Microtonal Composition

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This chapter deals with certain theoretical and programming aspects of microtonal computer-aided composition. After a brief introduction on the problem of microtonality, Carlos Agon will describe a few computer tools developed in *OMicron*, an integrated OpenMusic extension dealing with the musical notation of microtonal systems. The study of microtonality raises very interesting theoretical issues, particularly those which concern combinatorial problems. Moreno Andreatta will discuss these issues taking as a point of departure a description of the algebraic method, also implemented in OpenMusic (*MathTools* package). In the third part Alain Bancquart will discuss certain aspects of microtonal composition used in his piece *Amour grand terrible champ critique* and some corresponding models represented with the theoretical and computational tools described in the previous sections.

Historical introduction

The division of the octave into twelve equal semitones and the hierarchical tonal organization of intervals are relatively recent phenomena which date from the end of the seventeenth century. Before that time, according to physical rules, the semitones were unequal and as a consequence modulations were not feasible in all keys. The great variability of the constitutive element of the sound space led sixteenth-century musicians and scholars to propose new divisions of the octave. For instance, the music-theorist Nicola Vicentino (1511-1576) invented two instruments, the Archicembalo and the Archiorgano with thirty-one keys for one octave. Vicentino wrote a few four-part and five-part madrigals where he used quarter-tone accidentals on certain notes of the chord and most frequently on the thirds. The result was "neutral" harmonies which sound neither major nor minor and which lend a surprising expressionist flavour to this modal music. The idea that the sound universe is not necessarily divided into twelve is therefore not new. It derives from the proclivity for rational practice that took over in the eighteenth century at a time when people were eager to establish stable scientific and intellectual laws.

Every system, however perfect, is one day corroded to the point when it becomes obsolete. That is what happened to the tonal system at the turn of the twentieth century. In fact, the "amelioration" of the tonal system brought about by the increasing generalisation of chromaticism eroded its very foundations. Concurrently, the rhythm which derived from tonal harmony and which gave the tonal organisation its remarkable stability evolved around the same lines. Irrational values appeared notably in Scriabin's works (1871-1915), combined with an emancipation from the tonal system with the use of modes (and particularly of mode 2, dear to Olivier Messiaen) and chords composed of superimposed fourths that he called *synthetic chords*.

Thus, twentieth-century musicians were faced with the necessity to rebuild a sound universe deprived of its organisation and, in a certain way, of its meaning. Two paths were taken. On the one hand some promoted the equality of the twelve tones in a music which is no more based on hierarchical principles. This led Schoenberg (1874-1951) to devise the twelve-tone series that paved the way to what we call the "serial technique". On the other hand, three composers developed a more radical approach: Alois Haba (1893-1973), Juan Carrillo (1875-1965), and Ivan Wyschnegradsky (1893-1979).

Alois Haba is a Czech composer who wrote two theoretical works: *The Harmonic Foundations of the Quarter-Tone System* (1922) and *Neue harmonielehre* (1927), in which he proposes a new theory for ultra-chromatic composition. Despite the interest that such novelty may have raised, Haba's music is in fact close to Dvorak's style and not really innovative.

Juan Carrillo is a Mexican composer of Indian descent who first trained as a violinist. His practical experience certainly motivated his interest and systematic research about microtonal intervals. As early as 1927 he devised the construction plans for fifteen pianos using intervals from the third tone to the sixteenth tone. These instruments were built in 1958 and are now on display at the Carrillo Foundation in Mexico City. Although he was an ingenious inventor, Carrillo was not really a convincing composer. It is noteworthy that he acknowledged Claude Debussy's influence, in a way foreshadowing the works by Maurice Ohana in third tones, works which are also deeply indebted to Debussy.

Wyschnegradsky was a Russian composer born in Saint Petersburg. His early works were influenced by Wagner and, above all, by Scriabin. He also borrowed from the latter the idea that music is redemptive and possesses "a theurgical force of an incommensurable power that will transform mankind and the whole cosmos" (Marina Scriabin). Throughout his whole life Wyschnegradsky's thought was driven by this mystical conception. Micro-intervals were instrumental in his attempts to get closer to sound continuum.[1] He explored all the possible densities, reaching the density of 96 which corresponds to the sixteenth tone. His exploration ventured beyond the realm of pitches: he understood that the relation between space and time was crucial. His theoretical work is important, be it only because it mentions the idea of rhythmic micro-intervals. Wyschnegradsky proposed to rearrange the sound space into what he called "non-octaviant spaces", which correspond to micro-intervallic scales organised as cycles and which exceed or do not meet the octave.

Wyschnegradsky's extensive musical works contain very beautiful passages, such as the pieces for two or more pianos tuned with a quarter-tone difference (see notably *Cosmos* for four pianos, *Study on rotating motions* for two pianos, *Integrations* for two pianos), three string quartets and his last opus, a string trio.

In addition to these three pioneers, we must also mention Pierre Boulez's cantata on a poem by René Char, *Le Visage nuptial*. The first version of this work is for small orchestra and was composed between 1946 and 1947. The second version, for soprano, contralto, female choir and big orchestra, was composed between 1951 and 1952. Three out of the five movements in this version are in quarter tones. Due to the fact that this work is extremely difficult to perform, Boulez composed a third version between 1985 and

[1] See his major theoretical work, *La loi de la pansonorité*, published seventeen years after his death [8].

1989, in which he got rid of the quarter tones. Despite the interpretative difficulties, the second version represents the most accomplished microtonal composition during what can be termed the exploratory period of new scales. Boulez's use of quarter-tone series may explain the performance problems linked to this piece, but it represents a bridge between the two practices that came after the end of tonality: the serial technique on the on hand and the microtonal technique on the other.

Micro-intervals in OpenMusic: *OMicron*

Carlos Agon

OMicron is the fruit of a collaboration between composers, musicologists and computer-scientists. It is an OpenMusic extension designed for the representation, the manipulation, the notation and the rendering of microtonal structures.

In OpenMusic pitch representation is closely linked to the MIDI standard. It is useful to recall that in this system pitches are expressed by whole numbers (e.g. C3 = 60, D3 = 62, etc.) A MIDI pitch unit is equal to a semitone. The *midicent* (that is, C3=6000, D3=6200, etc.) is taken as the unit in OpenMusic to accommodate microtonality (a semitone equals 100 midicents). According to this convention a quarter tone is equal to 50 midicents, an eighth tone to 25 midicents, and more generally a $1/n$ interval to $200/n$ midicents. Several tools were developed in OpenMusic to allow working with midicents. Conversion tools (see Figure 1) were particularly important.

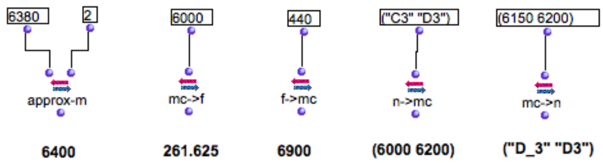

Figure 1. Conversion tools for midicents in OpenMusic. The function *approx-m* produces an approximation in midicents to the closest tempered division of the octave ; with *mc->f* the midicents are converted into frequencies (Hz) and with *f->mc* it is the opposite; with *n->mc* a note symbolically described by a letter and a number is converted into midicents, and with *mc->n* it is the opposite (quarter tones are expressed by '-', and third tones by '+'; the subdivisions smaller than a quarter tone are expressed as a difference between the closest quarter tone, e.g. midicent 8176 corresponds to Bb4-24).

There exist other more general functions for the structuring and modification of midicents (for instance *om+* for transposition, and so on).

Notation of the microtonal intervals

Since we are trying here to notate equal divisions of the whole tone, the problem of micro-intervallic notation can be reduced to finding symbols for the accidentals representing the various intervals within a tone. This section describes the accidentals used in OpenMusic.

Of course, this account remains an open proposal that needs to be improved in practice. The development of such accidentals was motivated by three factors: respect of and adaptation to existing notational systems (at least for the quarter tones and the eighth tones), readability (the signs should not take too much space in the score), and a certain logic by which the signs give information about the notated interval.

Figure 2 shows the accidentals corresponding to quarter tones. Note that flats are excluded from this notational system: the symbolic logic here is exclusively incremental.

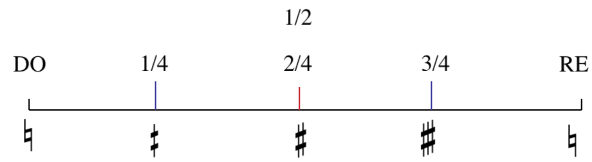

Figure 2. Notation for the subdivision into quarter tones.

The notation for the subdivisions into eighth tones (Figure 4) and sixteenth tones (Figure 4) is derived from this first subdivision.

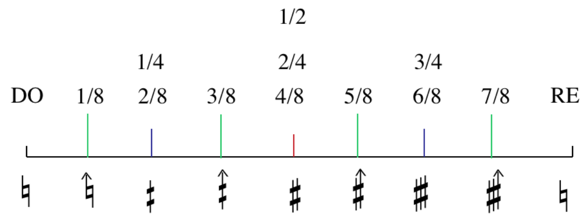

Figure 3. Notation for the subdivision into eighth tones.

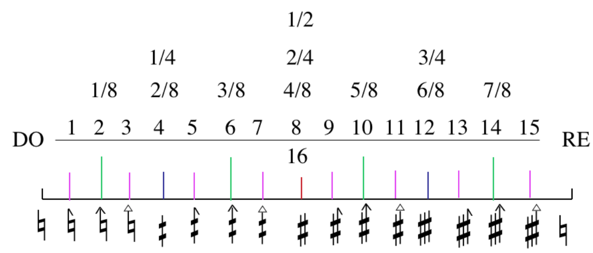

Figure 4. Notation for the subdivision into sixteenth tones.

The "arrows" used in these two systems express a notational logic (for instance in the eighth-tone subdivision, a quarter tone with an arrow equals 3/8). In the sixteenth-tone subdivision system, the three lines constituting the triangular tip of the arrow express the three successive stages from one quarter tone to the next (see Figure 5).

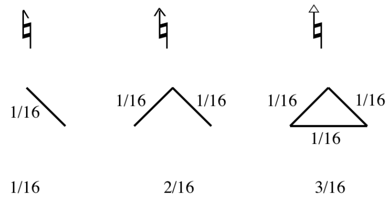

Figure 5. Use of arrows to express sixteenth tones.

A new convention must be introduced to extend the notation to subdivisions that are not to the power of 2. Roman numerals turned over on a vertical line are used to express certain subdivisions in a given scale. Figure 6 gives an example with third tones.

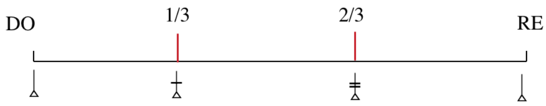

Figure 6. Notation for the subdivision into third tones.

Arrows are used in these scales to express subdivisions into two or into four. Figures 7 and 8 give an example of the construction of sixth and twelfth tones from the scale in Figure 6. Note the use, when possible, of symbols derived from the tempered scale or the scales with subdivisions by 2 described above, in order to avoid multiplying the number of symbols.

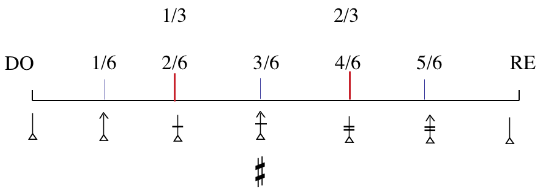

Figure 7. Notation for the subdivision into sixth tones.

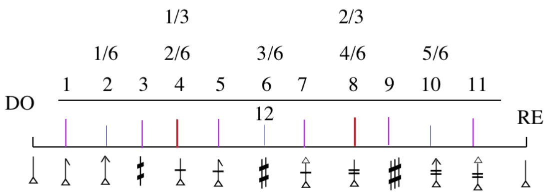

Figure 8. Notation for the subdivision into twelfth tones.

Similar methods were used for the scales with intervals of 1/5, 1/10, 1/7 and 1/14 of a tone.

The use of micro-intervallic structures and representation with sound analysis and synthesis

The generalisation of the use of sound analysis and synthesis in contemporary music practices establishes significant links between these domains and that of microtonality. Indeed, the emancipation of music from the tonal scales described in the historical introduction to this chapter must also be put in relation with the new technologies of sound synthesis and reproduction, which allow for efficient implementations of the corresponding principles. These changes in the formal conception of music were difficult to perform with traditional instruments: because of the limitations of their instruments or their instrumental technique, very few instrumentalists are able to interpret musical pieces with subdivisions that go beyond the quarter tone (Juan Carrillo's pianos are unique examples of microtonal instruments). Moreover, if instrumental technique and design evolve with microtonal practice, reading microtonal scores is a more problematic obstacle. With sound synthesis, the pitch can be controlled up to one Hertz. From this point of view, it helps overcome the difficulties and encourages the development of a microtonal-oriented musical thought. Be the music instrumental or electronic, it allows the composer to implement his formal procedures, and to hear immediately the results of his research.

In order to render the sound of micro-intervallic structures in OpenMusic, a Max/MSP application was created (*MicroPlayer*, see Figure 9).[2] The *use microplayer* option in the OpenMusic score editors transforms the musical objects into messages transferred to the *MicroPlayer* via the OSC protocol. This application allows for a more accurate sound rendering of micro-intervals than the traditional MIDI synthesizers (which are limited to semitone subdivisions).

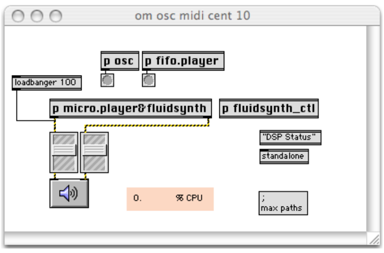

Figure 9. *MicroPlayer*: a Max/MSP application for the rendering of micro-intervallic structures.

Conversely, it is possible to conceive of microtonal music practice as a thriving force behind the advances of electroacoustic musical composition. Inserting microtonality into sound synthesis processes leads to questioning the very concept of pitch and to replacing it among the other descriptions of the sound timbre. If musical sound is traditionally founded on a conception in which timbre is a parameter of the same kind as pitch, duration and intensity, the use of sound synthesis establishes more ambiguous relationships between these different parameters. Within a timbre, the ratios and the organisation of the frequency components (be they harmonic or not) structure the sound identity.

[2]The *MicroPlayer* application was created by Gilbert Nouno.

In this sense, during the creation and superimposition of microtonal musical objects, the "written" pitch will interfere with the "spectral" pitch contained within the sound timbre since they both evolve within the same range of values.[3] The connection that microtonality establishes between pitch and timbre can be seen in other compositional domains (duration, intensity), and reaffirms the idea of a total composition of sound in which timbre is not dissociated from other musical parameters but represents a synthesis of all of them.

In addition, although since the era of the first synthesizers many composers and music theorists have foreseen the large scope of possibilities opened by synthesis, it is now obvious that, despite the significant technologies at hand (i.e. the variety of methods and models, the power and the processing speed of computers, and the quality of the rendering), those possibilities are far from having been satisfactorily explored. The difficulty encountered in the development of electronic music is often linked to problems of representation and control in the technologies employed. In a way, each attempt to formalize the musical concepts that come into play in sound synthesis, be it specific or not, is therefore a form of progress. This is the case with microtonality: notational systems and systems for handling pitch that are not subjected to the diatonic scale allow the development of a thought structured around the concept of microtonality. This thought can only enrich (in the domains of pitch and frequency) the theory and practice of electronic music.

Even though it executes high subdivisions, the notational system of *OMicron* remains within the representational context of a traditional score, which is familiar to musicians. Thus, putting sound synthesis in relation with such a context allows one to envision an

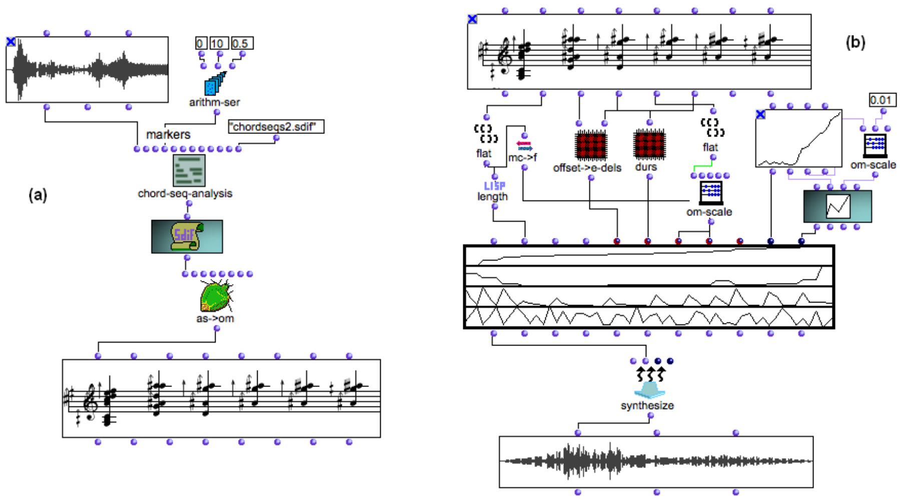

Figure 10. The use of microtonal structures and notation in (a) sound analysis and (b) sound synthesis processes carried out in OpenMusic.

[3]Note that this remark also holds when the musical structures are not microtonal. Such interference makes the complexity of orchestration obvious. However, it is particularly apparent when we talk about micro-intervals.

extended space for the formalisation of electroacoustic music. The musical score editors can represent pitch accurately and symbolically enable one to observe and manipulate this parameter in accordance with one's compositional and musical strategy. Figure 10 - a) shows the example of the symbolic visualisation of sound analysis data using the *OMicron* notational system described above. In parallel Figure 10 - b) illustrates how microtonal structures (which may stem from various compositional procedures) can be used in OpenMusic to parameterize sound synthesis.[4]

Algebraic modelling of micro-intervallic systems

Moreno Andreatta

This section deals with some theoretical aspects of the combinatorial study of microtonal systems, for now envisioned independently from their possible compositional application. The algebraic approach is particularly useful for the systematic study of microtonality. We will concentrate on a few algebraic and computing tools that offer an adequate description of the combinatorial properties of some microtonal systems.

Microtonality as defined above is a particular case of pitch organisation in a tempered space in which the minimum unit (the tone) is divided into a whole number of parts m. This implies a division of the octave into a number n of equal parts, n being a multiple of 6 (e.g. n=18 for the subdivision into third tones, n=24 for the subdivision into quarter tones, and so on). Note that certain micro-intervallic systems with interesting musical properties have been discarded: for instance divisions of the octave into a prime number of equal parts and in particular the enneadecaphonic system (division of the octave into 19 parts) or the Vicentino system (n=31). Divisions of the octave into a number n of equal parts, with n being an odd number or not a multiple of 6, have therefore been put aside. However, these micro-intervallic systems, which do not enter our definition of microtonality, can also be easily formalized in terms of algebraic structures.

To manipulate our micro-intervallic structures with algebra, two complementary strategies are possible: one makes use of cyclical group structures and the other of ordered structures such as the sieves used by Xenakis. The two approaches meet, as Xenakis rightly noted when he described the relation between sieve theory and modular congruence theory as "annexation". I will focus in particular on the issue of the classification and enumeration of microtonal structures with invariance properties regarding transposition. These structures generalize Messiaen's notion of "modes of limited transposition" and raise interesting theoretical problems about their exhaustive calculation.

Circular representation of microtonal structures

Although there are many possible ways to represent microtonal spaces, we will principally use the circular representation. This representation helps formalize every musical chord (in a division of the octave into n equal parts) as a subset of a cyclic group $\mathrm{Z}/n\mathrm{Z}$ of order n.

[4]See [2] for a detailed description of the current research on sound analysis/synthesis in computer-aided composition.

Each chord with m distinct notes (modulo the octave) can be assigned a series of integers indicating its successive intervals (intervallic structure), and represented geometrically as an m-polygon inscribed in a circle. This intervallic structure is an invariant in the algebraic sense of the word for with a single formula it identifies a chord and its transpositions. The transpositions are rotations of the inscribed polygon according to an angle with a value that is a multiple of an elementary angle, which in turn corresponds to the "minimum" step in the division of the octave. The cyclic group Z/12Z of order 12 of the traditional temperament can be generalised and applied to the third-tone system (Z/18Z) or the quarter-tone system (Z/24Z), and so on until the sixteenth-tone division of the octave (Z/96Z).

In OpenMusic, the circular representation is available through the class *n-cercle*. A *n-cercle* is instantiated with a number corresponding to the division of the octave (n) and a list that gives the position of the m successive points of the intervallic structure. The *c2chord* function can then convert a circular representation (*n-cercle*) into a *chord* object, given a base pitch (third argument of the function) and a unit (a value in midicents corresponding to one subdivision of the *n-cercle* – fourth argument of the function) which allows to interpret a circular representation in any division of the octave.[5] Figure 11 shows the reinterpretation of the chromatic scale in terms of a division of the octave into 96 parts.

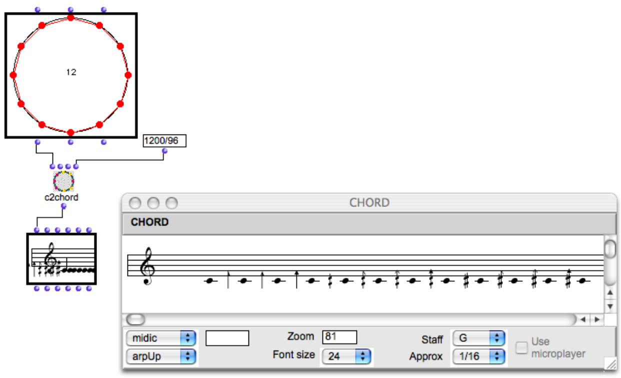

Figure 11. Circular representation of the chromatic scale and interpretation in a micro-intervallic space (division of the octave into 96 parts).

The circle of fifths can also naturally be generalised when one takes into account the "generative" intervals of microtonal systems. Contrary to the division of the octave into twelve parts (in which only one interval of fifth generates the chromatic total), in a division of the octave into n equal parts several intervals can generate circles of fifths. Each interval d which is prime with n can generate the microtonal space. This makes the generalisation of certain classical structures, and their application to any division

[5]In the case of multiple circular representations contained in a same *n-cercle* object (as will be seen in further examples), the *c2chord-seq* function allows to interpret this set of circular representations as a sequence of chords. The functions *chord2c* and *chord-seq2c* perform the reverse operations by converting respectively a *chord* or a *chord-seq* into a *n-cercle*. Interpretations of the circular representations in the rhythmic domain are also available with the *c2rhythm* and *rhythm2c* functions.

of the octave, difficult. For instance, it is not obvious to determine the equivalent of the traditional diatonic scale (shown in Figure 12) in quarter-tone or sixth-tone systems. Figure 13 shows the circular representation of these two generalised diatonic scales.[6]

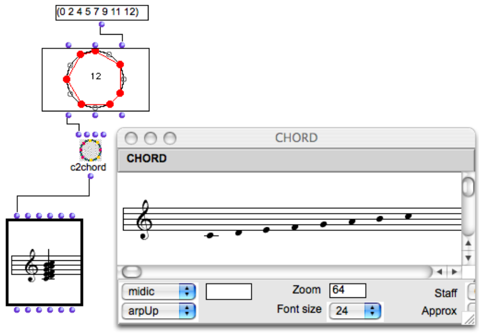

Figure 12. Representation of the traditional diatonic scale.

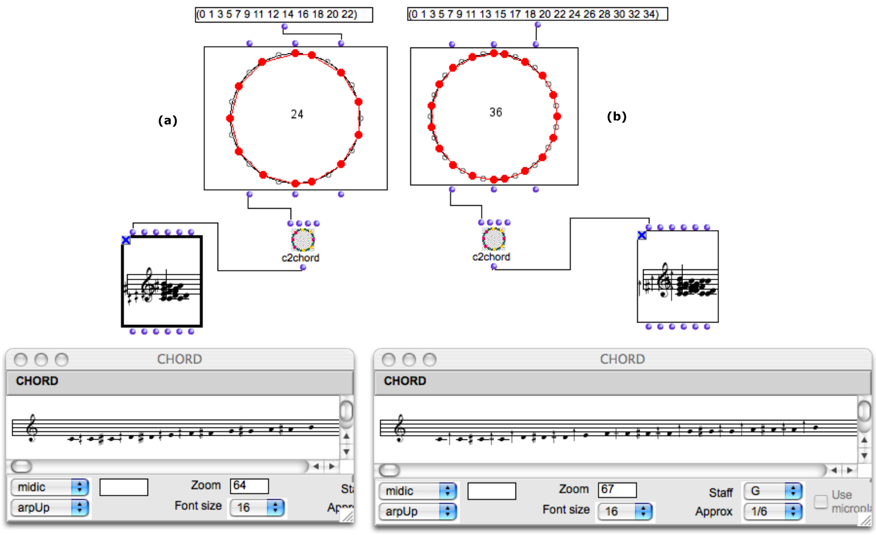

Figure 13. Representation of generalised diatonic scales (a) in a quarter-tone scale and (b) in a sixth-tone scale.

However, other traditional scales can be transferred to microtonality without any theoretical difficulties, for instance as in the case of Messiaen's modes of limited transposition.

[6]The use of algebraic tools for the application of diatonic theory to the micro-intervallic system cannot be discussed here. It is a flourishing branch of mathematical music theory which recently underwent remarkable developments thanks to the weighty use of "strong" (group structures) and "weak" (Christoffel's words) algebraic methods. See for instance [5].

Calculation of microtonal scales of limited transposition

It is important to recall that by definition a mode of limited transposition is a subset A of Z/12Z for which there exists a transposition T_d with $d \neq 0$ verifying the equation: $T_d(A) = A$. In order to calculate the catalogue of "microtonal" modes of limited transposition based on this definition, it is necessary to access the space of the subsets of Z/nZ, n being the size of the space. Note that the cardinality of the space of the possible micro-intervallic structures (that is, of the subsets of Z/nZ) increases exponentially with the number n of divisions of the octave into equal parts. An application of Burnside's lemma yields an explicit formula for the calculation of the cardinality of the set of chords which have a property of transpositional invariance for all the divisions of the octave into n equal parts.[7] For each divisor d of n, the number M_d of chords with an invariance property regarding the transposition T_d of d semitones, is given by the following formula:

$$M_d = d^{-1} \sum \mu(n/k) 2^k$$

in which the summation process is carried out with the integers k dividing 12 and in which μ is Möbius' function.[8] For instance, in the case of $n = 12$, we obtain 9 chords corresponding to Messiaen's modes of a period equal to 6 because:

$$6M_6 = 2\mu(6) + 4\mu(3) + 8\mu(2) + 64\mu(2) + 64\mu(1) = (-1)2 + (-1)2 + (-1)2 + 2 = 54$$

This gives $M_6 = 9$. The nine musical structures with this transpositional invariance property regarding the tritone interval are given in Figure 14.

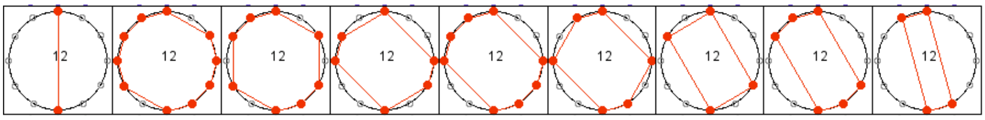

Figure 14. Catalogue of Messiaen's modes of limited transposition with an invariance property regarding the transposition to the tritone T_6.

This calculation, implemented in OpenMusic in Figure 15, gives us the number of modes of limited transposition for microtonal systems of size n (with $6 \leqslant n \leqslant 96$).

Note that the enumeration formula does not tell us anything about the problem of the actual construction of these modes. In other words, resolving a problem about the enumeration of musical structures does not imply resolving the problem of their explicit calculation. In addition, in the particular case of symmetrical structures, a calculation strategy based on the simple definition of transpositional invariance is not operational for it rapidly leads to a combinatorial explosion. Two other strategies that I will now describe are more efficient.

[7] See for instance [6] or [3].

[8] Möbius' function $\mu(x)$ is by definition equal to 0 if x is divisible by a square or to $(-1)^q$ if x is the product of q distinct prime numbers.

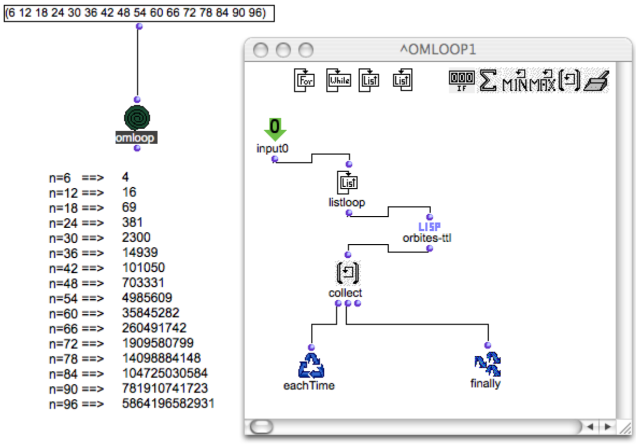

Figure 15. A few enumeration results for the generalised modes of limited transposition.

Calculation of the catalogue of the modes of limited transposition by chords multiplication

The idea of chords multiplication was introduced by Boulez as a serial process with which he could make the material proliferate while keeping an intervallic coherence in the derived series [1]. This operation is formally equivalent to the concept of "transpositional combination",[9] and is defined algebraically by Anatol Vieru [7] who uses the term "composition" between intervallic structures. A first definition of the operation of "composition" involves the relation between an intervallic structure and a note. To "compose" an intervallic structure with a note means simply to recover the scale which is represented by such a structure starting from the selected note. Figure 16 shows the composition of the intervallic structure (1 2 3 1 2 3) and the note C arbitrarily represented by the integer $\{0\}$ (the operation of "composition" is represented with the symbol $\bullet$).

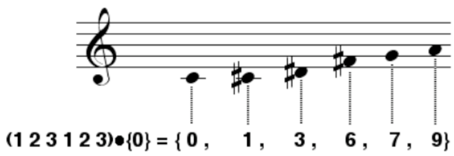

Figure 16. Composition of an intervallic structure with a pitch class.

At a higher level of abstraction, it is possible to "compose" an intervallic structure with a mode, that is to say, a set of residual classes with more than one element. One has to compose the intervallic structure with the elements constitutive of the mode and to take the union of the results. Figure 17 shows the intervallic structure (6 6) which corresponds to two notes forming a triton "composed" with the mode $\{0, 1, 3\}$.

[9] See for instance [4].

Formally the result is:

$$(6\ 6) \bullet \{0,1,3\} = ((6\ 6) \bullet 0) \cup ((6\ 6) \bullet \{1\}) \cup ((66) \bullet \{3\}) = \{0,6\} \cup \{1,7\} \cup \{3,9\}$$
$$= \{0,1,3,6,7,9\}$$

Note that the result of the "composition" is the same mode of limited transposition as the one in the previous figure.

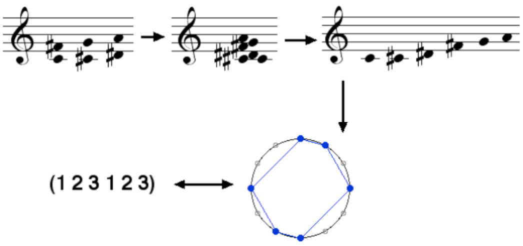

Figure 17. A Messiaen's mode resulting from a "composition" of the intervallic structure (6 6) and the mode $\{0,1,3\}$.

The third stage in the definition of the operation of "composition" focuses on the intervallic structures. Contrary to what is described above, the result of the operation is an intervallic structure. This very clearly exemplifies Vieru's will to explore more deeply one of the most fundamental dualities in music: the sound/interval duality.

Composing one intervallic structure with another means composing one of the two structures with any of the modes represented in the other. For instance, composing the structure (6 6) with the structure (1 2 9) consists in taking a mode represented in one of the structures (for example the mode $\{0,1,3\}$ in the second intervallic structure) and composing the other structure with this mode. The resulting mode is then transformed into the corresponding intervallic structure. In the given example, this operation simply corresponds to the calculation of the intervallic structure of the mode $\{0,1,3,6,7,9\}$, that is (1 2 3 1 2 3). This can be expressed formally as:

$$(6\ 6)\bullet(1\ 2\ 9) = (1\ 2\ 3\ 1\ 2\ 3)$$

This operation is well-defined for it does not depend upon the selected representative. In other words if another mode is chosen, for instance $\{1,2,4\}$ with the same intervallic structure (1 2 9) as the mode $\{0,1,3\}$, the result of the composition is the same (up to transposition!) (see Figure 18).

In algebraic terms, the operation of "composition" is a law of internal composition within the set of intervallic structures and it gives this space the structure of a commutative monoid with a unitary element. The multiplication of chords is an extremely powerful operational tool for the calculation of the exhaustive catalogue of the modes of limited transposition. To use it, it is necessary to introduce a family of intervallic structures that can be expressed with one single interval, or to use a more mathematical term, that are associated with modes generated by an element of the cyclical group Z/12Z.

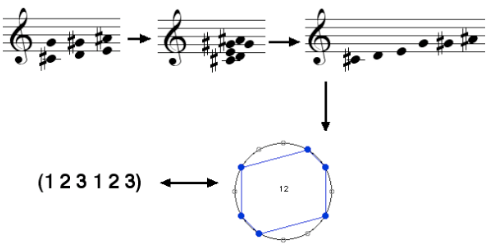

Figure 18. Change of representative for the calculation of the composition of the intervallic structure (6 6) with the intervallic structure (1 2 9).

The form of these structures is therefore $A = (a\ a\ a\ ...\ a)$, in which a is repeated a finite number of times (with a maximum of 12 when the octave is divided into 12 equal parts). Such structures, called "idempotent", correspond to modes or chords that are well-known in music (from the triton to the chromatic total). Figure 19 shows the catalogue of idempotent structures in divisions of the octave into 12, 18 or 24 parts.[10]

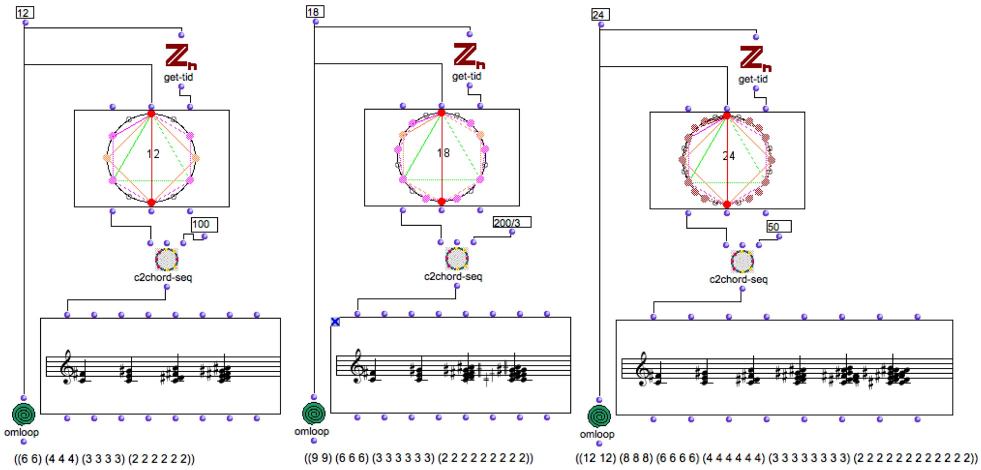

Figure 19. Idempotent structures. The function *get-tid* returns the collection of all the idempotent structures for a division of the octave into n parts.

In the construction of modes of limited transposition idempotent structures play the role of prime numbers in the construction of integers. Every integer can be decomposed into the product of powers of prime numbers. Thus, every mode of limited transposition can be obtained by "composing" a mode of some sort with an idempotent structure. The catalogue of Messiaen's modes of limited transposition can therefore be calculated in maximum $6c$ operations, where c is the cardinality of the catalogue of chords up to transposition. The algorithm is not optimal because the integer c grows exponentially with the size n of the tempered space.

[10]The chromatic total was left aside for these three divisions.

In fact, an application of Burnside's lemma produces:

$$c = n^{-1}\phi(d)2^{n/d}$$

where the sum is done on the dividers of n and $\phi(d)$ is Euler's totient function.[11] Yet, the algorithm is constructive and enables the composer to obtain modes of limited transposition with supplementary properties regarding the multiplicity of intervals. For instance, if one is looking for an eight-tone scale with the temperament Z/18Z and with intervals of one third, two thirds and three thirds tones, it is possible and even easy to build it by multiplying chords starting from the idempotent structure (9 9) and the intervallic structure (1 2 3). The result is shown in Figure 20.

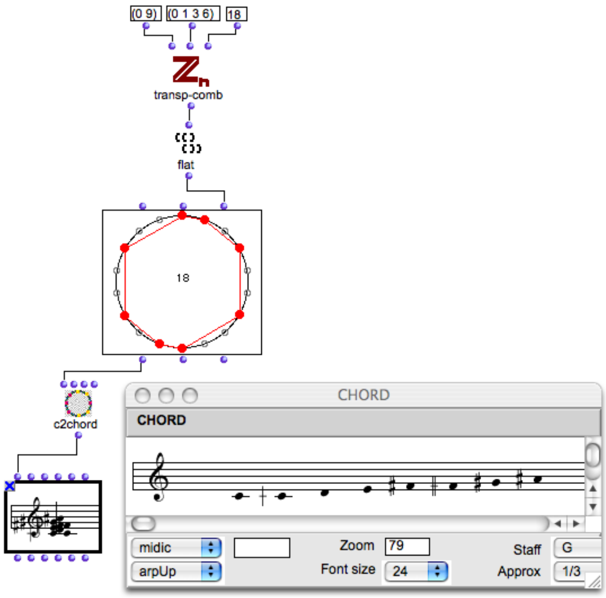

Figure 20. Construction of a mode of limited transposition with third tones. The *transp-comb* function performs the "composition" (or "transpositional combination") of two subsets of Z/nZ.

Calculation of the catalogue of the modes of limited transposition with the sieve theory

As for the previous case, the operation of chord multiplication can be formalised with Xenakis' sieve theory. In effect, if a_b is the set $\{b,\ b+a,\ b+2a,\ \ldots\}$ modulo n, where n is as usual the size of the microtonal space, it is possible to express the previous scale as a set union of the four sets 9_0, 9_1, 9_3 and 9_6. The corresponding construction implemented in OpenMusic is illustrated in Figure 21.

[11] By definition the totient $\phi(d)$ of a positive integer d is equal to the number of positive integers inferior to d and prime with d.

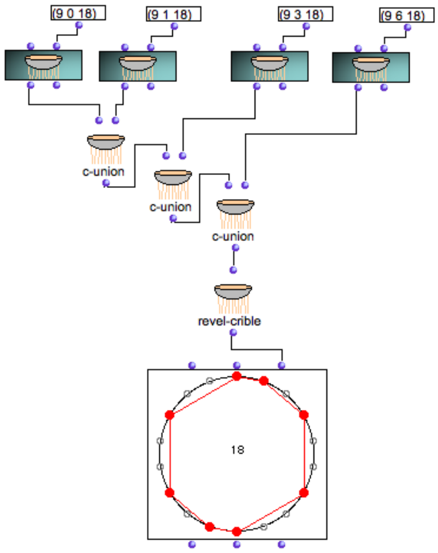

Figure 21. Representation of a quarter-tone mode of limited transposition with Xenakis' sieve theory. In OpenMusic sieves are represented by the class *crible* and can be manipulated by the set operations *c-union*, *c-intersection*, *c-complement*. The function *revel-crible* then allows to interpret the sieve as a list of residual class values that can be connected to a *n-cercle* in order to obtain a circular representation.

Calculation of the catalogue of modes of limited transposition via the intervallic structure

Resorting to the intervallic structure helps reduce to a minimum the complexity of the calculation of the modes of limited transposition for every division of the octave into n equal parts. The property can be expressed as follows:

> A subset A of Z/nZ is one of Messiaen's modes if and only if its intervallic structure is redundant, i.e. equal to the concatenation of a same pattern.[12]

As in the case of the multiplication of chords, this property gives birth to a constructive algorithm for a catalogue of modes of limited transposition for every division of the octave into n equal parts. In addition, as was mentioned above, with this algorithm the problem of the classification of this type of structure can be solved in polynomial time. Figure 22 shows the result corresponding to a division of the octave into third tones (i.e. $n = 18$).

[12] As in every theorem which has a necessary and sufficient condition (if and only if), one of the two implications is easier to demonstrate. In this case, it is easy to show that the redundancy of the intervallic structure is necessary. In other words, if a set M is a mode of limited transposition, then its intervallic structure will be of the type (a_1 a_2 ... a_k ... a_1 a_2 ... a_k), the m factor of the transposition T_m makes the subset equal to $a_1 + a_2 ... + a_k$ invariant. The reader can practice and verify that the condition for the intervallic structure is sufficient.

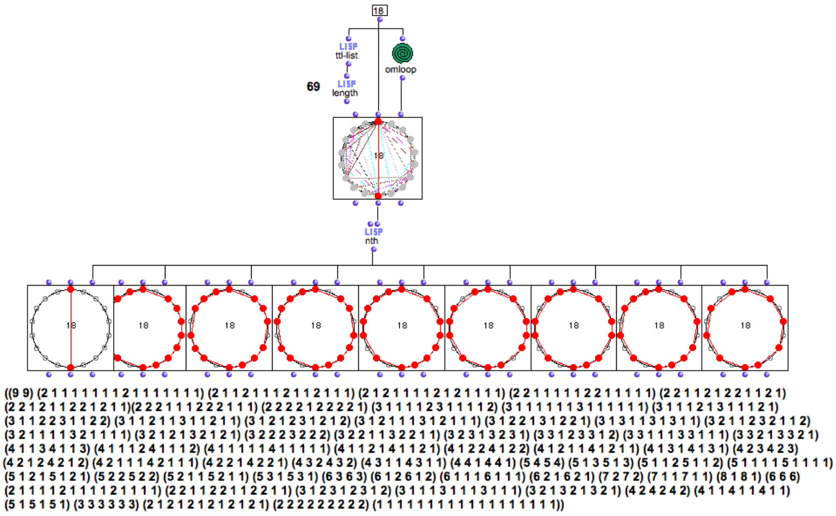

Figure 22. Catalogue of the 69 modes of limited transposition expressed with their intervallic structures. The first 9 solutions of the catalogue are presented with circular representations.

Non-octaviant scales

To conclude this introduction to the algebraic tools used for the study of microtonality, I should like to propose an example of musical structures which at first sight resist a circular representation. In fact, as we shall see, circular representation does not univocally determine the size of the microtonal space. In other words, scales evolving in spaces of sizes that are not of a multiple of the octave can easily be obtained from any circular representation. Take for instance the third-tone mode of limited transposition analysed above. From the same circular representation, it is possible to interpret this scale in a microtonal space where the basic unit is an interval of three quarter tones. The result is a scale with a period that is not a multiple of the octave (see Figure 23).

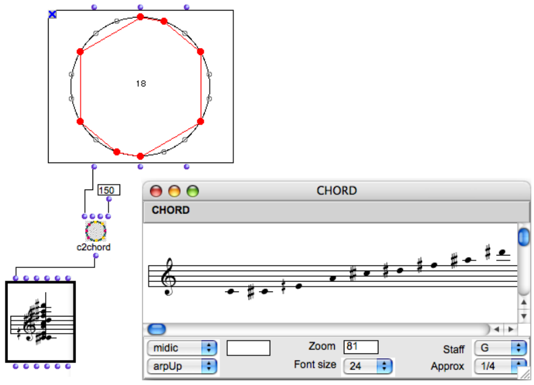

Figure 23. Non-octaviant scale.

Following the same principle, it is possible to interpret any scale in any micro-intervallic space. Therefore, it is possible to determine micro-intervallic interpretations of traditional chords, such as major or minor chords. Figure 24 shows these chords interpreted in the normal space, then with a minimum unit equal to a quarter tone and a sixteenth tone.

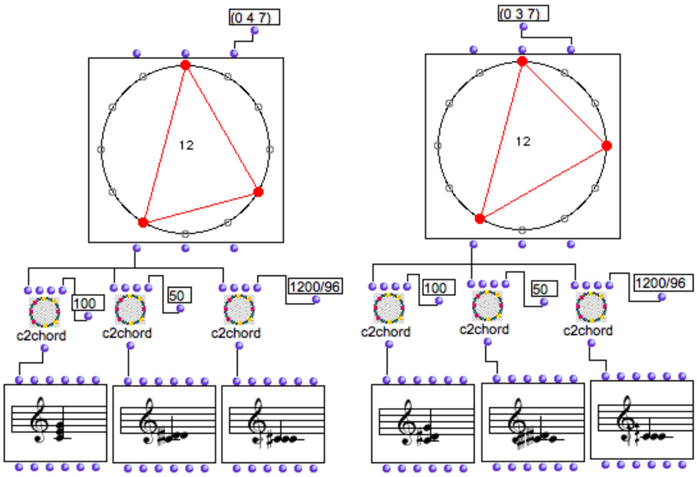

Figure 24. Micro-intervallic interpretations of major and minor chords from a same circular representation.

Amour grand terrible champ critique

Alain Bancquart

The use of quarter tones in musical composition has been popular since the end of the 60s. I have composed many pieces using quarter tones. If my contribution to this technique may seem relatively interesting, it is for two principal reasons. On the one hand, contrary to other composers who are marginally interested in micro-intervallic systems, I radically adopted microtonality. On the other hand, following Wyschnegradsky's ideas I have tried to work in the domains of both pitch and time.

Working with time is in a certain way more convenient than working with pitch. Indeed, the pitch space is limited when it comes to the divisions of the tone, notably by the perceptive ability of the ear. On the contrary, time can always be, at least theoretically, divided up or lengthened, provided one ignores the division of time into seconds as the reference interval. It is during the composition of my works with sixteenth tones that, thanks to OpenMusic, I became aware of the significance of this method.

Numerical notation for micro-intervals

While working with sixteenth tones OpenMusic offered me the possibility to construct and hear the structures I was elaborating. To do so I had to use a special language: files had to be written in numbers and letters, and time calculated in milliseconds. Thus, I decided to use a numerical notational convention with the format date/duration/pitch. The pitches in sixteenth tones are expressed with the help of tempered pitches to which are added the corresponding number of midicents:

```
c3 ; c3+12 ; c3+25 ; c3+37 ; c3+50 ; c3+62 ; c3+75 ; c3+87;
c#3 ; c#3+12 ; c#3+25 ; c#3+37 ; c#3+50 ; c#3+62 ; c#3+75 ; c#3+87 ; d3 ; ...
```

The first sounds of the fourth part of the piece *Amour grand terrible champ critique* for percussion and electronics are given as example:

```
0 2616 f2
2616 5000 f2+75 2616 5000 g2+37 2616 5000 g2+50 2616 5000 a2+25 2616 5000 a2+50
7616 800 f1+25
8416 2384 f2 8416 2384 f3+75 8416 2384 g3+37 8416 2384 g3+50
10770 4200 a1+87 10770 4200 b1+75 10770 4200 c2+87 10770 4200 f2 1077 4200 f3+75 1070 4200 g3+37
14970 1816 a3+12 14970 1816 g3+37
21786 77 g2+12 21786 77 g2+62
21863 273 b2+87 21863 273 c3+37 21863 273 d3+75
22136 1400 b2+12 22136 1400 d3+37 22136 1400 f3+12 22136 1400 g3+75 22136 1400 g3+25
23536 4000 c2+37 23536 4000 d2+50 23536 4000 f2+37 23536 4000 g2+25 23536 4000 c3 23536 4000 d4+25
23536 4000 e4+87
27536 196 f3 27536 196 g3+75
27732 1127 a2+75 27732 1127 c3+75 27732 1127 f3
28859 2610 f3 28859 2610 a3+87 28859 2610 c4+25 28859 2610 d4+12 28859 2610 f4
```

This type of notation may at first seem unpractical. After getting used to it, it becomes as easy to manipulate as any other musical notation. It has the important advantage of an absolute precision. Moreover, and this is the most important aspect, this notation enables the composer to completely avoid a regular pulse, and to construct time sets free from any dependence to a reference unit (the second) or to a tempo. The relation with the quarter note or the whole note, so indispensable in the traditional music theory, disappears. Time is free from all constraints and its variation is infinite: it truly becomes micro-intervallic and totally irrational. Thus the research of the computer-scientists and engineers who designed the software preceded that of the composer.

Some modelling tools to work with sixteenth tones

The algebraic and geometrical representations described by Moreno Andreatta in the previous section are precious tools for the analysis of microtonal composition. Figure 25 shows the example of the numerical system used to encode a sixteenth-tone scale and its implementation in OpenMusic with the help of the circular representation.

An excerpt from the piece *Amour grand terrible champ critique* can therefore be easily modelled with the circular representation. This representation notably gives a more precise idea about the pitch distribution in the tempered space (see Figure 26).

The circular representation of another excerpt from the piece, corresponding to the second voice, is shown in Figure 27. Note that contrary to the first voice, the second one makes full use of the chromatic total. Finally, Figure 28 shows the fusion of these two voices in a circular representation, and in the corresponding score.

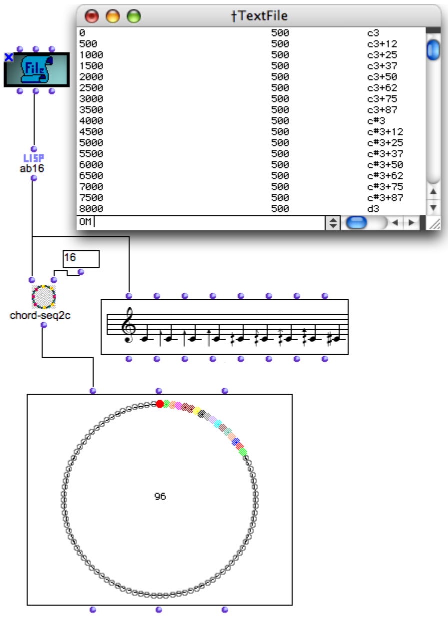

Figure 25. System used to divide the octave into 96 parts (sixteenth tones), and the associated circular representation.

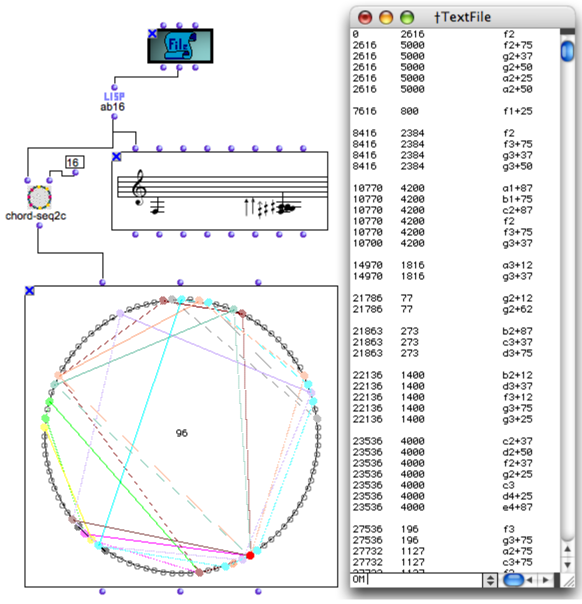

Figure 26. Excerpt from the piece in numerical notation and the circular representation associated to the first voice.

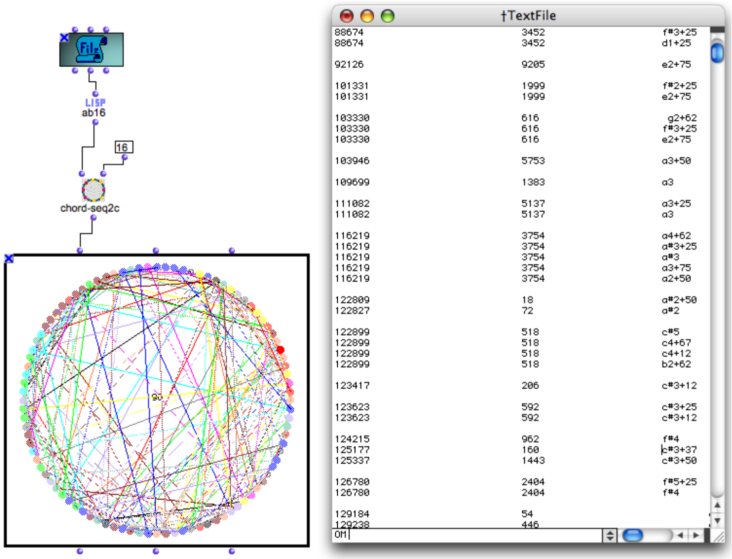

Figure 27. Excerpt from the piece in numerical notation and the circular representation associated to the second voice.

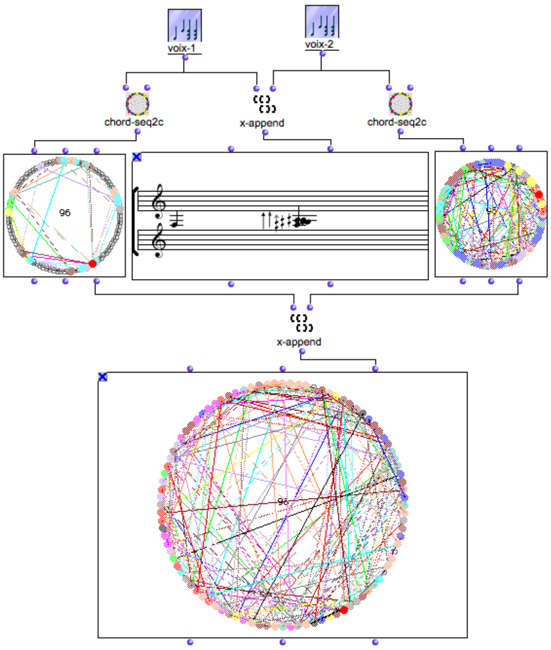

Figure 28. Fusion of the two voices from Figures 26 and 27 with circular representations and traditional notation.

A hypothesis for the generalization of micro-intervallic and microtemporal spaces

A great step forward was indeed taken in the research and formalization of microtonal structures but it is only the beginning of a much more adventurous study. The knowledge and practice of all the audible divisions of the tone are effectively equivalent to the use of irrational rhythmic values. However, the work carried out on rhythm using milliseconds remains to be done for pitch. One has to take as a minimum unit one thousandth of a tone which, like the millisecond, is obviously a purely theoretical unit. It would then be possible to construct scales that are emancipated from the division of the space into tones, non-iterative scales with degrees that are not tone divisions but freely defined distances without any reference to the tone (in the time domain, this would mean excluding references to quarter notes or full notes). It would be possible to conceive space-times in which the series of numbers used to define the durations could also be used to define the pitches. The result would be music with homogeneous prime materials (I am speaking here about the constitution of music material and not about compositional processes).

As an experiment, let us imagine the following series of durations: 2079 189 1512 597 945 378 2457 756 2268 1134 171 1323. The duration of the series is 13809 milliseconds. The operation transferring these numbers to pitch can take for instance a point of departure C3+3 sixteenth tones. The range of this scale is then of 13809 thousandth tones, that is 13 tones + 809 thousandth tones. It is easy to expand or narrow down this range. If these numbers are divided by 2.27, the result is the following series: 915 83 666 262 416 166 1082 333 999 499 75 582, with a range of 6078 "millitones", that is 6 tones + 78 thousandth tones. If this series is subjected to the treatment I often employ for the durations, a treatment I call "duration anamorphosis" and which consists in subtracting the numbers from one another, the result is the following series:

> 2079 189 1512 597 945 378 2457 756 2268 1134 171 1323 1890 1323 915
> 348 567 2079 1701 1512 963 1152 408 219 1512 378 189 189 159 1293 207
> 18 945 159 756 786 30 927 153 537 30 768 774 507 129 609 378 480 603
> 102 123 21

Note that the size of the intervals gradually decreases to reach a minimum interval of 18 millitones, equivalent to a fifty-sixth of a tone! It would probably be desirable to discard those intervals that are not perceptible (since it is a non-iterative series, I got rid of the repetitions). The range of this new scale is 24595 millitones, that is 24 tones (6 octaves) + 594 millitones divided up into 52 intervals of all sizes. The combinatorial possibilities of this type of scale are infinite. The composer would then have to invent new organisational methods, but such an activity lies outside the scope of this chapter.

Eventually...

The use of such an environment as OpenMusic in microtonal composition considerably enhances the composer's work, be it for the formalization, calculation or the representation.

Before undertaking research on sixteenth tones, I had worked on the organisation of a free time. OpenMusic enabled me to go further and to rapidly perform long and fastidious calculations. These organisational methods can be related to the "time-blocks" that Karim Haddad is working on,[13] and to many other efforts, all of which share a common desire for the conquest of the dimension of time, a focus that will no doubt nourish future research.

The *OMicron* system, as well as the mathematical tools for the geometrical representation of musical structures available in OpenMusic, allow us to notate, represent and hear all the micro-intervals from the third tone to the sixteenth tone. It is now possible to play with odd densities hitherto unheard of. In this context, the notations proposed in *OMicron* are clear and logical. Although there is a tendency to minimize the role of notation as a tool for immediate reading (in many cases microtonal music can solely be realized by electronic means), the latter in fact remains an important aspect for it offers a non-negligible formal support for musical thought and composition.

In the light of this new manner of conceiving the organisation of space and time, it will certainly become necessary to reconsider the notions of timbre, of spatialization, as well as all the other musical parameters. It will then be possible to integrate sound synthesis into the composition of the musical material.

Thus great progress has been made towards the perception of a different universe. Each of the various aspects of this universe remains to be studied, and this will be a long endeavour for the musicians. What we call "tomorrow" is in fact probably a close future, a future that will be filled with discoveries – and difficulties.

[13]See the chapter "*Livre Premier de Motets*: the Concept of TimeBlocks in OpenMusic" by Karim Haddad.

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