo 12 is an equivalence relation

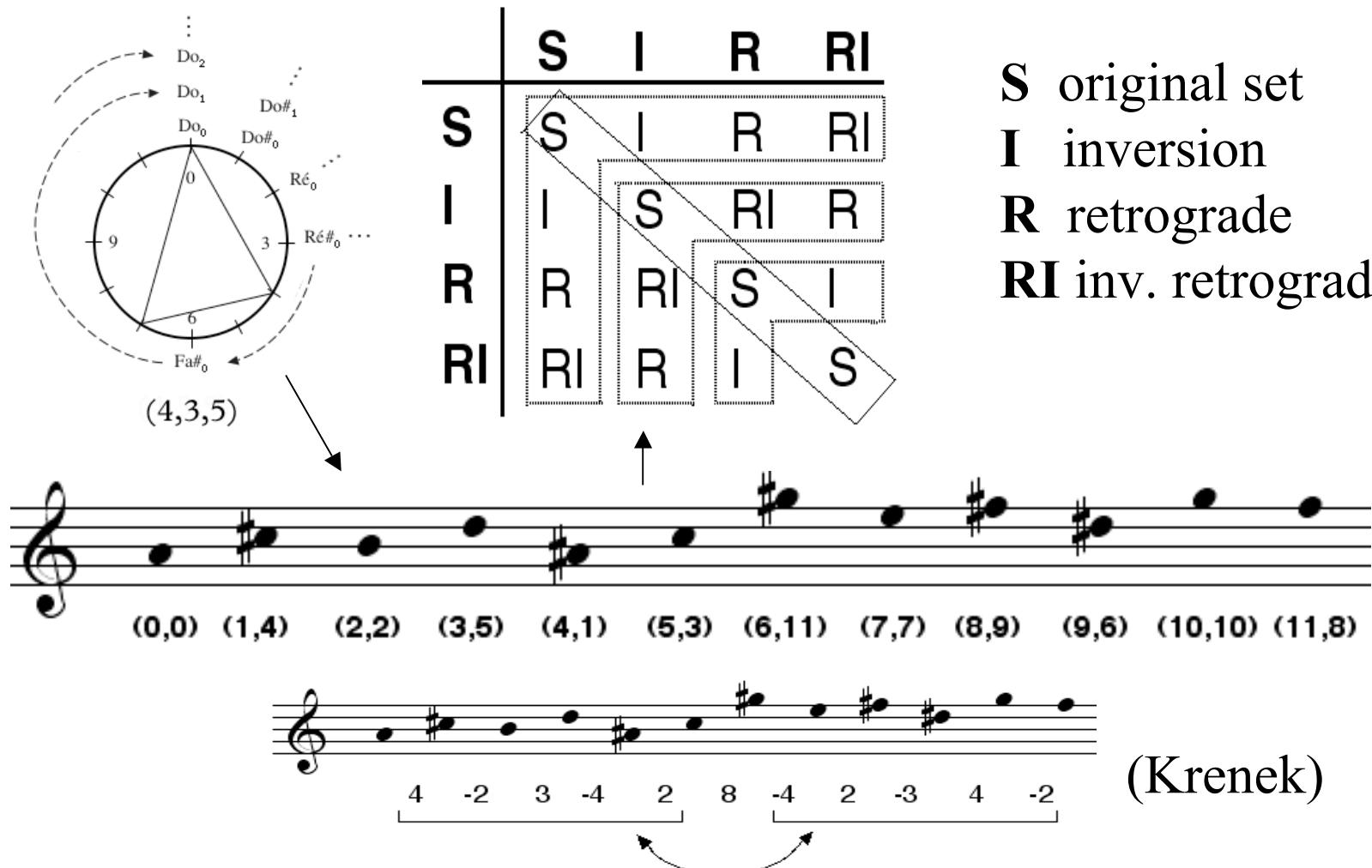
- Reflexivity: $a \sim a$
- Symmetry: $a \sim b \Leftrightarrow b \sim a$
- Transitivity: $a \sim b, b \sim c \Rightarrow a \sim c$

*The equivalence classes modulo 12 define a **group** structure*

- The operation is internal
- Existence of an identity
- Existence of an inverse
- Associativity

Twelve-Tone System and group theory (Babbitt)

The Twelve-Tone System, is a « *set of elements, relations between elements and operations upon elements* » (1946)



Twelve-Tone System and group theory (Babbitt)

S: $(a,b) \rightarrow (a,b)$

I: $(a,b) \rightarrow (a, 12-b \text{ mod. } 12)$

R: $(a,b) \rightarrow (11-a,b)$.

RI: $(a,b) \rightarrow (a, 12-b \text{ mod. } 12)$

$$\begin{array}{c} \downarrow \\ (11-a, 12-b \text{ mod. } 12) \end{array}$$



S(a,b) = (a,b)

I(a,b) = (a, 12-b mod. 12)

R(a,b) = (11-a,b).

RI(a,b)= R(a, 12-b mod. 12)=
 $= (11-a, 12-b \text{ mod. } 12)$

	S	I	R	RI
S	S	I	R	RI
I	I	S	RI	R
R	R	RI	S	I
RI	RI	R	I	S



Compositional consequences of the algebraic formalization...
«...directly derivable from the theorems of finite group theory »
(M. Babbitt: “Past and Present Concepts of the Nature and Limits of Music”, 1961)

- « Given a collection of pitches (pitch classes), the multiplicity of occurrence of any interval (an interval being equal with its complement since ordering is not involved) determines the number of common pitches between the original collection and the transposition by the interval »

$$\text{INJ}(A, B)(T_i) \underset{\text{(Lewin)}}{=} \text{IFUNC}(A, B)(i)$$

- The Hexachord Theorem**
An interval has the same molteplicity of occurrence inside of an hexachord and its complement

$$\text{INJ}(A, A')(f) \underset{\text{(Lewin)}}{=} \text{INJ}(A', A)(f)$$

Object/operation duality in Set Theory

Classical (Fortean) and transformational (Lewinian) approach

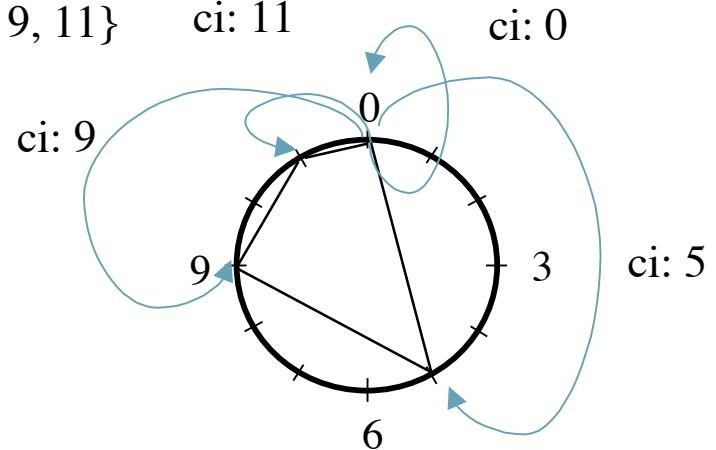
The IFUNC vector (Lewin,1959):

$$\text{IFUNC}(A, A) = [4 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]$$

↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑
ci 0 ci 1 ci 2 ... ci 11.

$$A = \{0, 5, 9, 11\}$$

Interval Classes and
Interval Content of a pcs



The IV vector (Forte, 1964):

$$\text{VI}(A) = [1 \ 1 \ 1 \ 1 \ 1 \ 1]$$

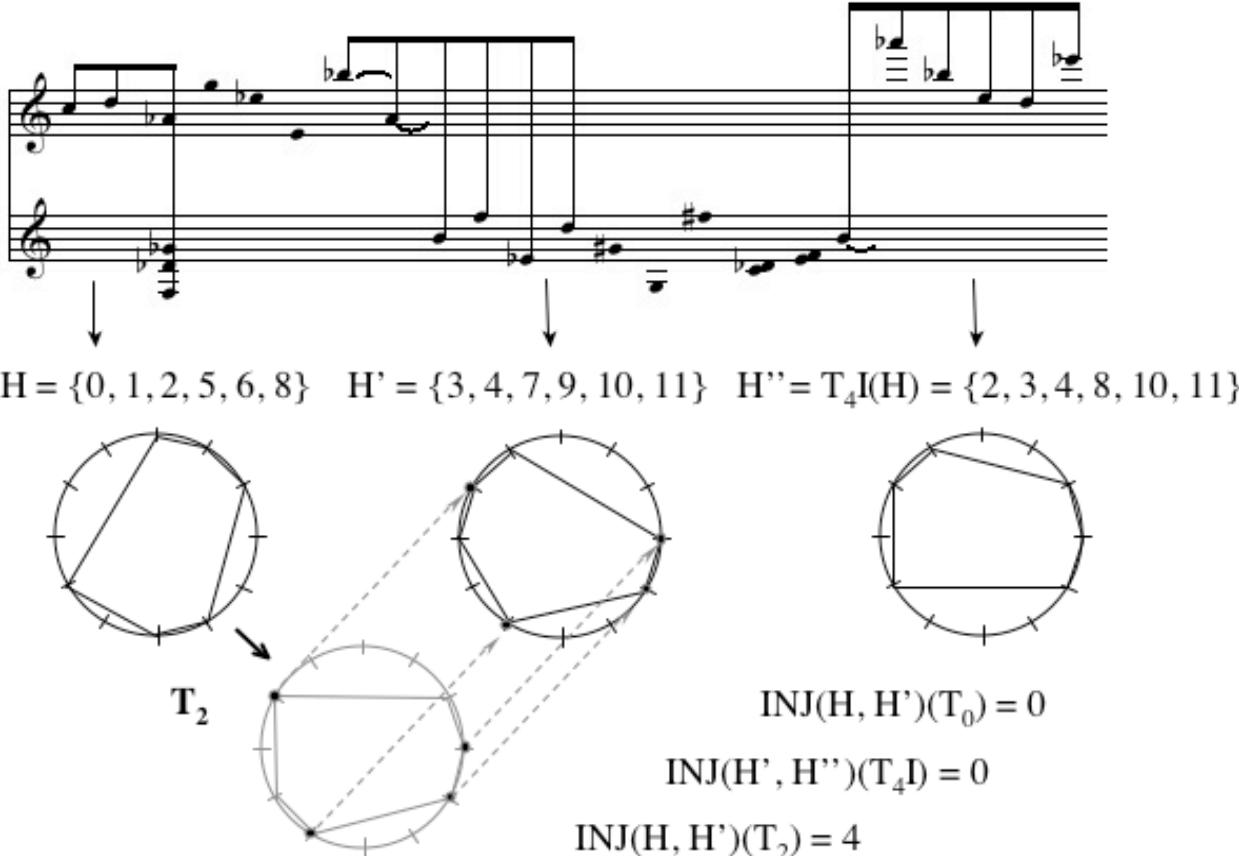
↑ ↑ ↑ ↑ ↑ ↑
ci 1 ci 2 ci 3 ... ci 6.

Ex. E. Interval classes



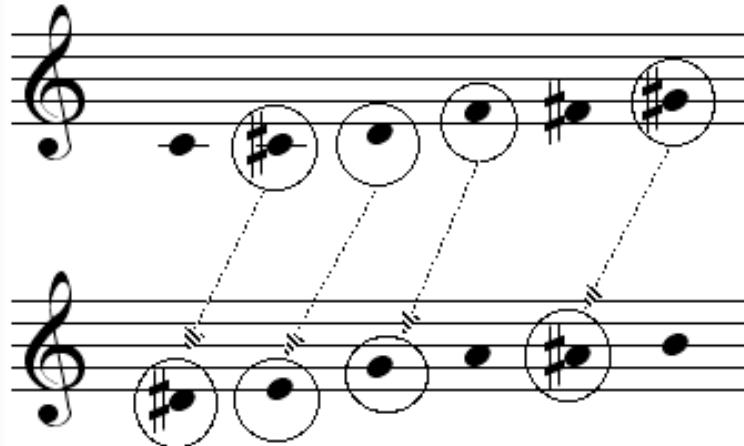
Injection function and the inclusion/complementary relation

A. Webern *Fünf Stücke* op. 10 no. 4, 1913 (Forte 1973 / Lewin 1987)



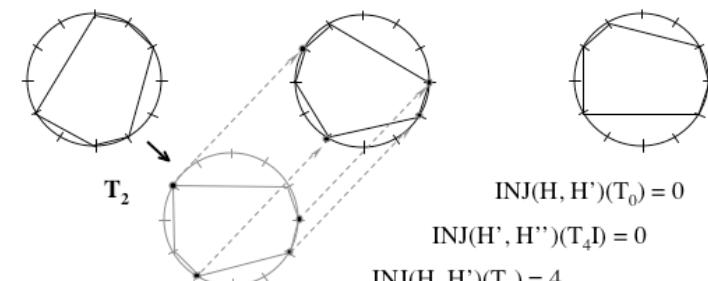
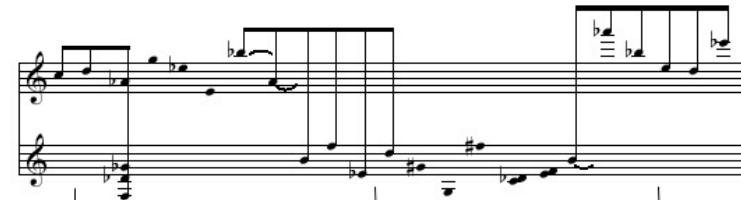
$\text{INJ}(H, H')(T_n) = \text{number of elements } a \in H \text{ such that } T_n(a) \in H'$

Injection Function and IFUNC



A

B



f transformation between A and B

$INJ(A, B)(f) =$ number
of elements a of A
such that $f(a)$ belongs
to B

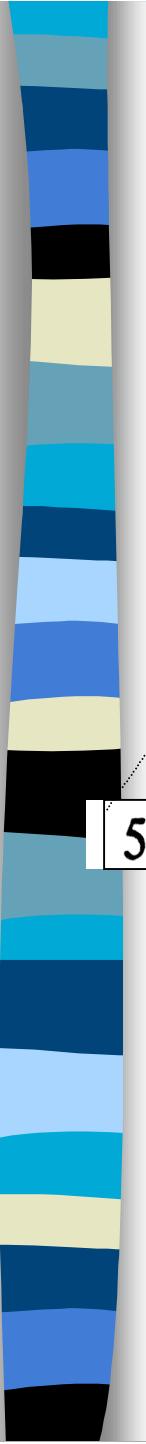
\longleftrightarrow

$IFUNC(A, B)(i) =$ number of
elements (a, b) of $A \times B$ such
that the **distance** between a
and b is equal to i

$INJ(A, B)(T_i)$

=

$IFUNC(A, B)(i)$



Classical « object-based » Set Theory: Forte's taxonomic approach

5-Z36

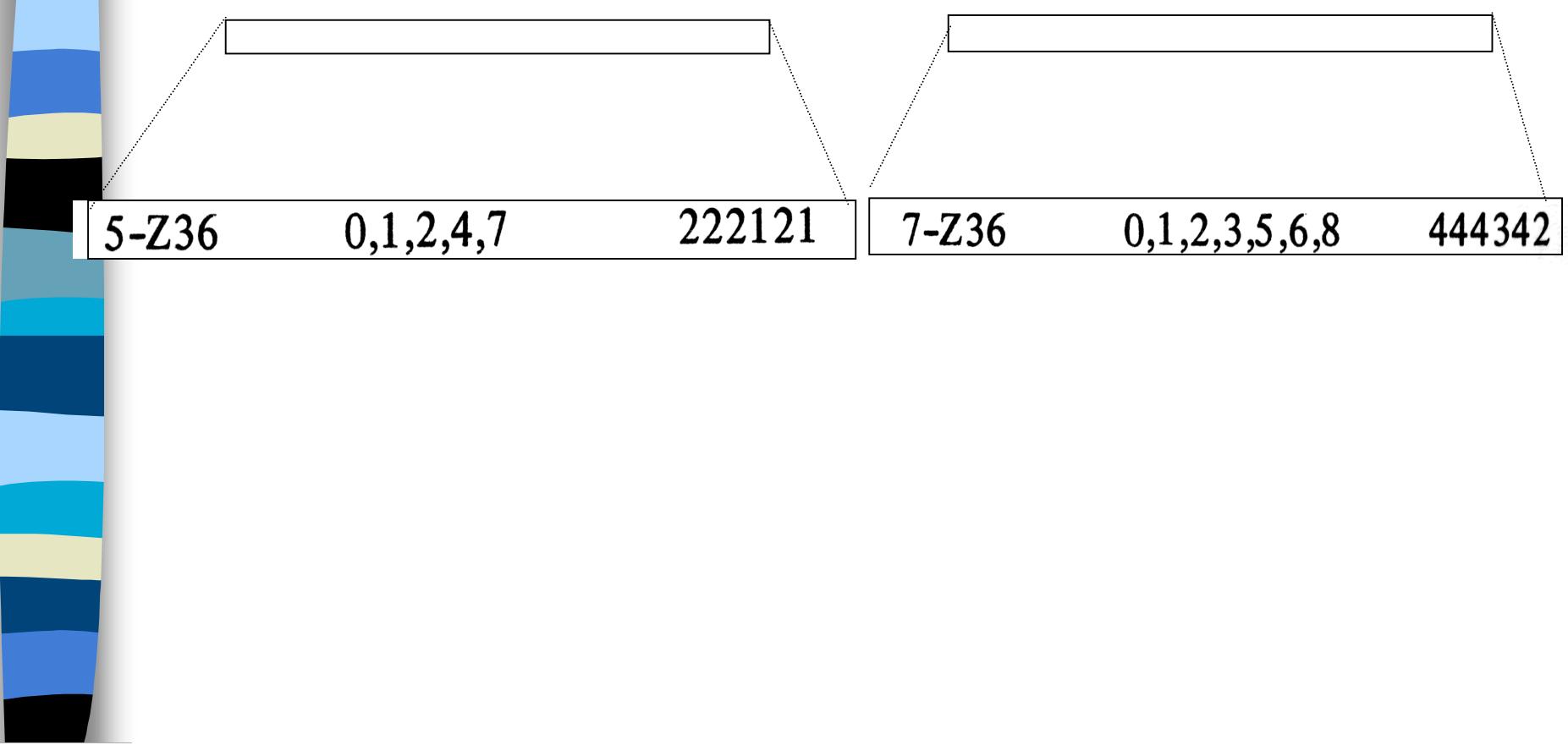
0,1,2,4,7

222121

7-Z36

0,1,2,3,5,6,8

444342





Statistical investigation

Table 2
Sets and Movements (Pentachords)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
5-1		*							*	*			*	
5-2	*	*											*	
5-3													*	
5-4	*									*	*			
5-5											*			*
5-6										*	*			*
5-7						*					*			*
5-8											*			*
5-9						*			*	*	*			*
5-10		*						*					*	*
5-11					*	*					*		*	
5-Z12	*				*								*	
5-13			*										*	
5-14				*						*				*
5-15					*	*					*			*
5-16			*	*	*						*			*
5-Z17	*												*	*
5-Z18				*				*		*	*			*
5-19	*					*					*			*
5-20	*							*					*	*
5-21	*		*							*				*
5-22	*		*					*						*
5-23	*	*	*								*			*
5-24	*			*										*
5-25	*			*						*	*			*
5-26	*			*	*				*	*				*
5-27				*							*			*
5-28	*					*					*			*
5-29	*					*					*			*
5-30						*	*							*
5-31	*	*	*	*	*	*				*	*			*
5-32	*	*	*	*						*	*	*		*
5-33														*
5-34	*	*						*		*				*
5-35	*	*		*										*
5-Z36						*	*							*
5-Z37							*							*
5-Z38						*	*							*

A. Forte : *The Harmonic Organization of « The Rite of Spring », New Haven, Yale University Press, 1978.*

Looking for a „primary harmony“ or *Nexus*

Ex. 104. *Sacrificial Dance*: R192

$6-z45$
 $6-27$
 $6-30$
 $6-z42$

$7-16$
 $5-31$
 $7-32$

Annotations on the score:

- 7 - 32
- 6-z45
- 6-z43
- 5-31
- 6-27
- 6-30
- Tutti
- 7-16
- 6-z42
- 6-z43

Ex. 88a. Sacrificial Dance: R142

Vn.
Va.
B.Cl.
Vc.

6-Z28

Ex. 89. Sacrificial Dance: R142+3

Ob.
E.H.
Vn.
Va.
Vc.
Hn.

6-Z42 6-Z42 6-Z42

Ex. 90. Sacrificial Dance: R144

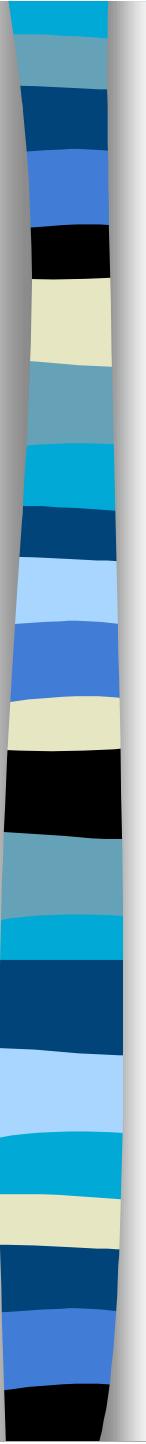
Fl.
Ob.
E.H.
Cl.
Tpt.
Vn.
Va.
Hn.
B.Tpt.
Cb.

6-Z29 5-20 6-Z19 6-Z45 5-20

7-32

$$\left. \begin{array}{c} 5-16 \\ 5-31 \\ 5-32 \end{array} \right\} \subset 6-27 \subset \left\{ \begin{array}{c} 7-16 \\ 7-31 \\ 7-32 \end{array} \right\}$$

$$5-31 \subset \left\{ \begin{array}{c} 6-z42 \\ 6-27 \\ 6-z28 \end{array} \right\}$$

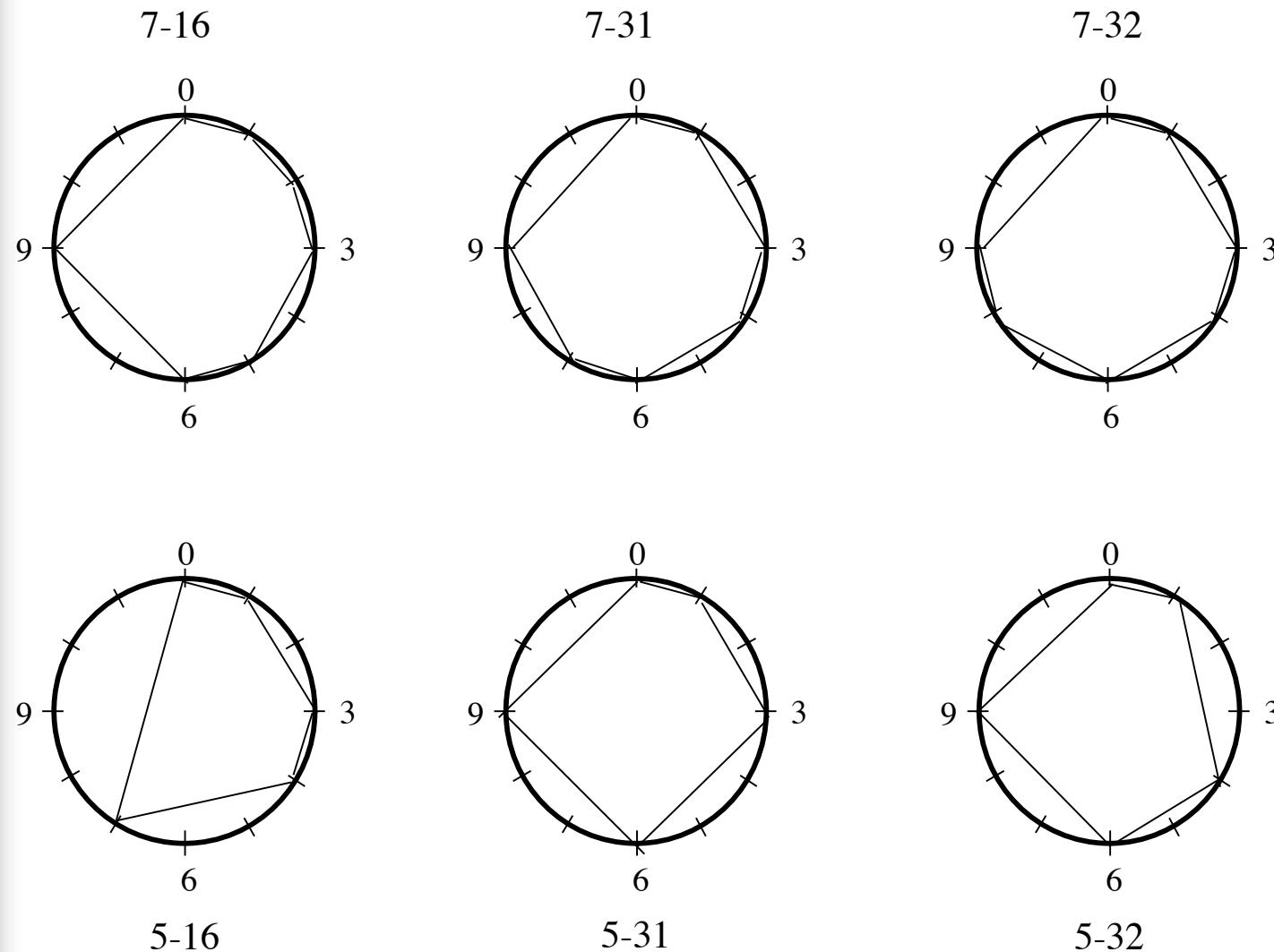


Complexes, subcomplexes and the *nexus rerum*

pcs

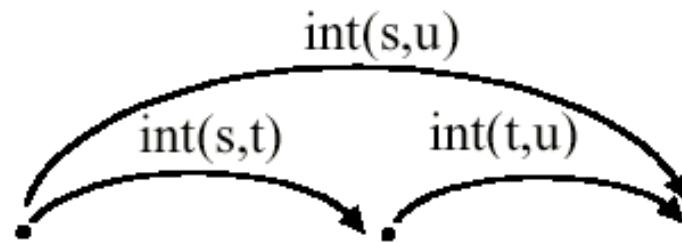
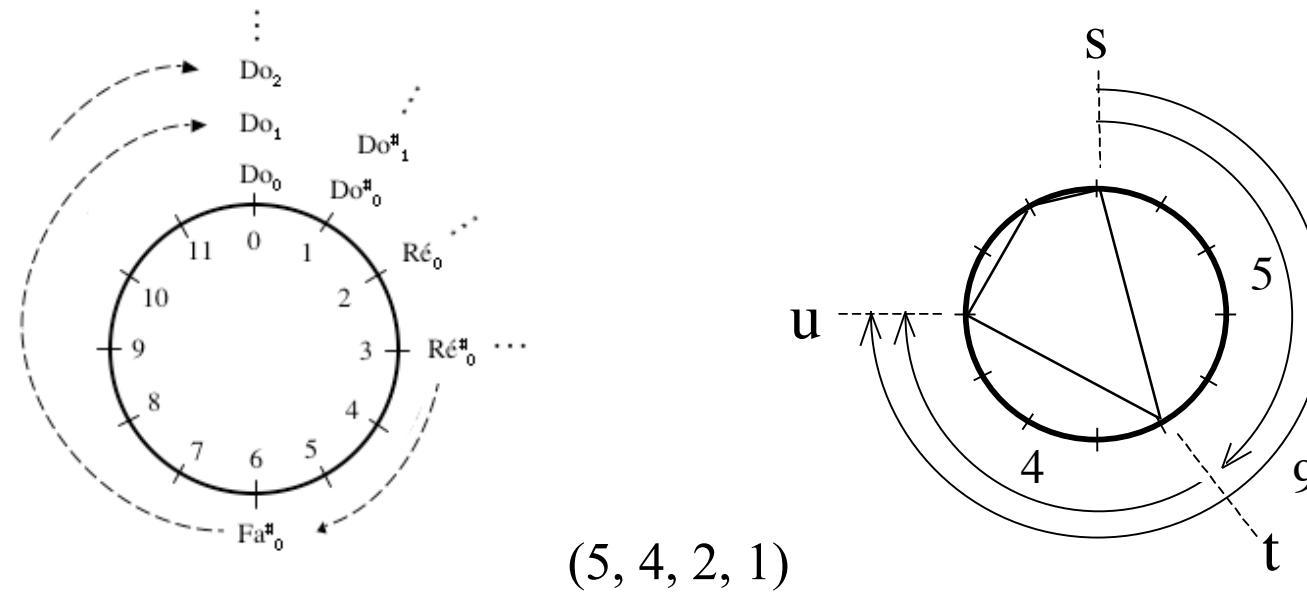


Compl.



David Lewin's Transformation(al) Theory/Analysis

Towards a generalized concept of musical interval



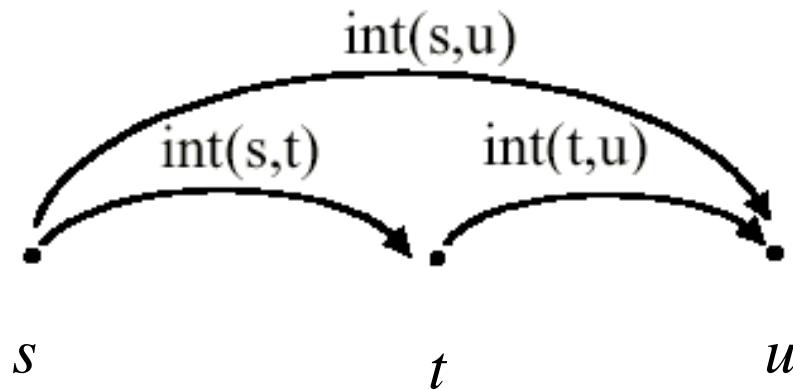
- *Generalized Musical Intervals and Transformations*, 1987
- *Musical Form and Transformation: 4 Analytic Essays*, 1993

Generalized Interval System (GIS)

S =collection of elements
 (G, \bullet) = group of intervals
int = intervallic fonction

$$\text{GIS} = (S, G, \text{int})$$

$$\text{int}: S \times S \longrightarrow G$$

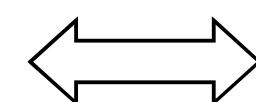


- Given three elements s, t, u in S :

$$\text{int}(s,t) \bullet \text{int}(t,u) = \text{int}(s,u)$$

- For each element s in S and for each interval i in G there exists a (unique) element u in S such that the distance between s and u is equal to i :

$$\text{int}(s,u) = i$$



Group action

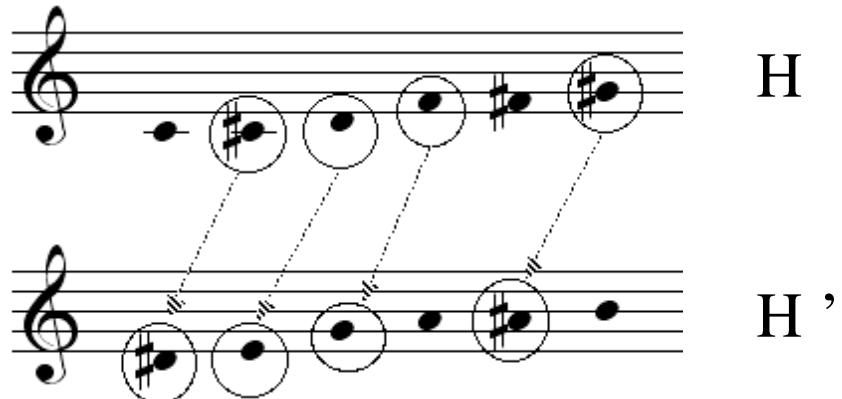
Interval Function IFUNC in a GIS

GIS = (S, G, int)

S set

H and H' subsets of S

$\text{IFUNC}(H, H')(i) =$
= number of (a, b) in $H \times H'$
whose elements have distance
equal to i i.e. $\text{int}(a, b) = i$



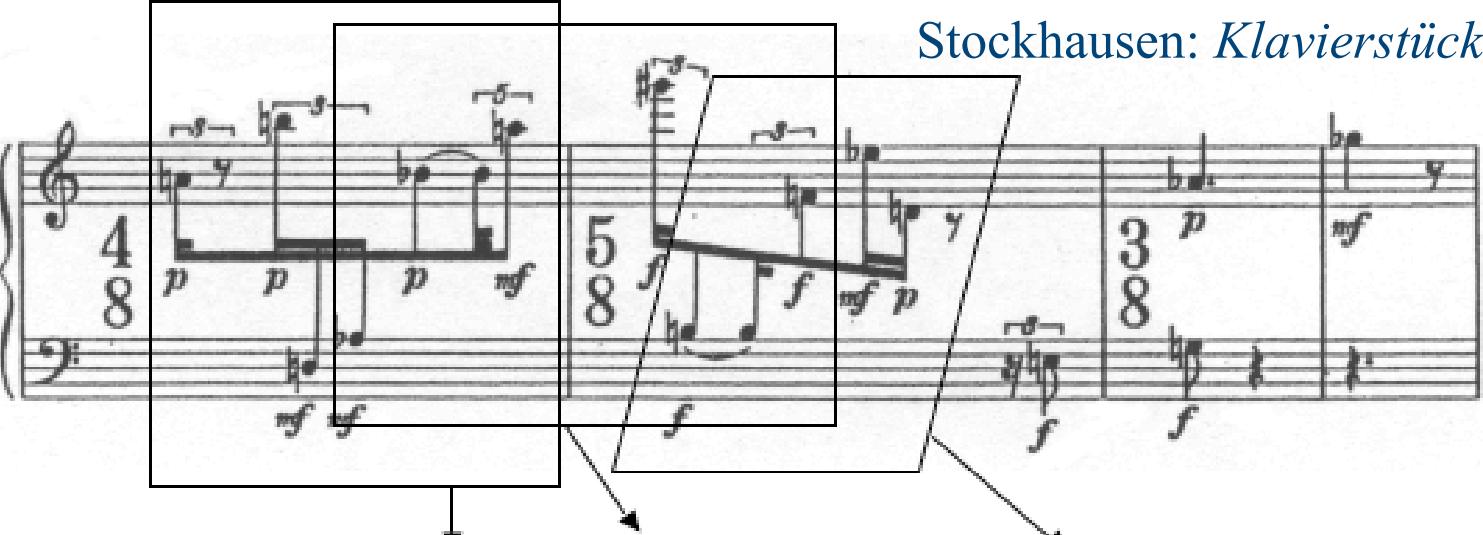
$$\text{IFUNC}(H, H')(2) = 4$$

$\text{INJ}(H, H')(T_n) = \text{number of elements } a \in H \text{ such that } T_n(a) \in H'$

Transformation theory/analysis

- *Generalized Musical Intervals and Transformations*, 1987
- *Musical Form and Transformation: 4 Analytic Essays*, 1993

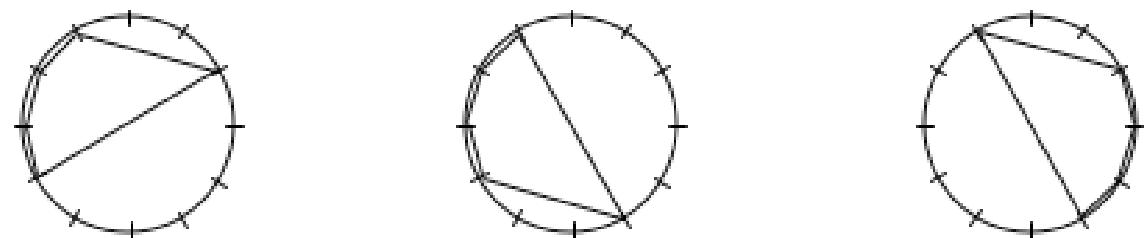
Stockhausen: *Klavierstück III*



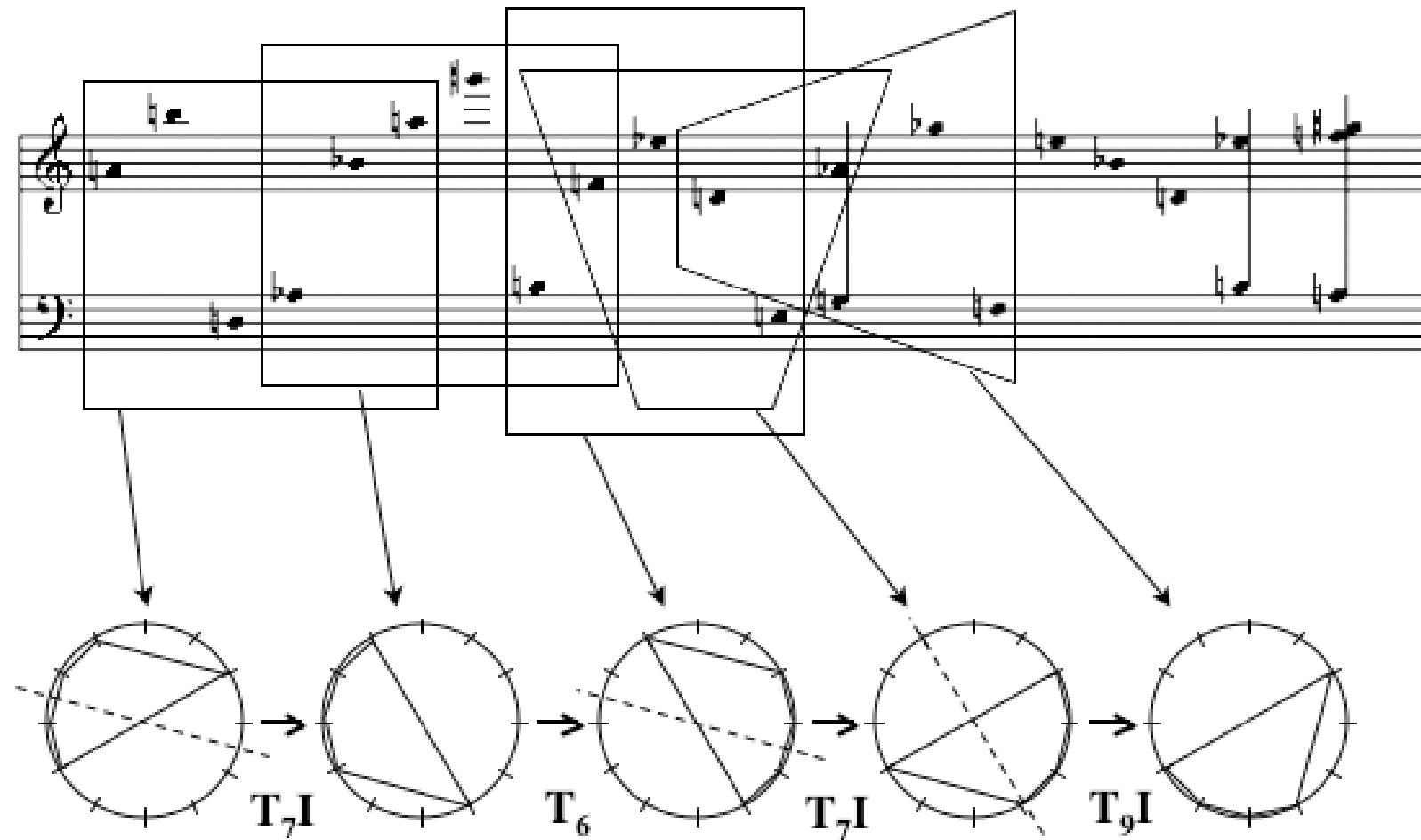
SI: (1, 1, 1, 3, 6) (6, 3, 1, 1, 1) (6, 3, 1, 1, 1)

IFUNC: [5 3 2 2 1 1 1 1 1 2 2 3] [5 3 2 2 1 1 1 1 1 1 2 2 3] [5 3 2 2 1 1 1 1 1 1 2 2 3]

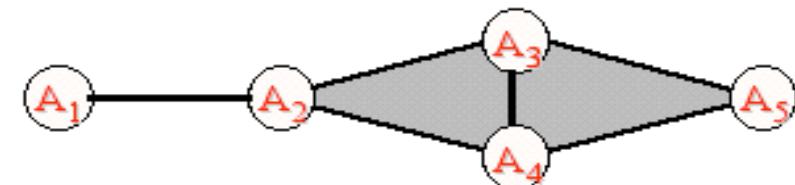
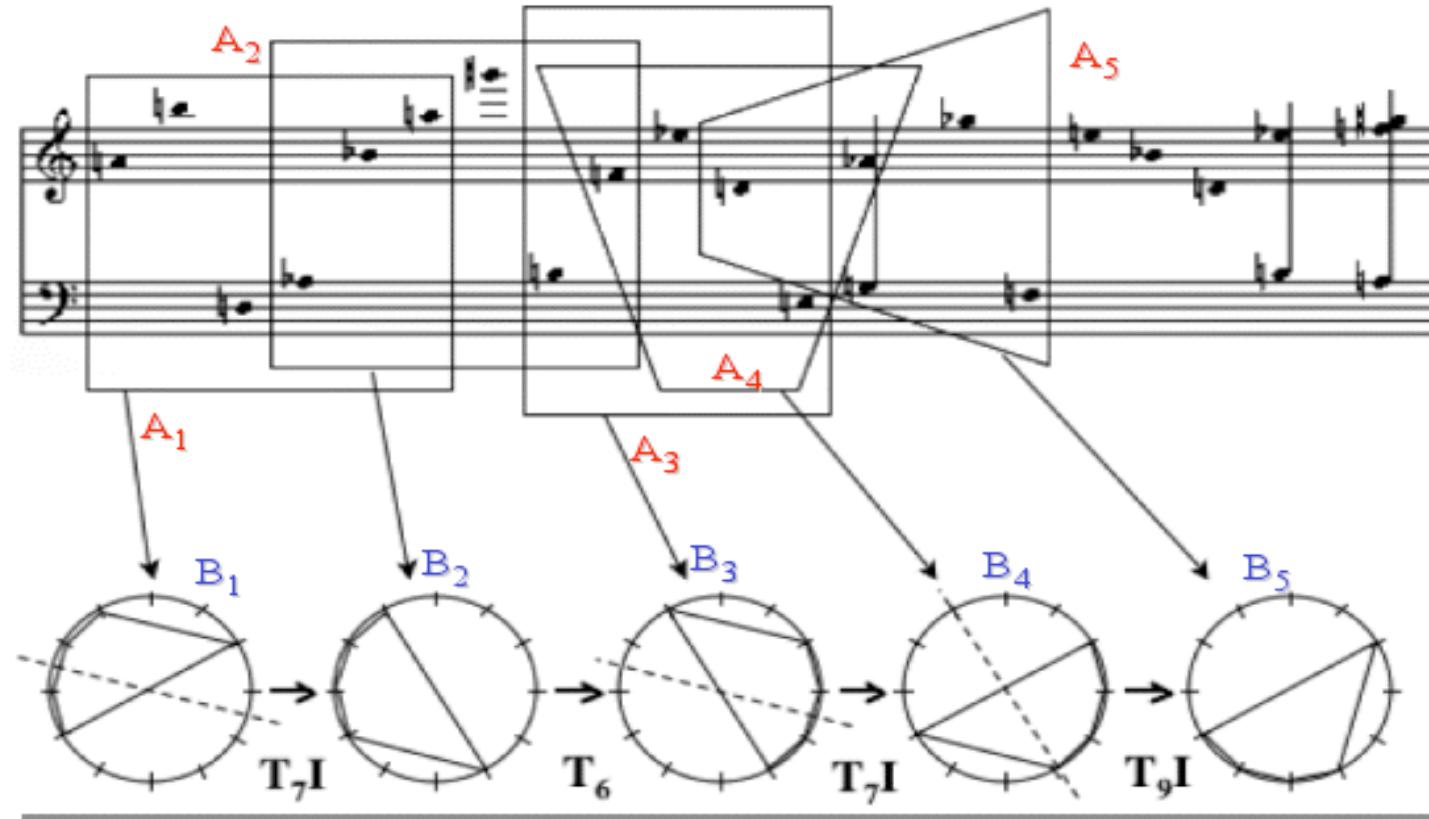
VI: [3 2 2 1 1 1] [3 2 2 1 1 1] [3 2 2 1 1 1]



Transformational progression

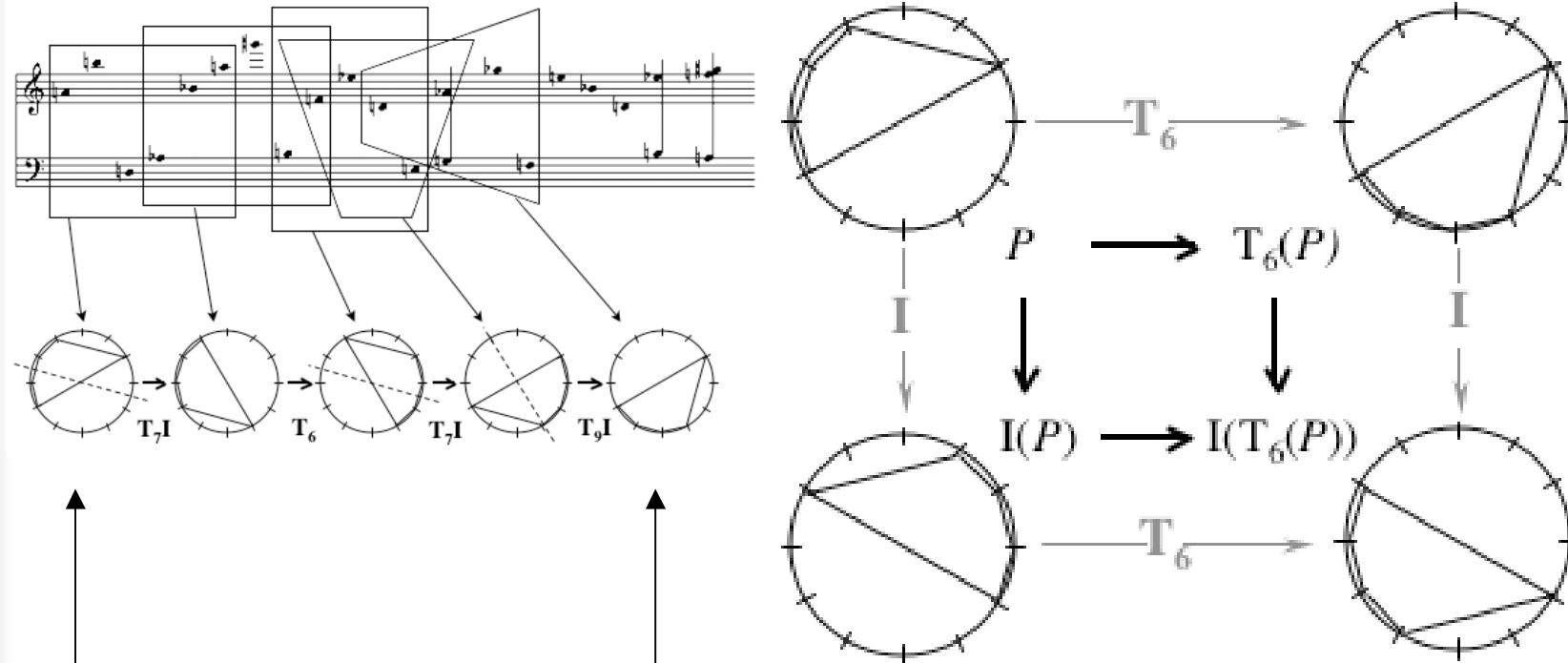


Atlas and the associated « simplicial complex »



Simplicial complexe (nerve)

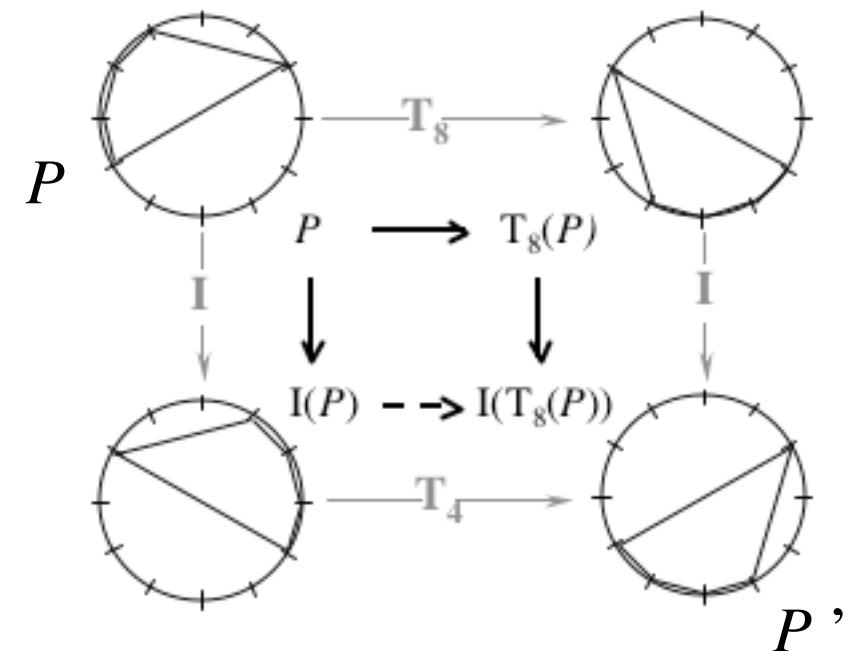
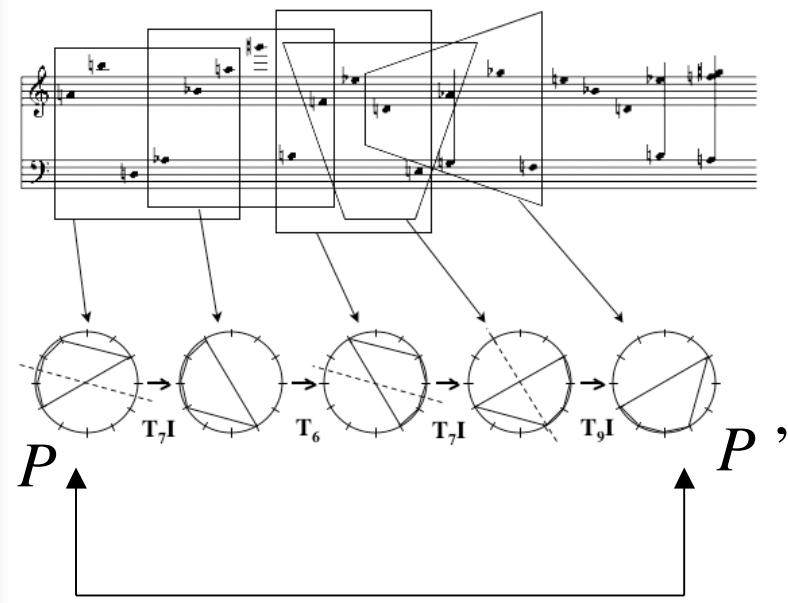
(Temporal) progressions *vs* transformational networks



$$IT_6 = T_6 I$$

commutativity

Non-commutative Transformational networks

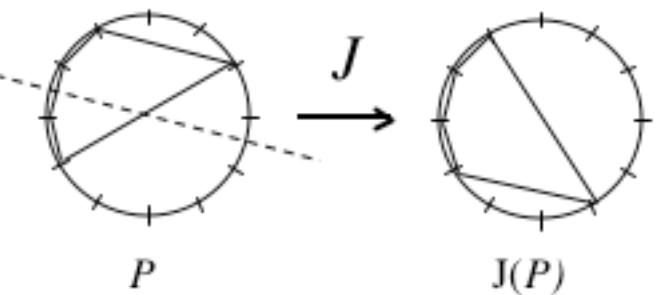
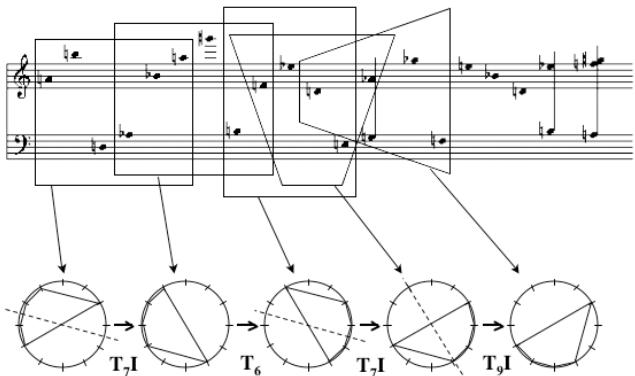


In general transposition does not commute with inversion

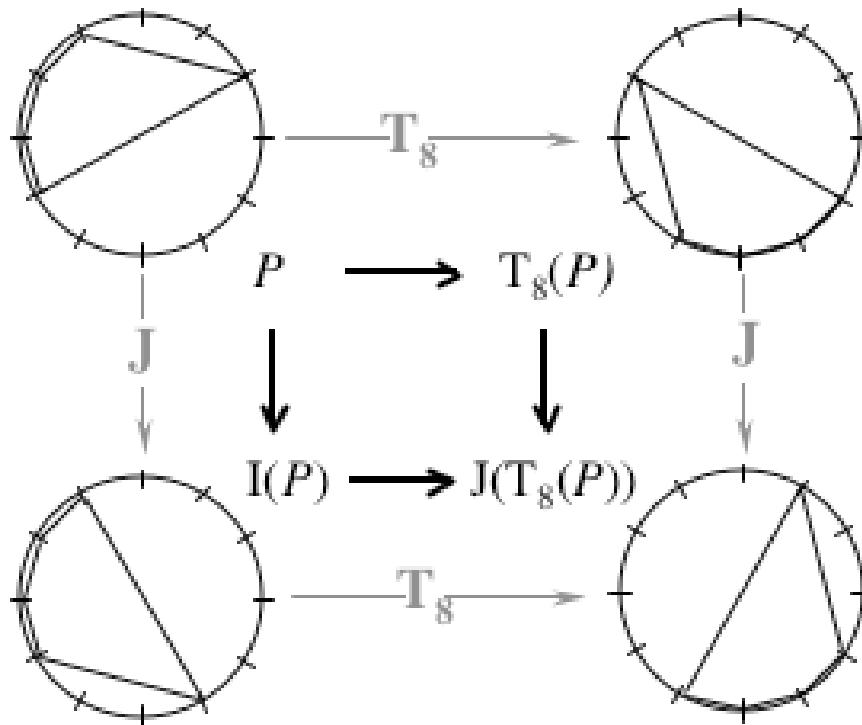
$$IT_8 \neq T_8 I$$

$$IT_n = T_{(12-n)} I$$

Contextual Transformations and commutative diagrams



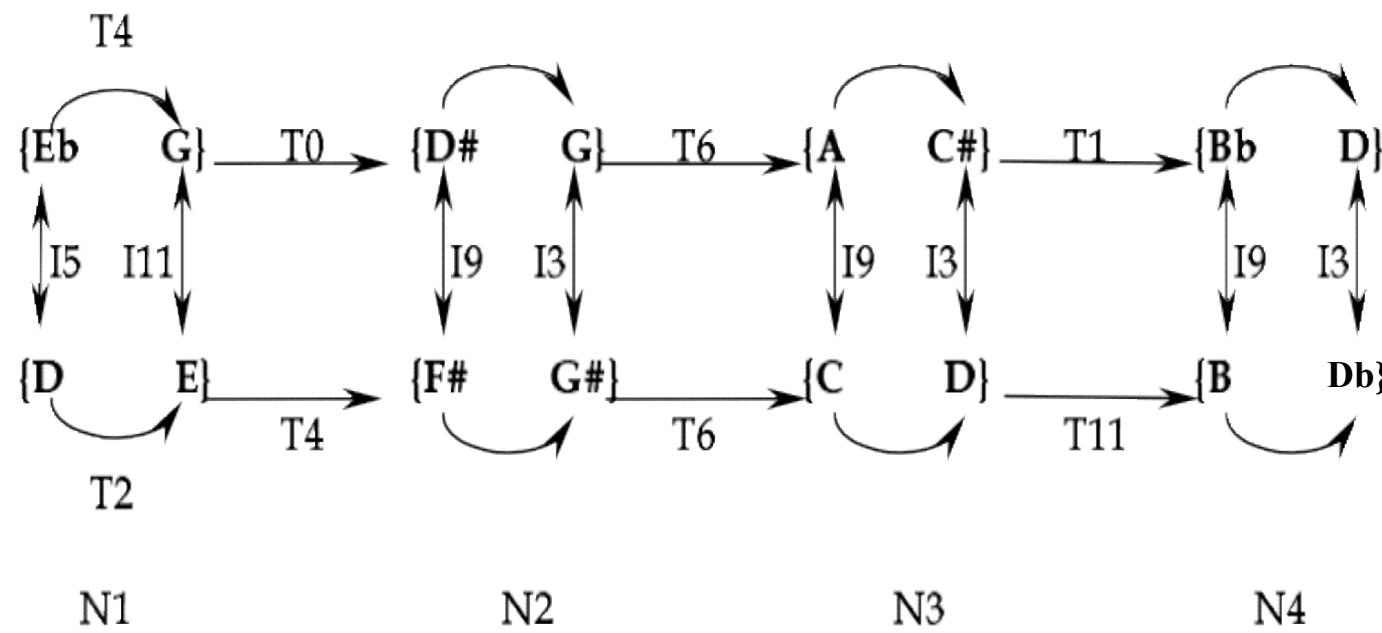
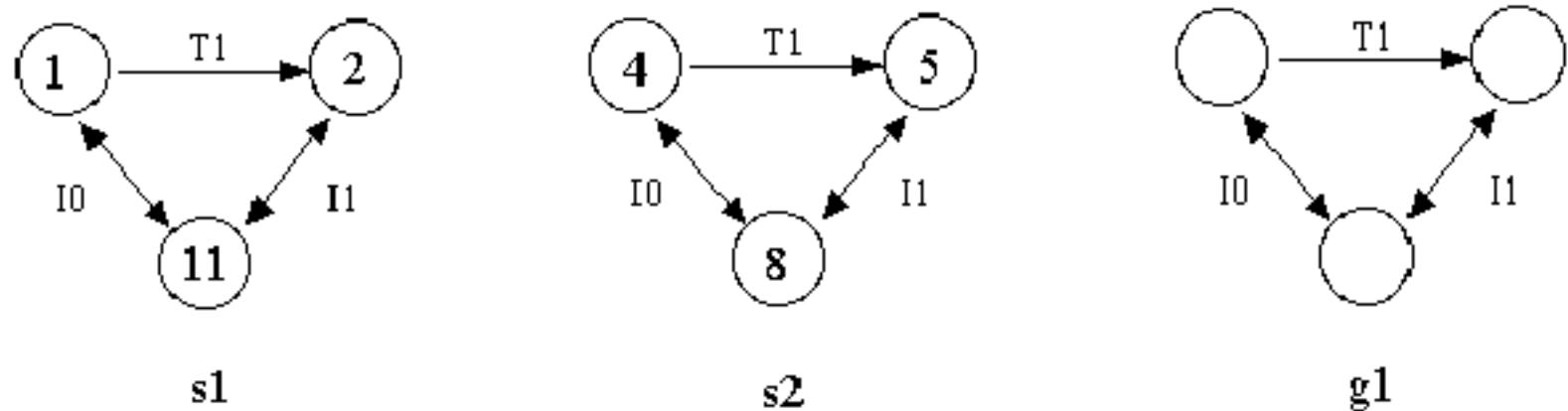
Contextual inversion



$$JT_8 = T_8 J$$

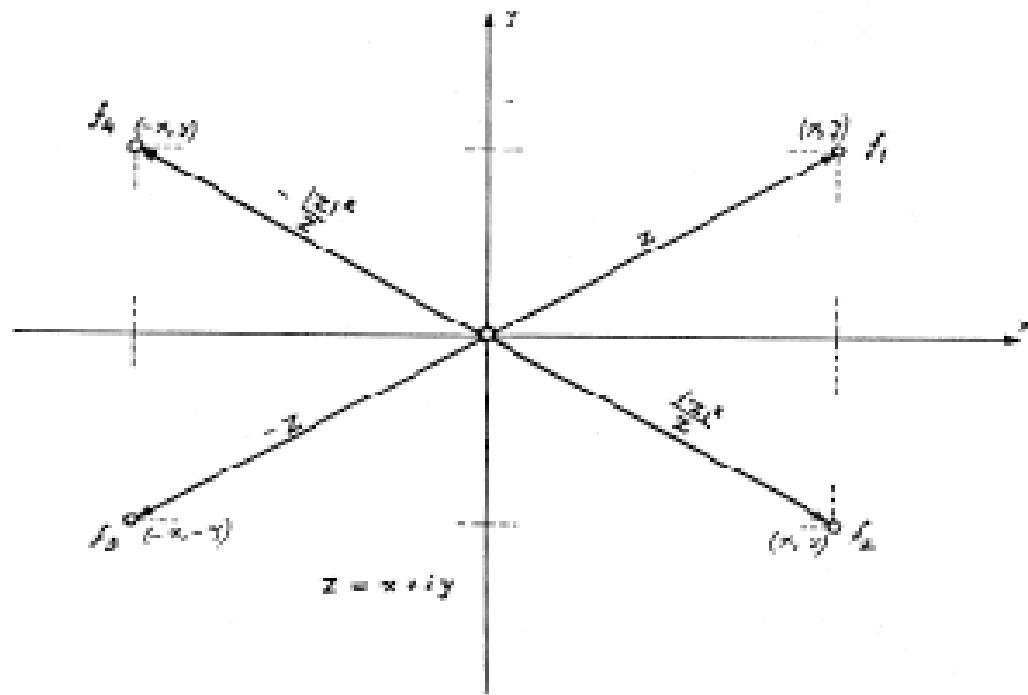
$$JT_n = T_n J$$

Klumpenhouwer Network (K-nets) and category theory



Twelve-Tone System and group theory: Xenakis' transformational approach to composition

	S	I	R	R
S	S	I	R	R
I	I	S	R	R
R	R	R	I	S
RI	RI	R	I	S



$$Z = x + yi$$

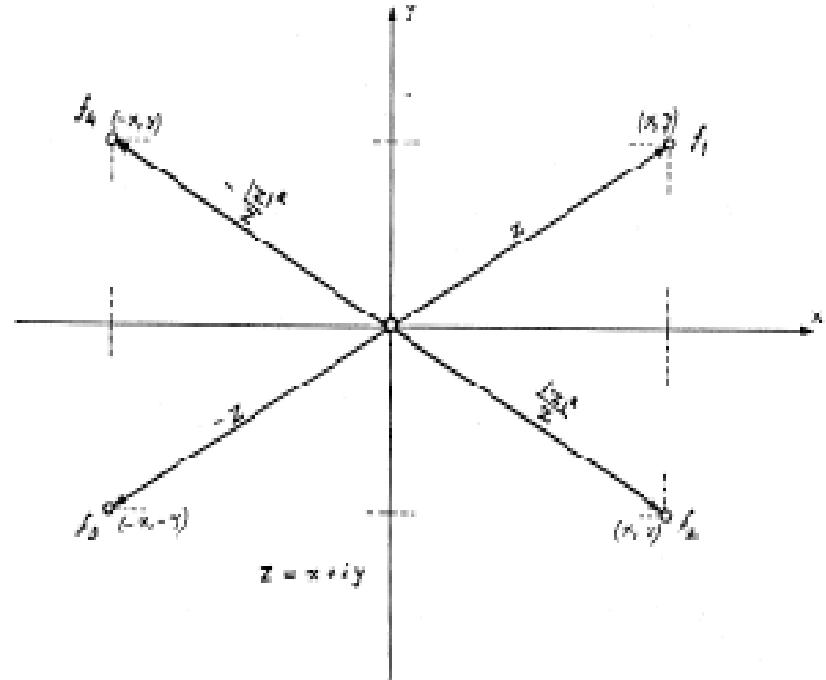
S $\longrightarrow f_1 = Z = x + yi = Z = f_1(Z)$ = original form

I $\longrightarrow f_2 = x - yi = |Z|^2/Z = \zeta_2(Z)$ = inversion

RI $\longrightarrow f_3 = -x - yi = -Z = f_3(Z)$ = inverted retrogradation

R $\longrightarrow f_4 = -x + yi = -(|Z|^2/Z) = f_4(Z)$ = retrogradation

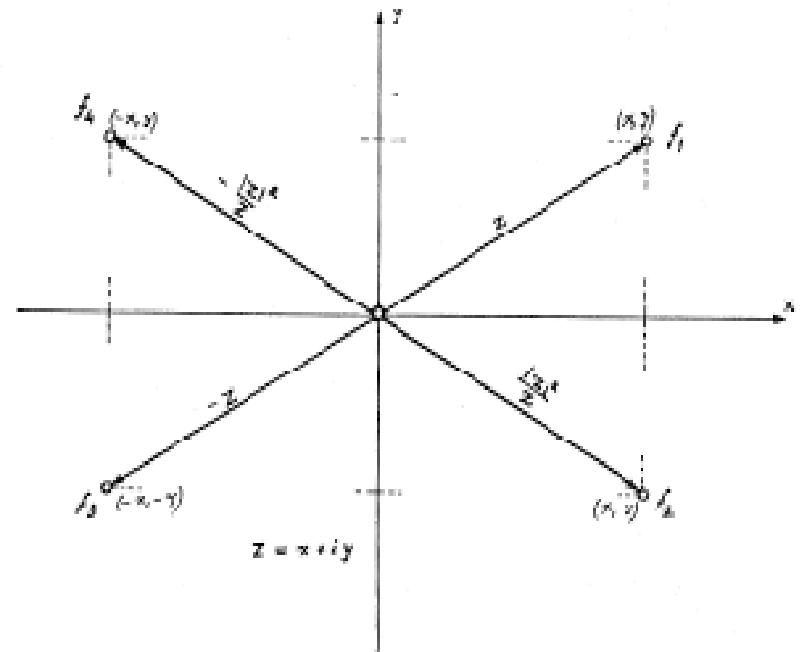
Geometrical/Compositional generalizations



« It is the Klein group (...). But we can imagine other kinds of transformations, as a continuous or discontinuous rotation [...] of any angle. This give rise to new phenomena, new events even by starting from a melody, since a simple melodic line becomes a polyphony »

(Xenakis/Delalande, p.93)

Geometrical and compositional generalizations



$$Z = x + yi$$

$f_1 = Z = x + yi = Z = f_1(Z)$ = original form

$f_2 = x - yi = |Z|^2/Z = f_2(Z)$ = inversion

$f_3 = -x - yi = -Z = f_3(Z)$ = inverted retrogradation

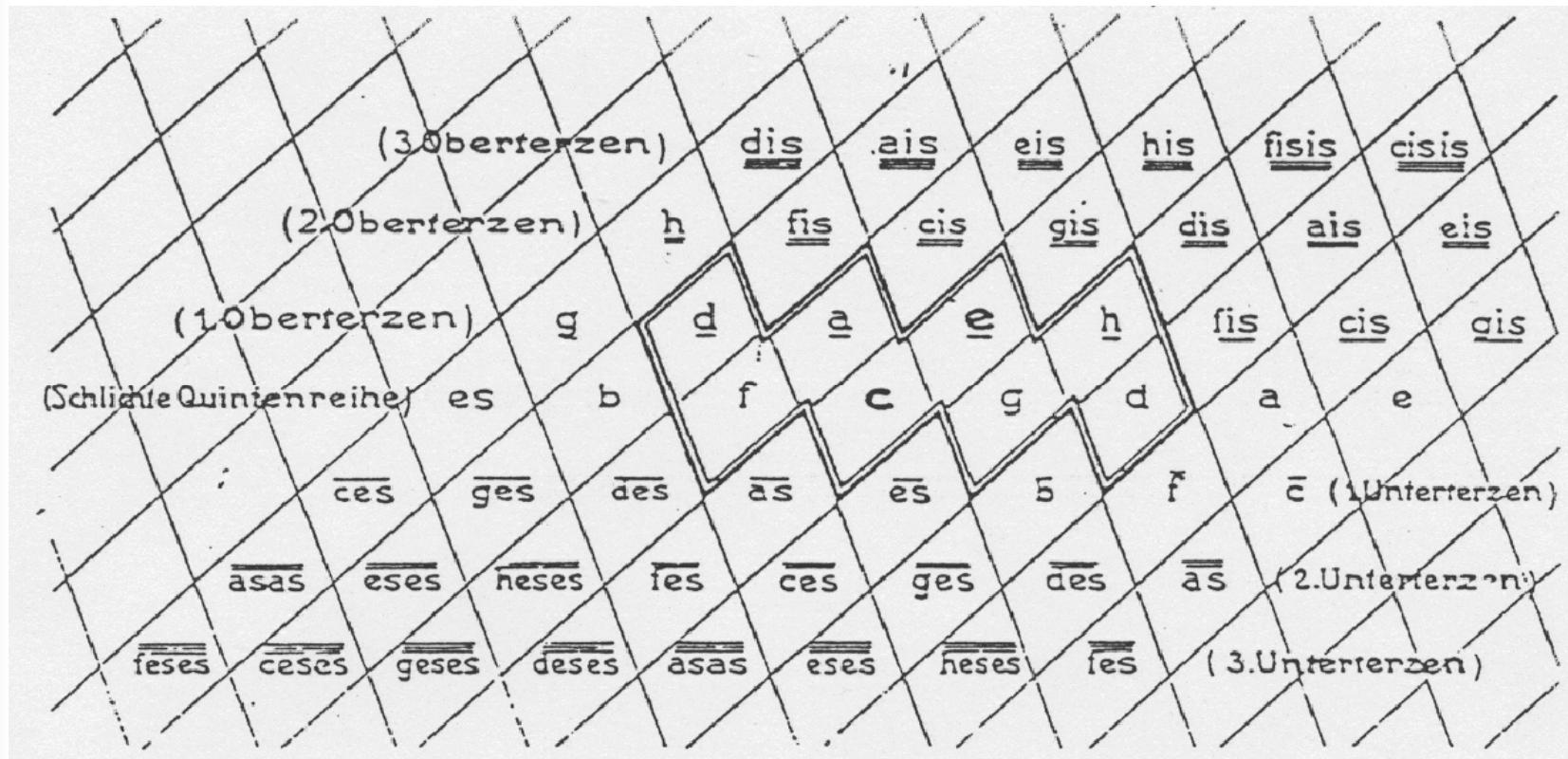
$f_4 = -x + yi = -(|Z|^2/Z) = f_4(Z)$ = retrogradation

« Let us assume that we have such a tree in the pitch versus time domain. We can rotate (transform) it; the rotation can be treated as groups. (...) We can use the traditional transformations of the melodic pattern: we can take the inverse of the basic melody, its retrograde and its retrograde inverse. There are of course many more possible transformations because we can rotate the object at any angle »

(Xenakis/Varga, 1996; p. 89)

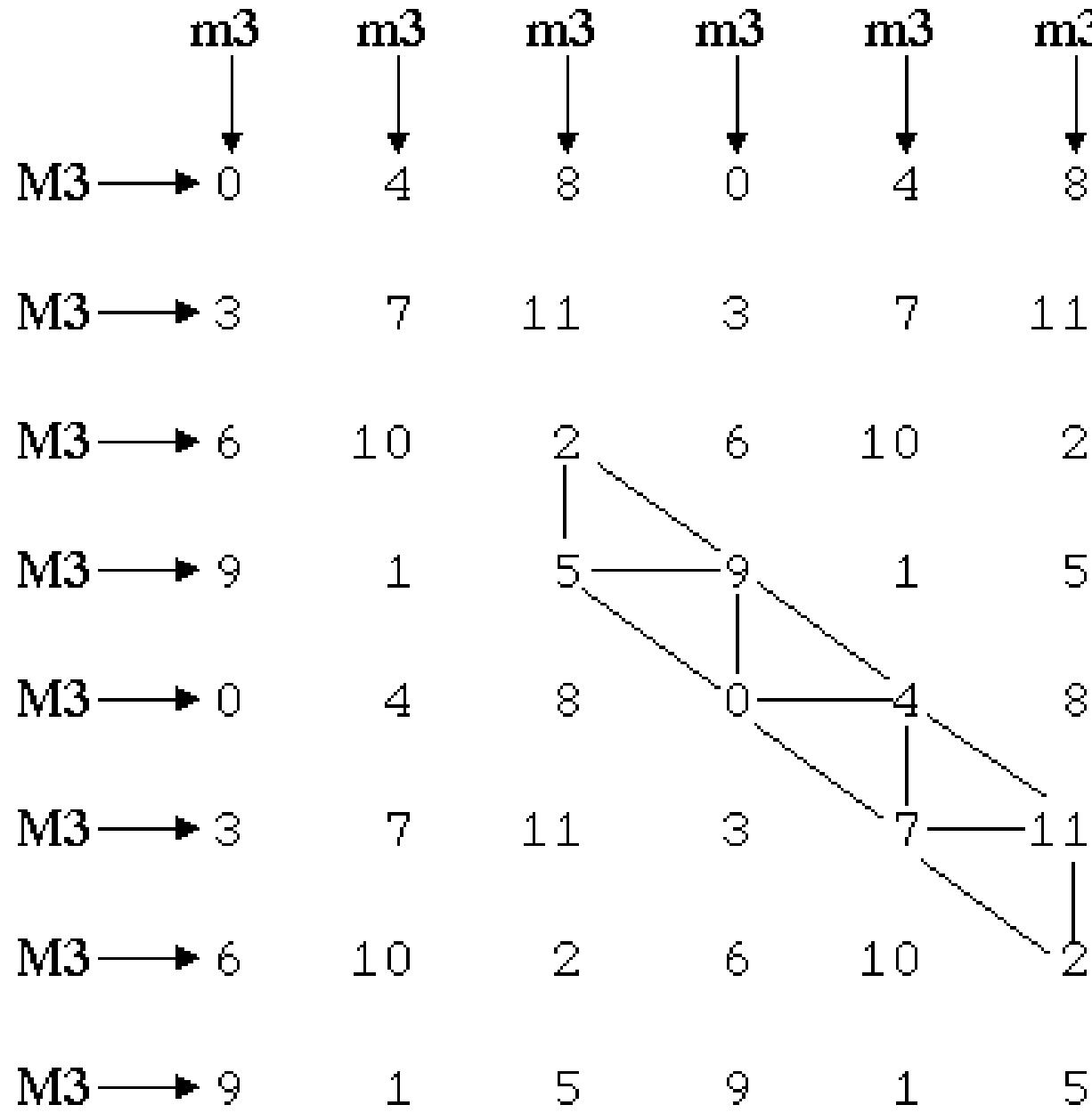
Geometrical representations: the *Tonnetz*

Hugo Riemann : « Ideen zu einer *Lehre von den Tonvorstellung* », 1914

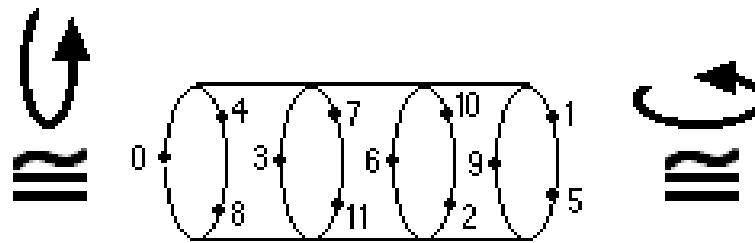
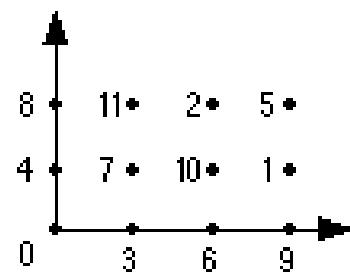


The ‘topological’ properties of the **diatonic structure**,
the only compact and convex subset of the *Tonnetz* (Longuet-Higgins, Balzano, Clough + Buffalo Group, Cohn...)

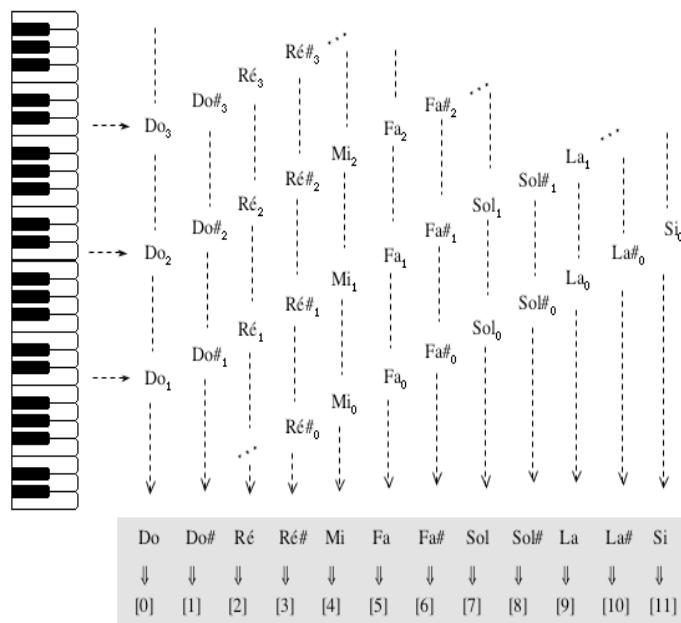
The *Tonnetz* (modulo the octave relation)



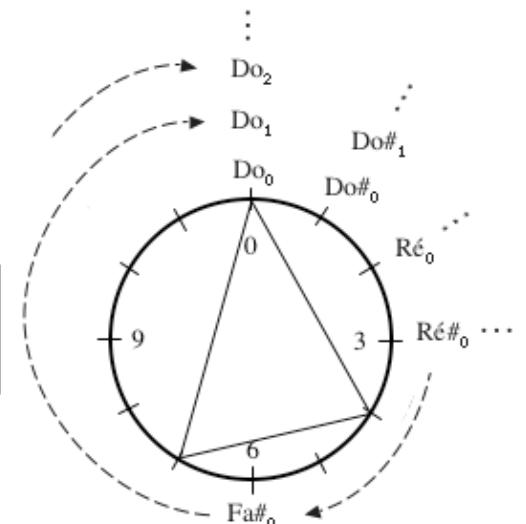
Formalization vs representation



Toroidal representation

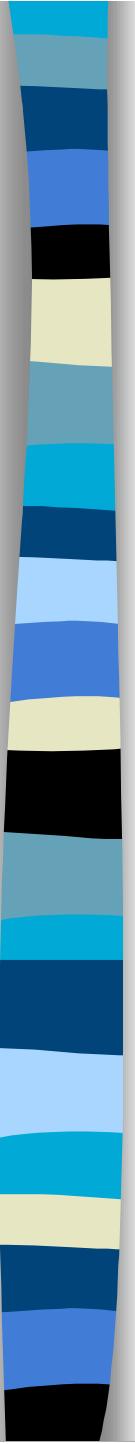


$$\mathbf{Z}_{12} = \mathbf{Z}_3 \times \mathbf{Z}_4$$



(4,3,5)

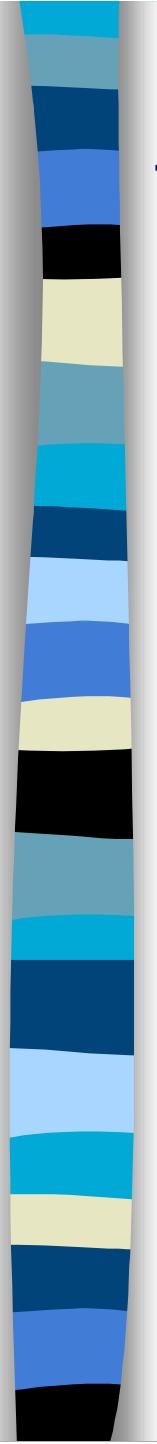
Circular representation



Towards a theory of *formalized music*

« Music can be defined as an organization of [...] **operations** and elementary **relations** between sound events [...]. We understand the place taken by the **mathematical set theory**, non only in the creative process but also in the analysis and better understanding of musical works of the past. Even a stochastic construction or an investigation of the history through stochastic methods may not be entirely exploited without using the king of sciences and of arts, I would say, which is logic or its mathematical form: **algebra** »

Iannis Xenakis : « La musique stochastique : éléments sur les procédés probabilistes de composition musicale », *Revue d'Esthétique*, vol. 14 n°4-5, 1961.



Axiomatization, group and sieve theory

« *Formalization and axiomatization are operational tools [guide processionnel] which are in general more adapted to the modern way of thinking* »

(Musiques formelles, 1963)

« ... we have now a universal formulation for what concerns the pitch **perception**: the space of melodic intervals has a **group** structure having the addition as an internal operation »

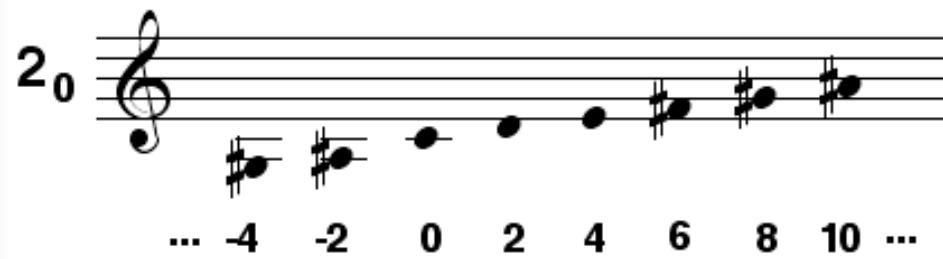
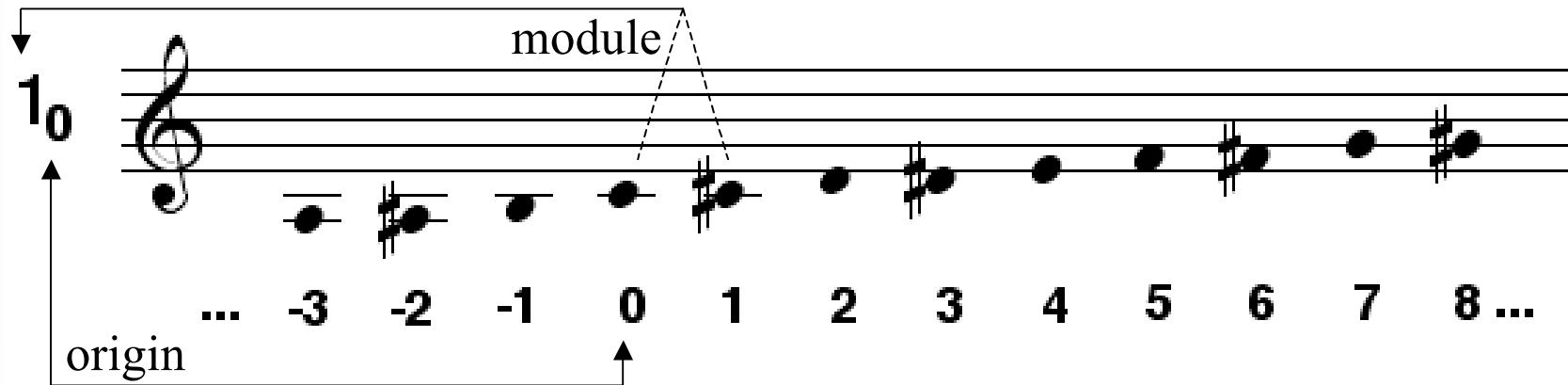
(« La voie de la recherche et de la question », *Preuves*, n° 177, nov. 1965)

« ...Nomos Alpha makes use of the sieve theory, a theory which annexes the residual classes and which is derived from an **axiomatics** of the universal structure of music »

(Score of *Nomos Alpha* for cello solo, 1966)

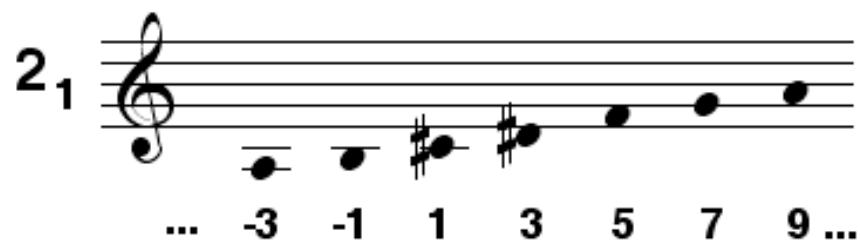
Sieve Theory

Algebraic formalization of musical structures (Xenakis)



$$1_0 = 2_0 \cup 2_1$$

$$2_0 \cap 2_1 = \emptyset$$



$$(2_0)^c = 2_1$$

$$(2_1)^c = 2_0.$$

Sieve Theory

Olivier Messiaen's « Modes à transpositions limitées »

1₀ 2₀

The image shows two musical staves. The top staff, labeled 1₀, has notes at positions ... -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8 ... with sharp symbols above the notes at positions 1, 2, 3, 4, 5, 6, 7, 8. The bottom staff, labeled 2₀, has notes at positions ... -4, -2, 0, 2, 4, 6, 8, 10 ... with sharp symbols above the notes at positions 0, 2, 4, 6, 8, 10.

$$T_m(A) = A$$

$$m \neq 0 \pmod{12}$$

(3₀)

Diminished chord

(4₀)

Augmented chord

(6₀)

Triton

6₀ ∪ 6₁

A musical staff with notes at positions ... -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8 ... with sharp symbols above the notes at positions 1, 2, 3, 4, 5, 6, 7, 8. Dotted rectangles highlight the notes at positions 0, 1, 2, 3, 4, 5, 6, 7, 8, which are labeled 6₀ and 6₁.

6₀ ∪ 6₃

?

A musical staff with notes at positions ... -4, -2, 0, 2, 4, 6, 8, 10 ... with sharp symbols above the notes at positions 0, 2, 4, 6, 8, 10. Dotted rectangles highlight the notes at positions 0, 2, 4, 6, 8, 10, which are labeled 6₀ and 6₂.

6₀ ∪ 6₂

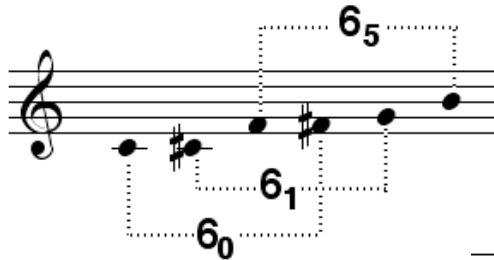
Sieves / Messiaen

Towards a complete catalogue

(1_0)	(3_0)
(2_0)	(4_0)
	(6_0)

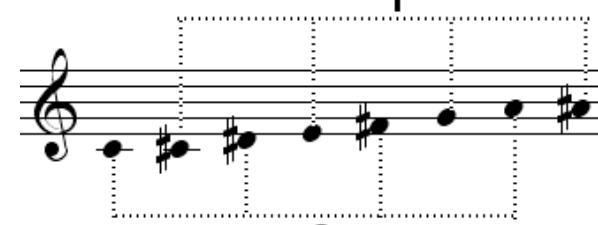
$6_0 \cup 6_1 \cup 6_5$

Mode n.5



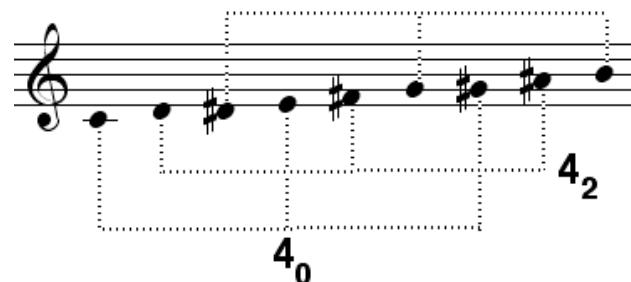
$3_0 \cup 3_1$

Mode n.2

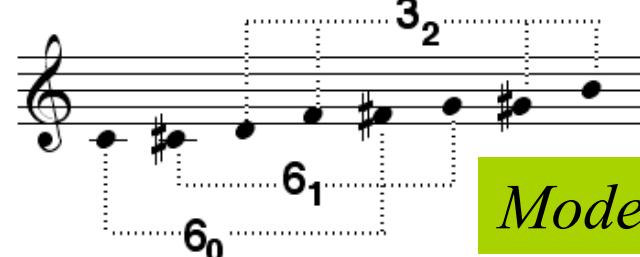


$4_0 \cup 4_2 \cup 4_3$

Mode n.3



$6_0 \cup 6_1 \cup 3_2$

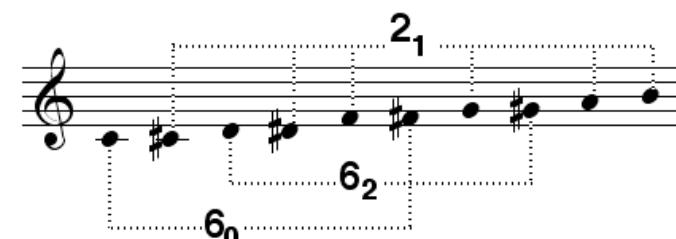


Mode n.4



$2_0 \cup 6_5$

Mode n.6



Mode n.7

$2_1 \cup 6_0 \cup 6_2$

(6₀)

(2₀)

6₀ ∪ 6₁

6₀ ∪ 6₁ ∪ 6₅

4₀ ∪ 4₂ ∪ 4₃

3₀ ∪ 3₁

(1₀)

(3₀)

(4₀)

6₀ ∪ 6₂

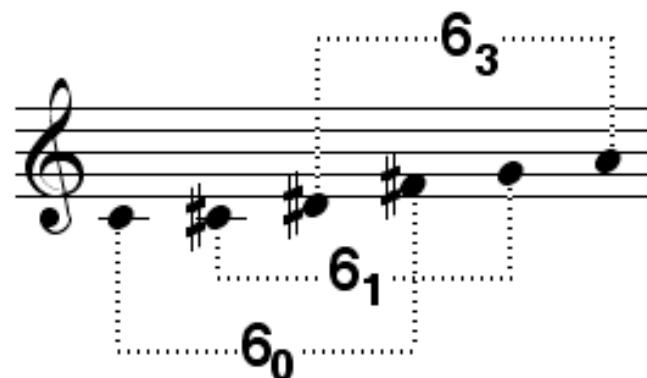
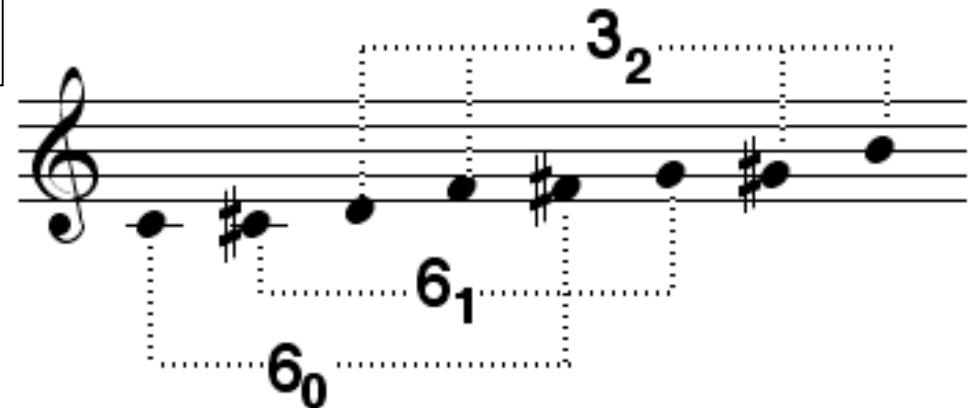
2₀ ∪ 6₅

2₁ ∪ 6₀ ∪ 6₂

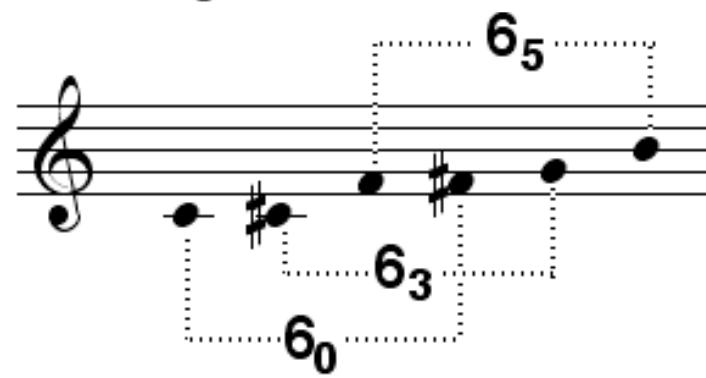
6₀ ∪ 6₁ ∪ 3₂

4₀ ∪ 4₁

Three forgotten modes



6₀ ∪ 6₁ ∪ 6₃



6₀ ∪ 6₃ ∪ 6₅

Pitch/Rhythm Isomorphism (Xenakis)

« [With the sieve theory] one can build very complex **rhythmic architectures** which can simulate the stochastic distribution of points on a line if the period is big enough »

(« Redécouvrir le temps », éditions de l'Université de Bruxelles, 1988)

$$A = (13_3 \cup 13_5 \cup 13_7 \cup 13_9)^c$$

$$B = 11_2$$

$$C = (11_4 \cup 11_8)^c$$

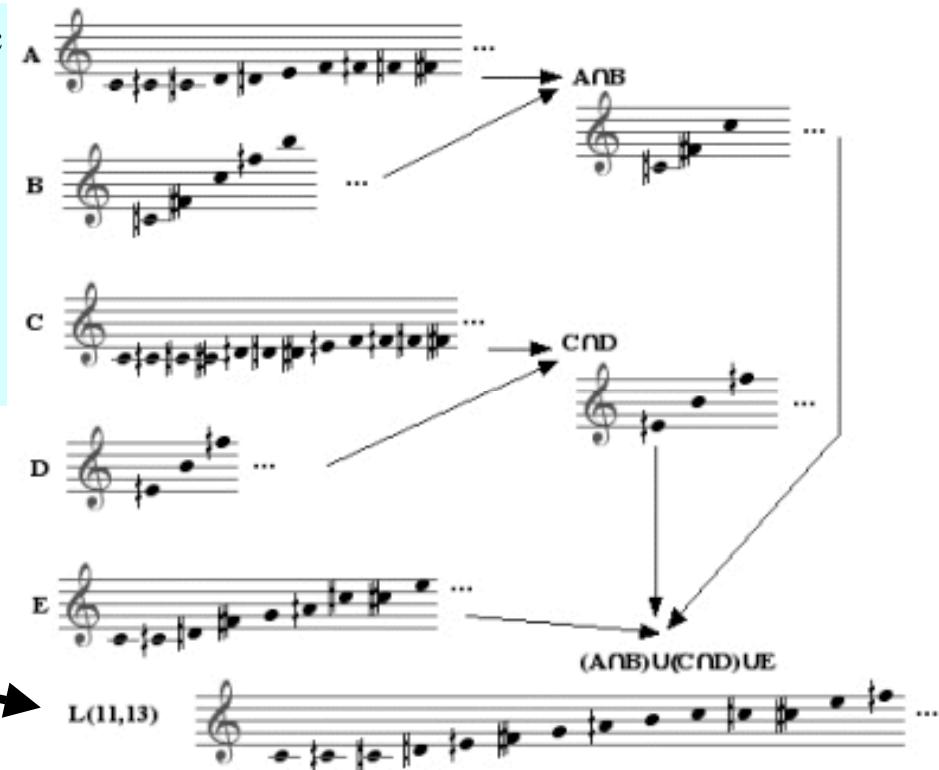
$$D = 13_9$$

$$E = 13_0 \cup 13_1 \cup 13_6$$

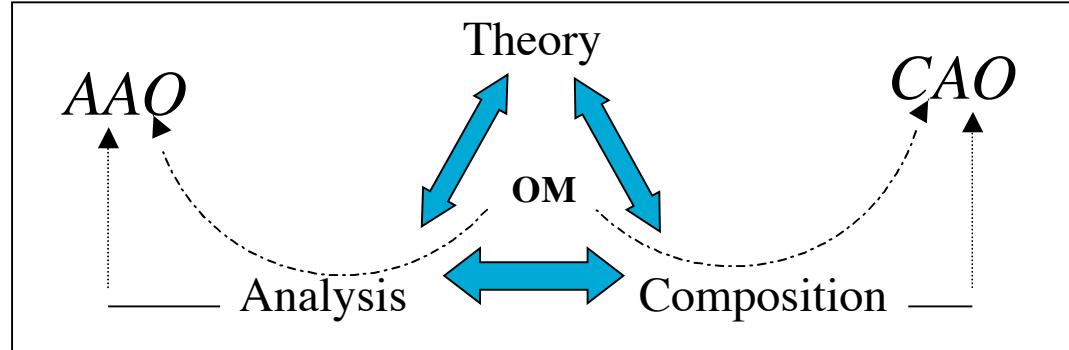
↓
(Nomos Alpha, 1966)

$$(A \cap B) \cup (C \cap D) \cup E$$

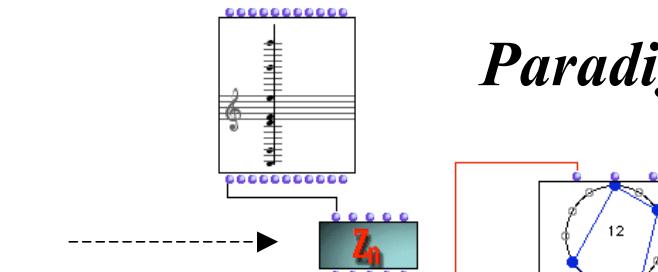
↓
1 1 4 3 4 1 5 3 2 2 1 5 3 4 1 5 7 1 5 3 4 1 5 7 1 5 3 3 1 1 5 3 1 3 1 5 2 1 4 1...



Sieve Theory, CAO, AAO & TAO

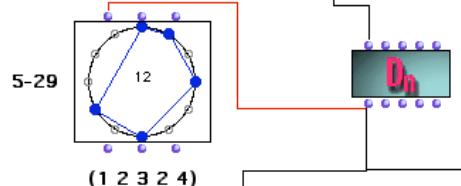


Cyclic group

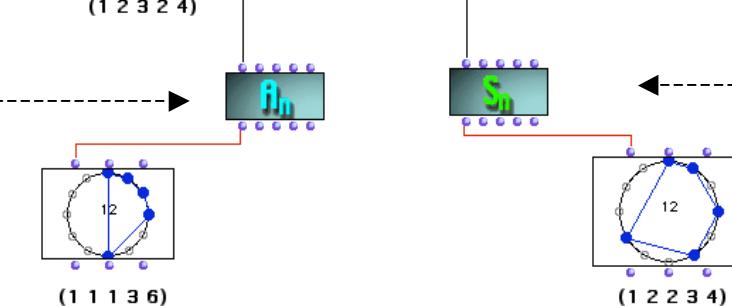


Paradigmatic architecture

Dihedral group
(*Set Theory*)



Affine group



Symmetric group

Paradigmatics and TAO

Algebraic-oriented catalogues of musical structures

Zalewski / Vieru / Halsey & Hewitt

Forte / Rahn

Morris / Mazzola

Estrada

158

208

351

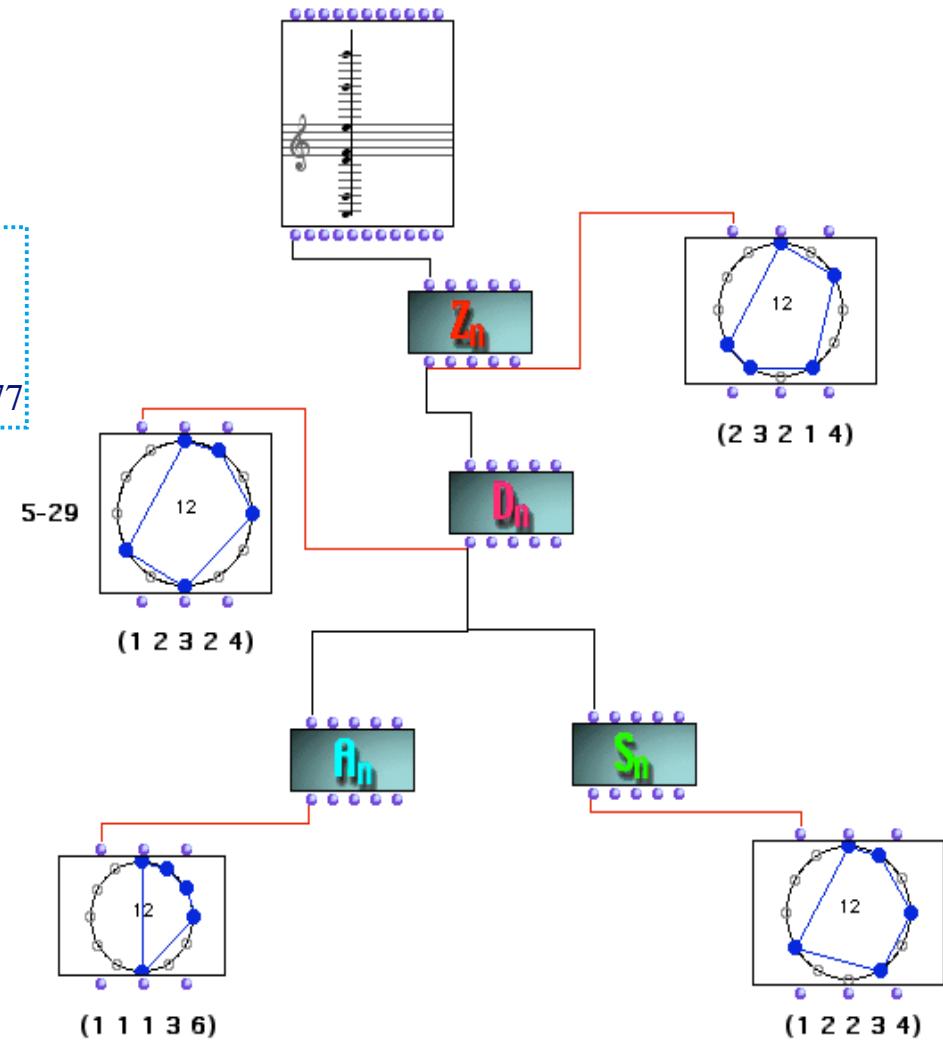
77

S_n

Aff_n

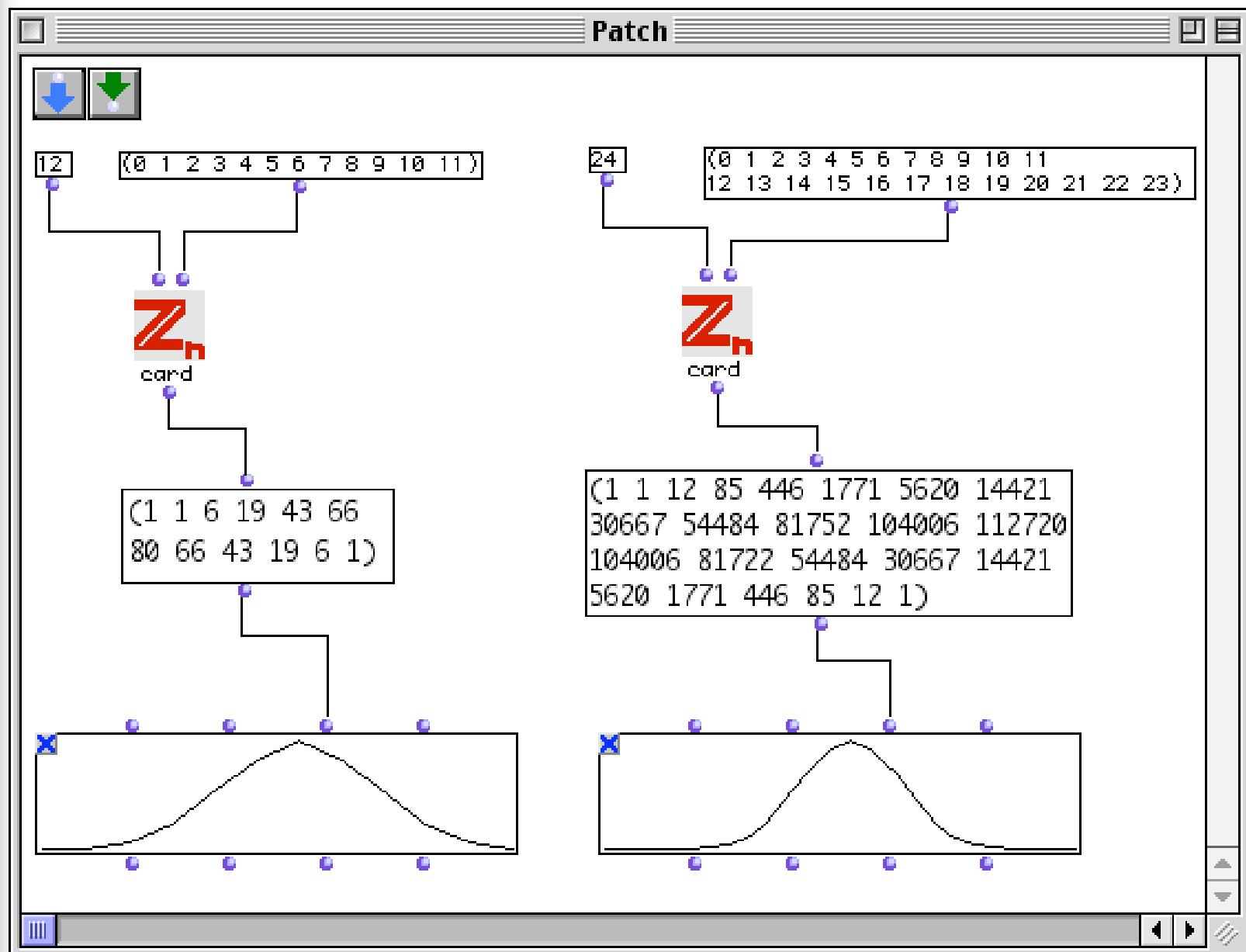
D_n

Z/nZ



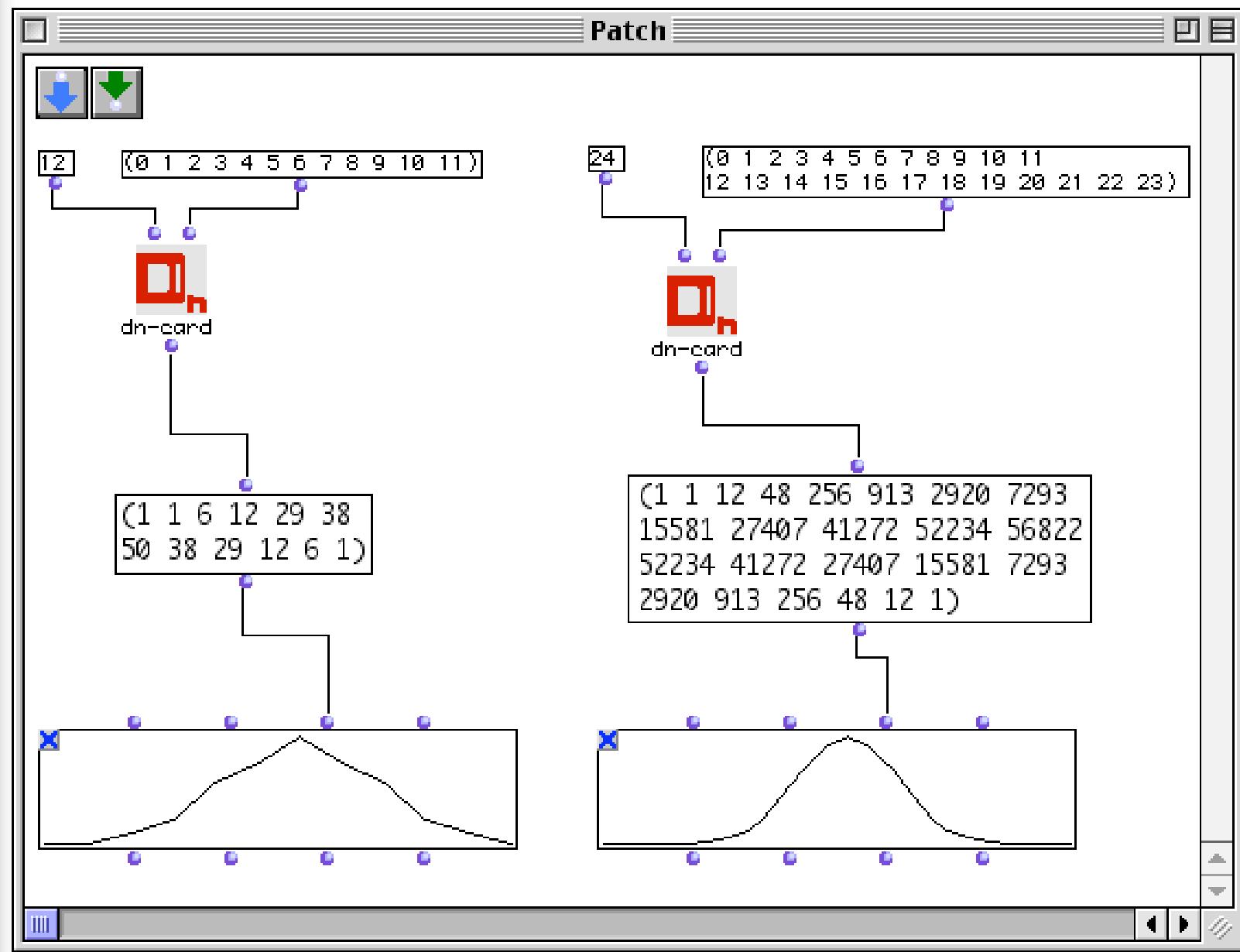
Computational aspects

Enumeration of transposition classes in a tempered space $\mathbb{Z}/n\mathbb{Z}$



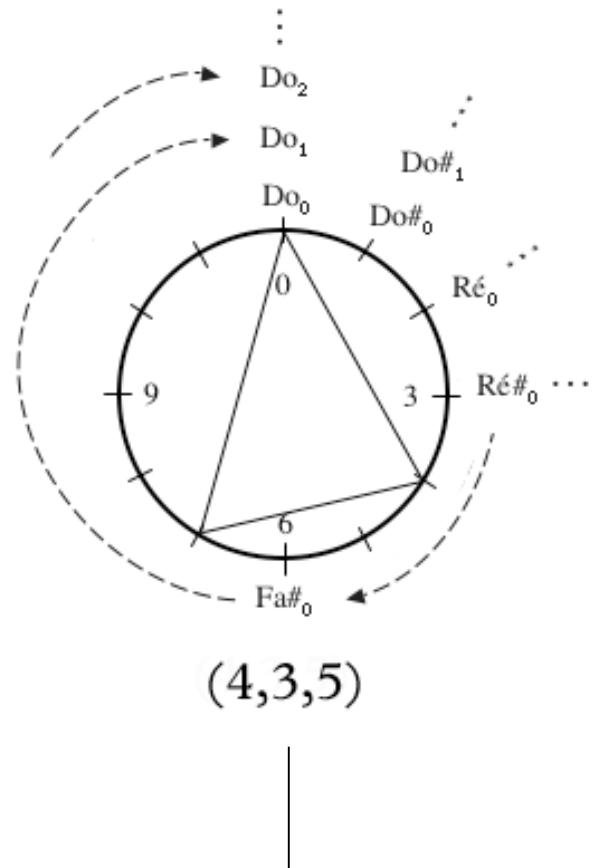
Computational aspects

Enumeration of transposition classes in a tempered space Dn



Anatol Vieru's modal theory

« A theory of relations between notes and intervals »



Intervallic structure

The congruence modulo 12 is an equivalence relation

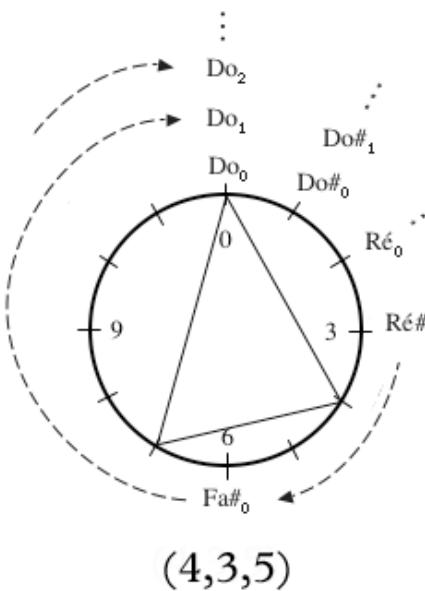
- Reflexivity: $a \sim a$
- Symmetry: $a \sim b \Leftrightarrow b \sim a$
- Transitivity: $a \sim b, b \sim c \Rightarrow a \sim c$

*The equivalence classes modulo 12 have a structure of **cyclic group** (of order 12) together with $+_{mod12}$*

- Internal law
- Existence of identity
- Existence of inverse
- Associativity
- There exists at least one element which generate the group

How many elements do generate the group $\mathbb{Z}/12\mathbb{Z}$?

Circular representation and intervallic structure

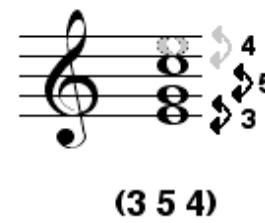


« *A mode is, by definition, a collection of equivalence classes modulo 12* »

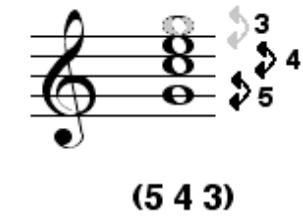
... Cartea Modurilor, 1980 (The Book of Modes, 1993)



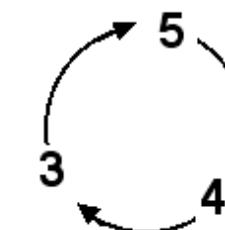
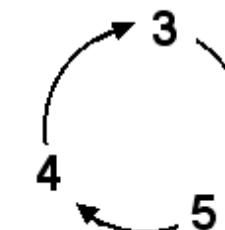
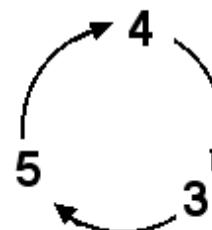
(4 3 5)



(3 5 4)

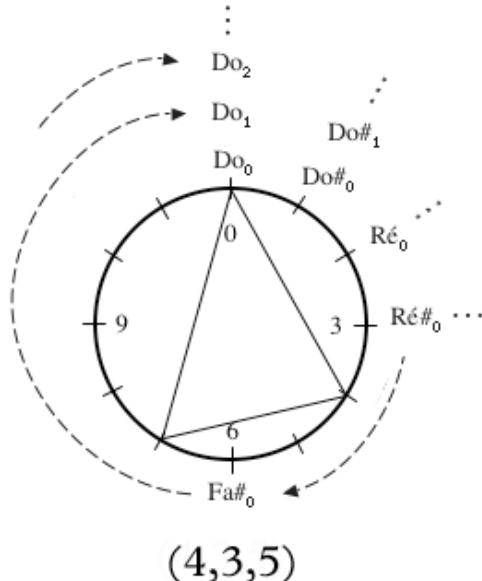


(5 4 3)



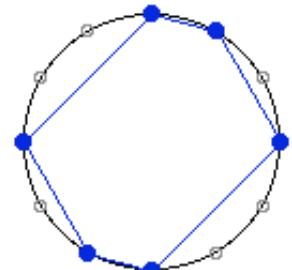
The « *inversions* » of a chord are all circular permutations on an intervallic structure

Circular representation, inversion and Messiaen

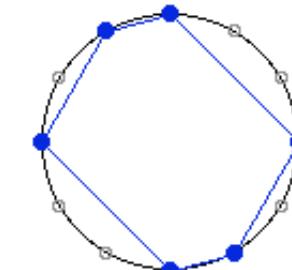


*The circular representation shows the properties of **symmetry** of a musical structure (mode, chord, scale, rhythm, ...)*

Messiaen's modes (or transp. symmetry)



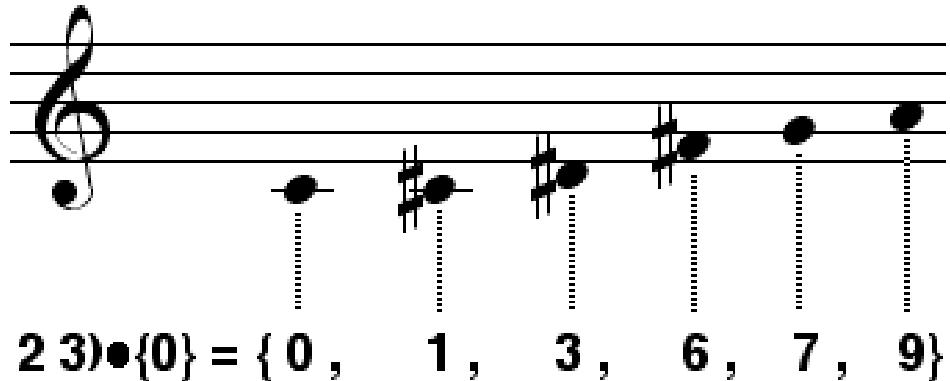
(1 2 3 1 2 3)



(3 2 1 3 2 1)

The composition « • » between modal structures

From an intervallic structure to the corresponding mode



$$(1 \ 2 \ 3 \ 1 \ 2 \ 3) \bullet \{0\} = \{0, 1, 3, 6, 7, 9\}$$

Composition between one intervallic structure and one mode

$$(6 \ 6) \bullet \{0, 1, 3\} = ?$$

$$= ((6 \ 6) \bullet \{0\}) \cup ((6 \ 6) \bullet \{1\}) \cup ((6 \ 6) \bullet \{3\}) =$$

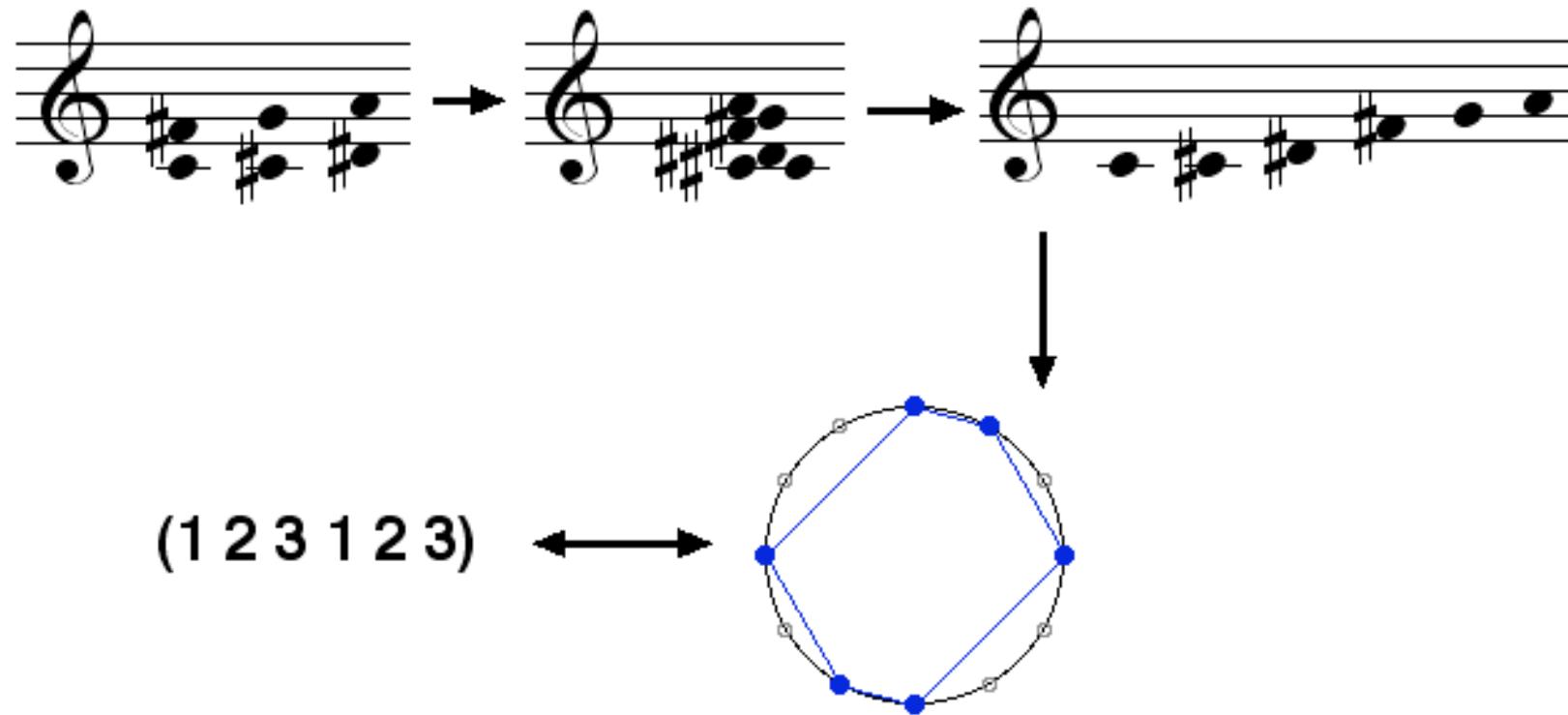
$$= \{0, 6\} \cup \{1, 7\} \cup \{3, 9\} =$$

$$= \{0, 1, 3, 6, 7, 9\}.$$

$$\boxed{6_0 \cup 6_1 \cup 6_3}$$

sieve

« • », chord multiplication (Boulez) & TC (Cohn)



Composition of two intervallic structures

$$(6\ 6) \bullet (1\ 2\ 9) = ?$$

The composition of two intervallic structures

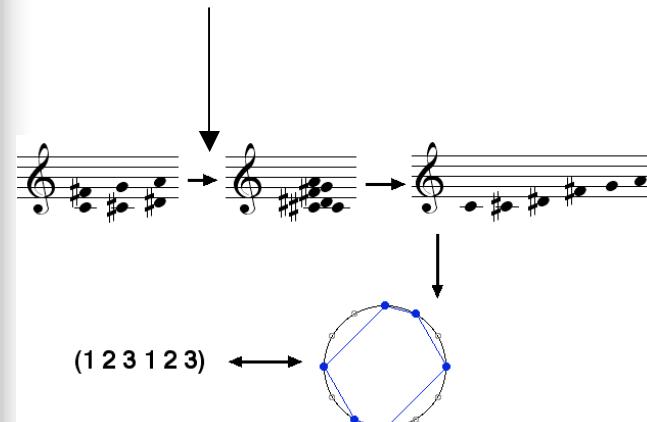
$$(6\ 6) \bullet (1\ 2\ 9) = ?$$

$$(6\ 6) \bullet \{0, 1, 3\} =$$

...

$$= \{0, 1, 3, 6, 7, 9\}$$

$$(1\ 2\ 3\ 1\ 2\ 3)$$



The composition between intervallic structures is well defined!

$$(6\ 6) \bullet \{1, 2, 4\} =$$

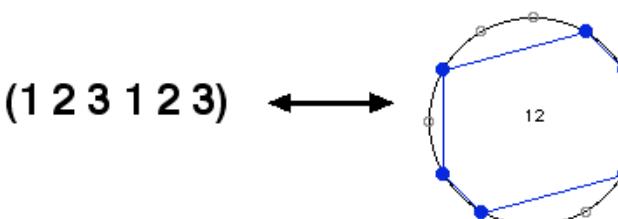
$$= \{1, 7\} \cup \{2, 8\} \cup \{4, 10\} =$$

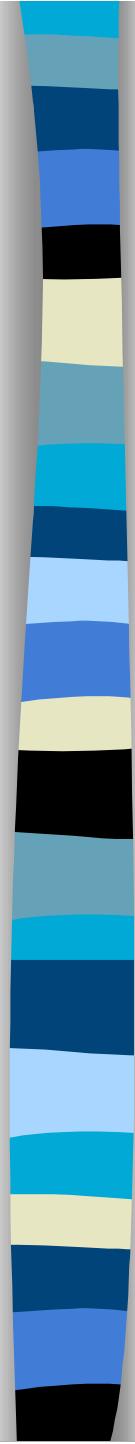
$$= \{1, 2, 4, 7, 8, 10\}$$

$$(1\ 2\ 3\ 1\ 2\ 3)$$



$$(1\ 2\ 3\ 1\ 2\ 3)$$





Some applications of the composition operation

- The construction of (generalized) Messiaen's modes
- The construction of rhythmic canons

$$(6\ 6) \bullet \{0, 1, 3\} = \dots = \{0, 1, 3, 6, 7, 9\} \rightarrow (1\ 2\ 3\ 1\ 2\ 3)$$

$$(6\ 6) \bullet \{0, 1\} = \dots = \{0, 1, 6, 7\} \rightarrow (1\ 5\ 1\ 5)$$

$$(6\ 6) \bullet \{a, b, c, \dots\} \rightarrow \text{Messiaen's mode}$$

$$A_1 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$$

$$A_2 = (2, 2, 2, 2, 2, 2)$$

$$A_3 = (3, 3, 3, 3)$$

$$A_4 = (4, 4, 4)$$

$$A_6 = (6, 6)$$

$$A_7 = (0)$$

Chromatic total (aggregate)

Whole-tone scale

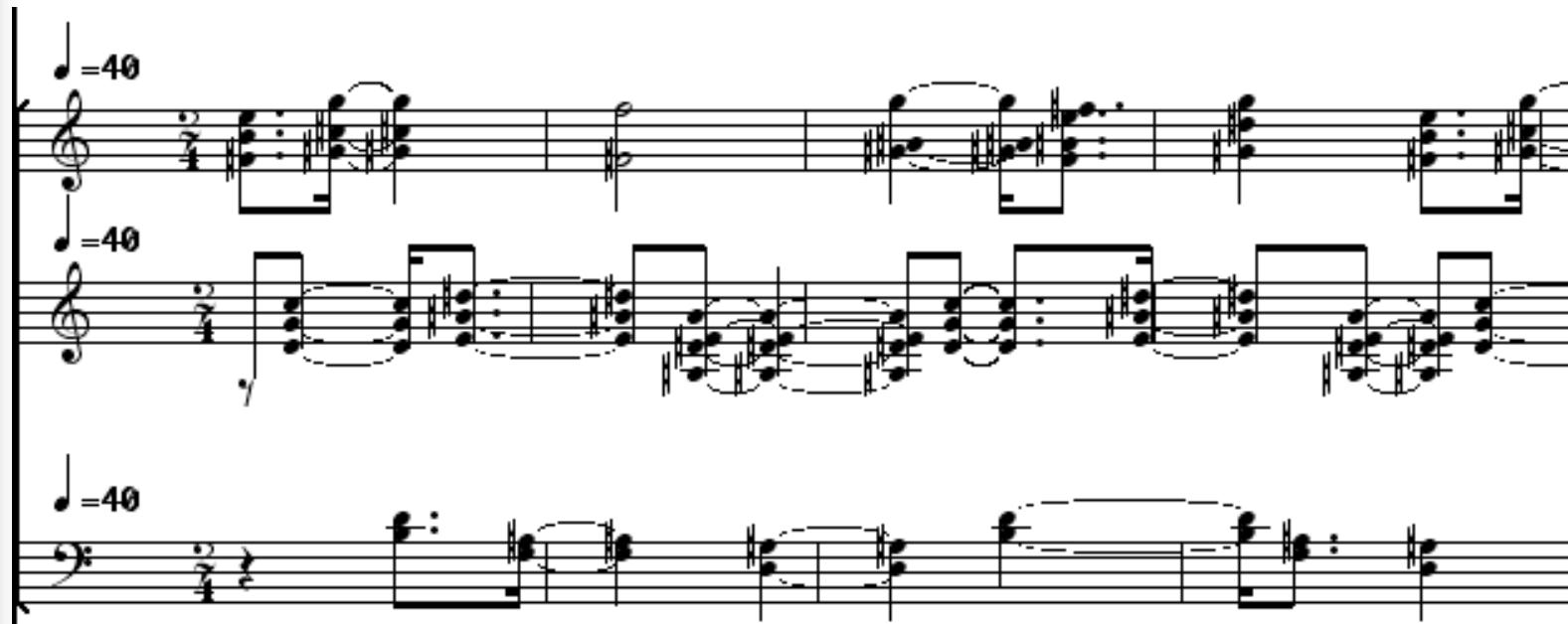
Diminished chord

Augmented triad

Triton

pc

The construction of rhythmic (tiling) canons



O. Messiaen: *Harawi* (1945)



« Remarquons [...] que les trois **rythmes non rétrogradables** divisent les durées en 5+5+7 durées, alors que les termes des trois ostinatos harmoniques contiennent toujours six sonorités pour le supérieur, et trois sonorités pour les deux autres. Ajoutons que les durées sont très inégales »

O. Messiaen : *Traité de Rythme, de Couleur et d'Ornithologie*, tome 2, Alphonse Leduc, Editions Musicales, Paris, 1992.

The construction of rhythmic (tiling) canons

The image shows two musical score snippets. The left snippet, labeled 'Harawi (1945)', consists of three staves: treble, bass, and alto. The right snippet, labeled 'Visions de l'Amen (1943)', consists of three staves: soprano, alto, and bass. Both snippets are set at a tempo of quarter note = 40. The music features complex rhythms and time signatures, typical of Messiaen's rhythmic canons.

Harawi (1945)

Visions de l'Amen (1943)

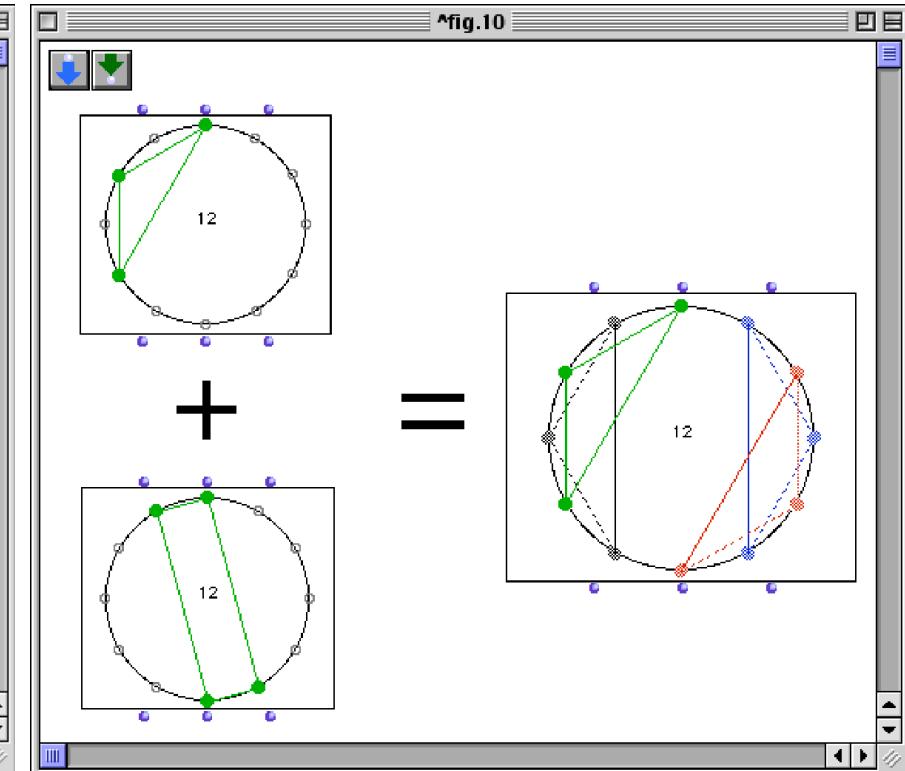
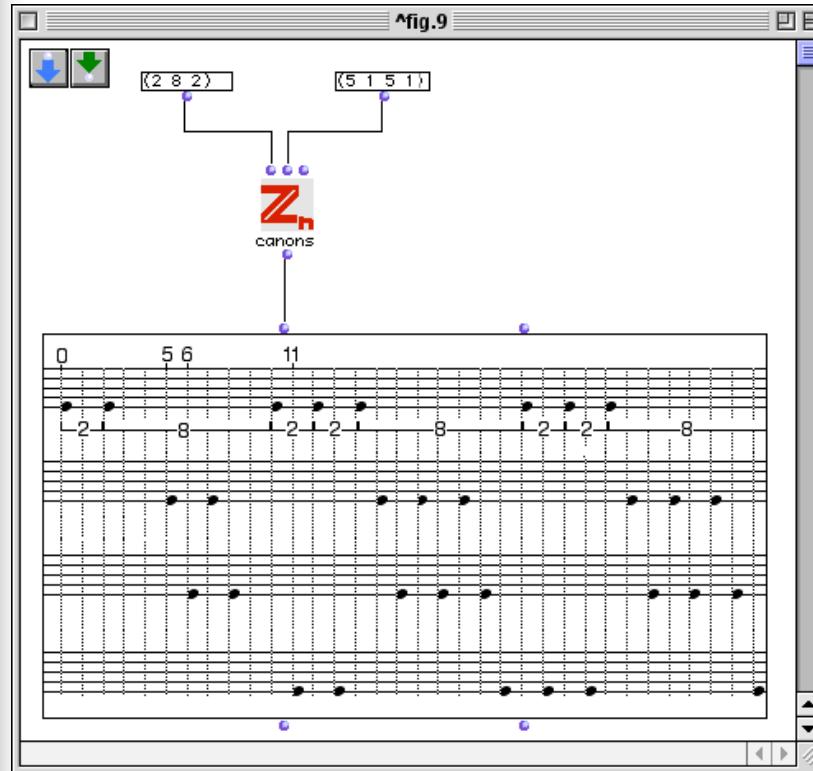
A rhythmic model diagram consisting of three staves, each with a treble clef. Vertical dotted grid lines align notes from all three staves. Blue dots represent one set of notes, while black dots represent another, illustrating the vertical organization of rhythmic patterns.

Rhythmic
model

« ...il résulte de tout cela que les différentes sonorités se mélangeant ou s'opposent de manières très diverses, **jamas au même moment ni au même endroit [...]. C'est du désordre organisé** »

O. Messiaen : *Traité de Rythme, de Couleur et d'Ornithologie*, tome 2, Alphonse Leduc, Editions Musicales, Paris, 1992.

Canons as Composition between modal structures

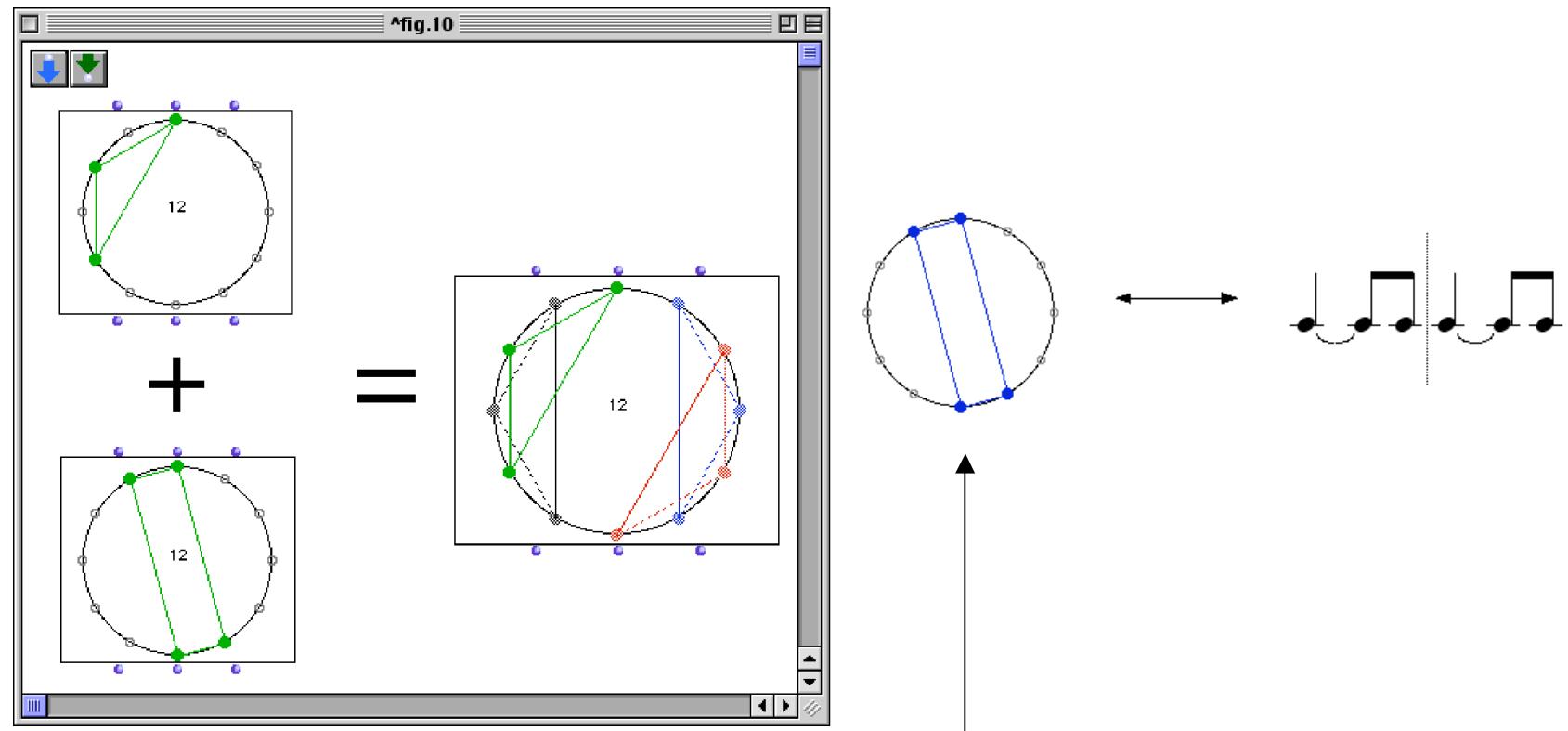


$$(2\ 8\ 2) \bullet (5\ 1\ 5\ 1) = Z/12Z$$



Transposition limited mode

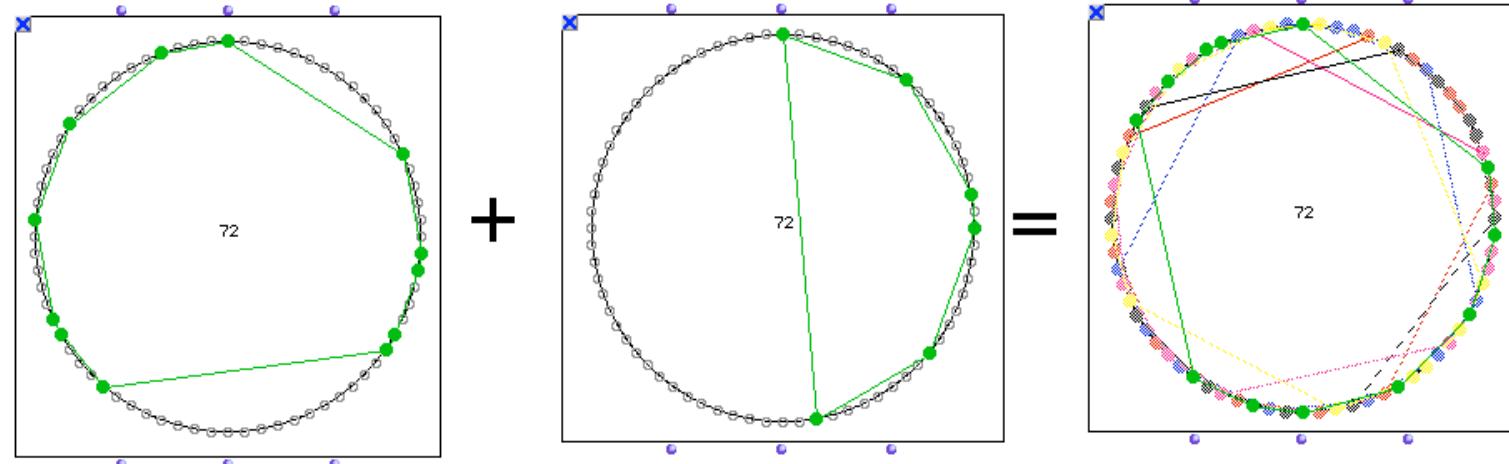
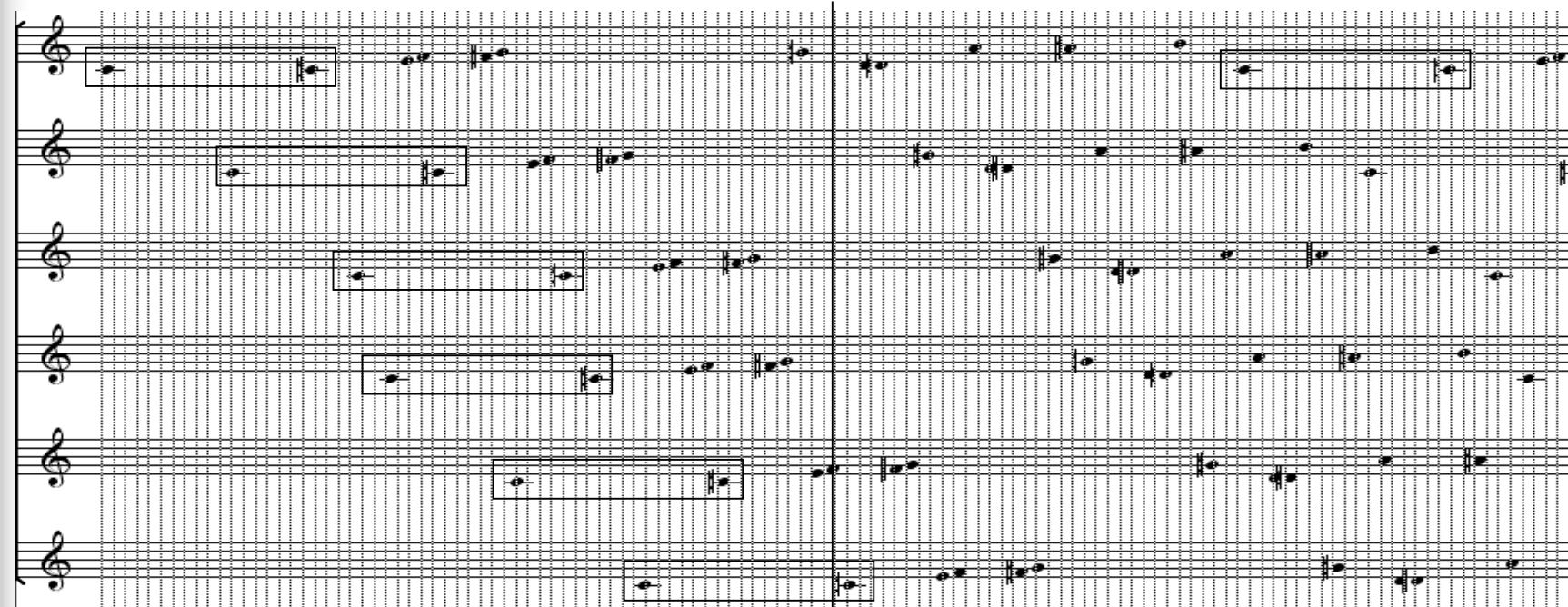
Rhythrical redundancy of a Messiaen's mode



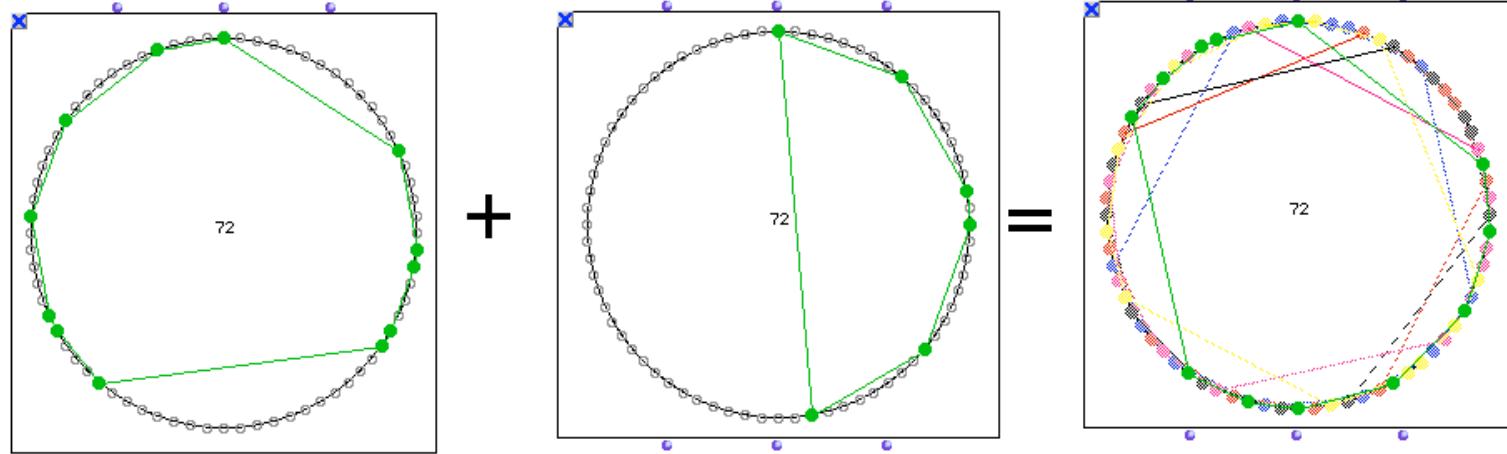
$$(2\ 8\ 2) \bullet (5\ 1\ 5\ 1) = \mathbf{Z}/12\mathbf{Z}$$

Messiaen's mode

RCMC-Canons (Vuza, 1992)

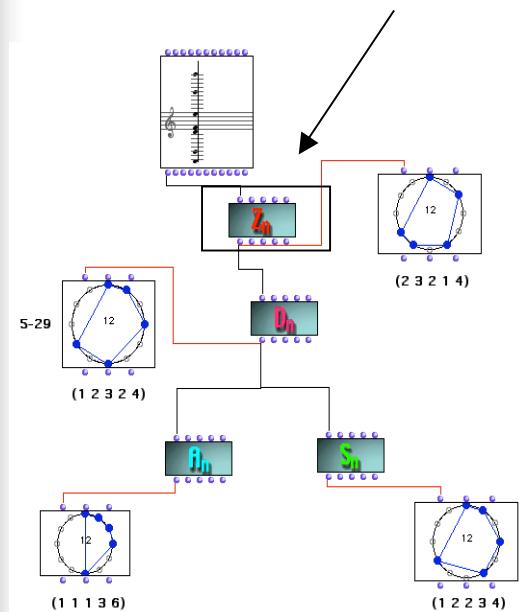
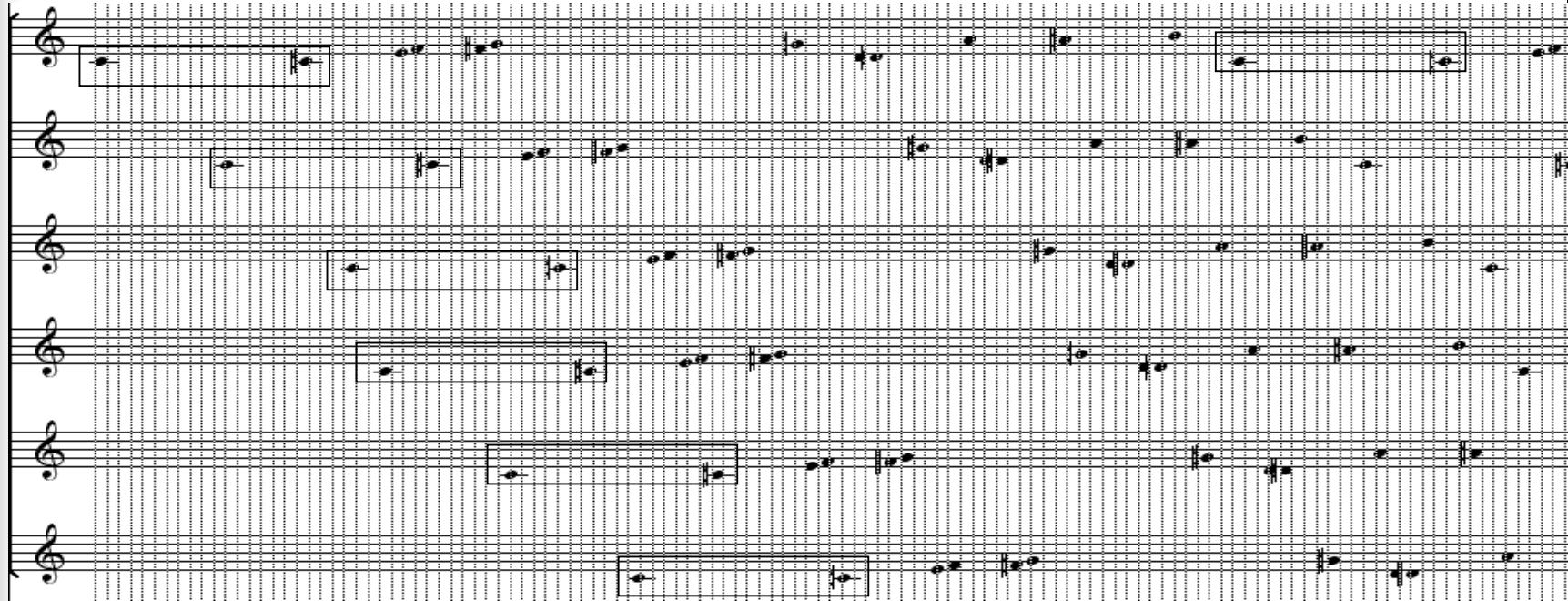


The family of RCMC-Canons



$n \in (72 \ 108 \ 120 \ 144 \ 168 \ 180 \ 200 \ 216 \ 240 \ 252 \ 264 \ 270 \ 280 \ 288 \ 300 \ 312 \ 324 \ 336 \ 360 \ 378 \ 392 \ 396 \ 400 \ 408 \ 432 \ 440 \ 450 \ 456 \ 468 \ 480 \ 500 \ 504 \ 520 \ 528 \ 540 \ 552 \ 560 \ 576 \ 588 \ 594 \ 600 \ 612 \ 616 \ 624 \ 648 \ 672 \ 675 \ 680 \ 684 \ 696 \ 700 \ 702 \ 720 \ 728 \ 744 \ 750 \ 756 \ 760 \ 784 \ 792 \ 800 \ 810 \ 816 \ 828 \ 864 \ 880 \ 882 \ 888 \ 900 \ 912 \ 918 \ 920 \ 936 \ 945 \ 952 \ 960 \ 968 \ 972 \ 980 \ 984 \ 1000)$

Computational and philosophical aspects



- Enumeration of all possible solutions
- Paradigmatic classification
- Reduction of voices and modulation
- Tiling structures and perception
- Local/global duality

The note/interval duality in Vieru's modal theory

The theory of periodic sequences (*Cartea Modurilor*, 1980)

$$f = \begin{matrix} 1 & 1 & 6 & 7 & 2 & 3 & 10 & 11 & 6 & 7 & 2 & 3 & 10 \dots \\ & \backslash & \backslash & & & & & & & & & & \end{matrix}$$

$$Df = \begin{matrix} & 7 & 1 & 7 & 1 & 7 & 1 & 7 & 1 \dots \\ & \backslash & & & & & & & \end{matrix}$$

$$D^2 f = \begin{matrix} & & 6 & 6 & 6 & 6 \dots \\ & & \backslash & & & \end{matrix}$$

$$D^3 f = \begin{matrix} & & & 0 & 0 & 0 \dots \\ & & & \backslash & & \end{matrix}$$

$$Df(x) = f(x) - f(x-1).$$

Reducible sequences: there exists an integer $k \geq 1$ s.t. $D^k f = 0$

$$f = \begin{matrix} 7 & 1 & 1 & 10 & 1 & 1 & 7 & 2 & 7 & 1 & 1 & 10 & 1 & 1 & 7 & 2 & 7 & 1 & 1 \dots \\ & \backslash & \backslash & \backslash & & & & & & & & & & & & & & & \end{matrix}$$

$$Df = \begin{matrix} 4 & 1 & 1 & 1 & 8 & 7 & 5 & 4 & 1 & 1 & 1 & 8 & 7 & 5 & 4 & 1 & 1 \dots \\ & \backslash & & & & & & & & & & & & & & & & \end{matrix}$$

$$D^2 f = \begin{matrix} 7 & 2 & 7 & 1 & 1 & 0 & 1 & 1 & 7 & 2 & 7 & 1 & 1 \dots \\ & \backslash & & & & & & & & & & & \end{matrix}$$

$$D^3 f = \begin{matrix} & & 1 & 8 & 7 & 5 & 4 & 1 & 1 & 1 & 8 & 7 & 5 & 4 & 1 & 1 \dots \\ & & \backslash & & & & & & & & & & & & & \end{matrix}$$

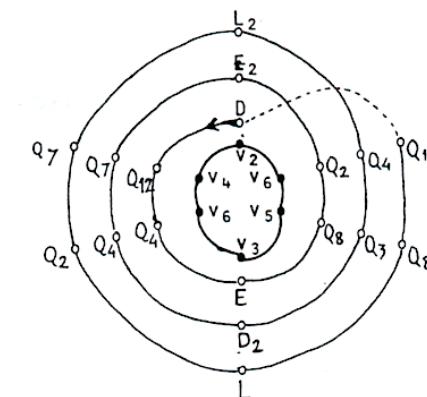
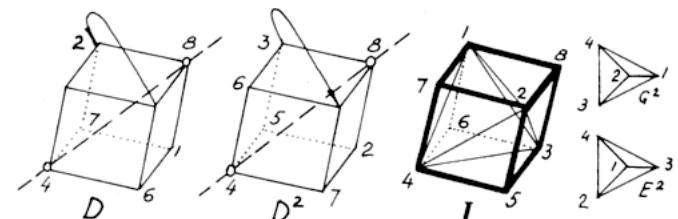
$$D^4 f = \begin{matrix} & & 7 & 1 & 1 & 10 & 1 & 1 & 7 & 2 & 7 & 1 & 1 & 10 & 1 & 1 & 7 & 2 & 7 & 1 & 1 \dots \\ & & & & & & & & & & & & & & & & & & & \end{matrix}$$

Reproducible sequences: there exists an integer $k \geq 1$ s.t. $D^k f = f$

OM implementation of *Nomos Alpha* (1966)

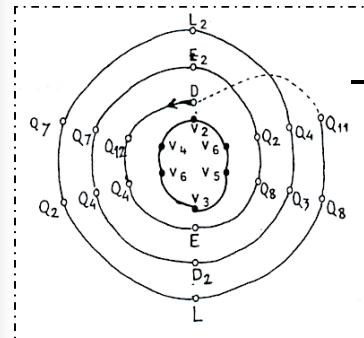
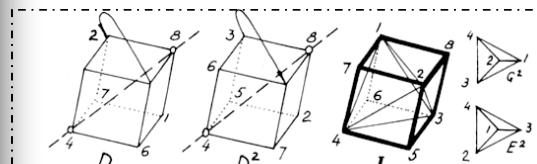
analysis  composition

- « *Musique symbolique pour violoncelle seul, possède une architecture « hors-temps » fondée sur la théorie des groupes de transformations. Il y fait usage de la théorie des cibles, théorie qui annexe les congruence modulo n et qui est issue d 'une axiomatique de la structure universelle de la musique ».*



 OpenMusic

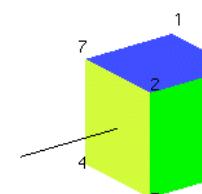
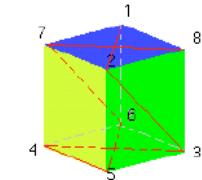
Group theory, cognition and perception



Cognition/Perception
- Group transformation
and Fibonacci process

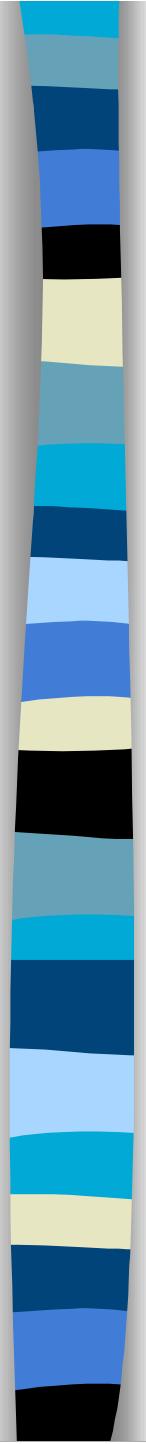
CUBE-TABLE1													
		CUBE-TABLE1											
		manual	loop	play	stop	clear							
I							I						
A							A						
B							B						
C							C						
D							D						
D ²							D ²						
E							E						
E ²							E ²						
G							G						
G ²							G ²						
L							L						
L ²							L ²						
Q ₁							Q ₁						
Q ₂							Q ₂						
Q ₃							Q ₃						
Q ₄							Q ₄						
Q ₅							Q ₅						
Q ₆							Q ₆						
Q ₇							Q ₇						
Q ₈							Q ₈						
Q ₉							Q ₉						
Q ₁₀							Q ₁₀						
Q ₁₁							Q ₁₁						
Q ₁₂							Q ₁₂						

q11 (8 5 6 7 4 1 2 3)
q8 (7 5 8 6 3 1 4 2)
l (1 3 4 2 5 7 8 6)
q2 (7 6 5 8 3 2 1 4)
q7 (8 7 5 6 4 3 1 2)
l² (1 4 2 3 5 8 6 7)
q11 (8 5 6 7 4 1 2 3)
q3 (8 6 7 5 4 2 3 1)
d² (3 1 2 4 7 5 6 8)
q4 (6 7 8 5 2 3 4 1)
q7 (8 7 5 6 4 3 1 2)
e² (4 1 3 2 8 5 7 6)
q2 (7 6 5 8 3 2 1 4)
q8 (7 5 8 6 3 1 4 2)
e (2 4 3 1 6 8 7 5)
q4 (6 7 8 5 2 3 4 1)
q12 (5 6 8 7 1 2 4 3)
d (2 3 1 4 6 7 5 8)



Epistemology
- Computer-aided model of the
compositional process

Object vs Operation



Group theory and cognitive sciences

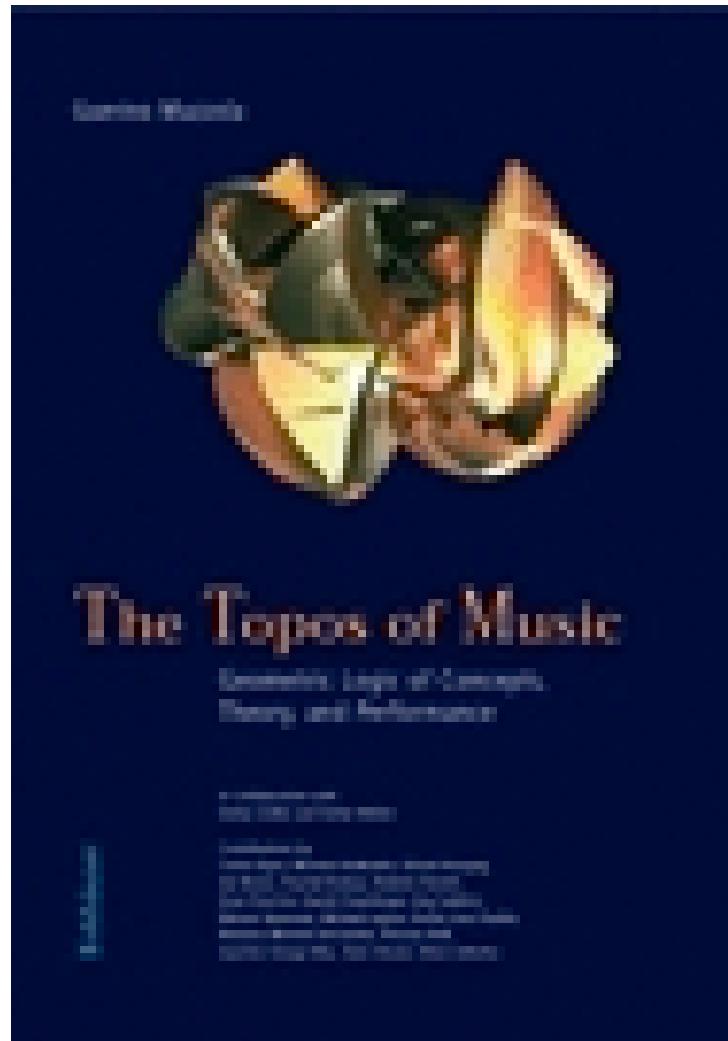
Group Theory has emerged as a powerful tool for analyzing cognitive structure. The number of cognitive disciplines using group theory is now enormous.

The power of group theory lies in its ability to identify organization, and to express organization in terms of generative actions that structure a space.

(Michael Leyton, The International Society for Group Theory in Cognitive Science)

See also: M. Leyton, *Generative Theory of Space*, Springer, 2002.

Category/Topos theory in music and musicology



Mazzola : *The Topos of Music. Geometric Logic of Concepts, Theory and Performance*, Birkhäuser, 2003

Ramificazioni cognitive e percettive di un approccio algebrico in musica

