

MATHEMATICS

Exploring Musical Space

Julian Hook

A musical score is a graph whose vertical axis represents pitch and whose horizontal axis represents time. Given this apparent simplicity, and the recognition since the time of Pythagoras that mathematical principles underlie many musical phenomena, it is perhaps surprising that our understanding of the mathematical structure of the spaces in which musical phenomena operate remains fragmentary. But as Tymoczko shows on page 72 (1), even the pitch domain is vastly more complex than it may first appear.

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Simply deciding what pitch structures are appropriate and when two of them should be considered “the same” can be difficult. Are notes sharing the same name but separated by octaves, such as two different C’s, to be regarded as the same? What about notes that differ only enharmonically, such as C-sharp and D-flat? (These are a single note on the piano but often serve two very different musical purposes.) Harmonic structures, moreover, are not single notes but chords. If two of the voices forming a chord exchange notes, is the resulting chord the same? Should a four-note chord with two C’s be reduced to a three-note chord with one C? Does the tuning of the instrument matter?

Musical sensibilities dictate different answers to these questions depending on the musical context and analytical objectives. Music theorists classify chords in categories such as major triads, grouping chords together with their translations in some appropriate space (in musical parlance, transposition), or in some cases also with their reflections (inversion). Western musicians are accustomed to a discrete view of pitch space, corresponding to the chromatic scale playable on the piano, but the general problem requires consideration of a larger space in which continuous pitch variation is possible. The various equivalence relations give rise to an assortment of quotient spaces (obtained by “gluing together” the points considered equivalent) and group structures acting on them. Tymoczko’s work presented here is part of an ambitious project (2) that characterizes these spaces in great generality and relates the geometry of the spaces to the musical behavior of the chords that inhabit them.

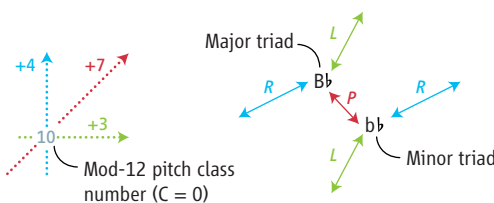
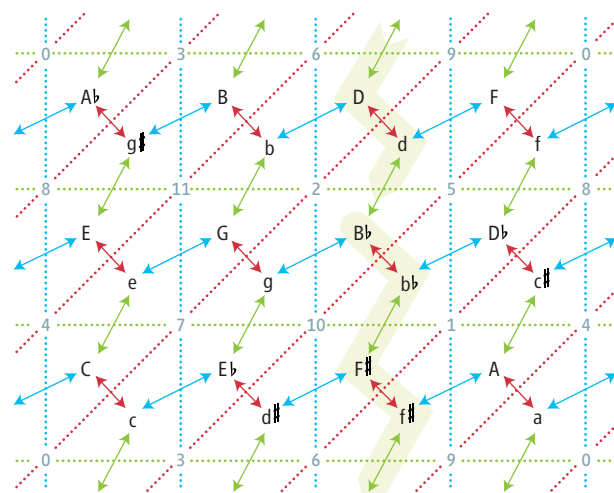
Mathematical music theory, although terra incognita to practicing musicians and even to many professional music theorists, has in recent years blossomed into a sizable and multifaceted industry. Pitch-class set theory (3), the study of a discrete 12-note quotient space, was developed as a means of confronting the analytical challenges posed by “post-tonal” music of the 20th century, whose harmonic materials are more varied and complex than those in most earlier music. Diatonic set theory (4, 5) investigates the subtle and beautiful relationship between the 12-note chromatic scale and diatonic scales such as the C major scale, with seven unequally spaced notes per octave (a scale type of great importance in many styles of music). Scale theory (6, 7) studies structural properties of scales and their subscales more broadly, allowing variation in both chromatic and diatonic cardinalities and occasionally engaging considerations of tuning and acoustics.

New mathematical approaches can elucidate abstract musical spaces and help our understanding of harmonic processes at work in musical compositions.

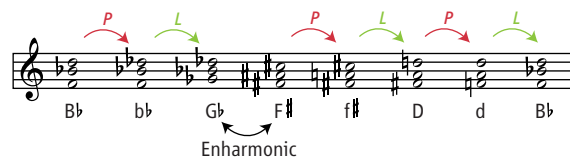
In the past two decades, transformation groups acting on musical spaces have proven to be enormously fruitful models in a variety of settings. Transformations are mathematical functions that describe relationships between chords (or other musical entities); they often form algebraic groups and bear intimate relation to musical notions of interval (8).

A particularly active area is neo-Riemannian theory, which synthesizes modern group-theoretic techniques with inspiration drawn from the work of the prolific German musicologist Hugo Riemann (1849–1919) and his contemporaries. In its basic form (9, 10), neo-Riemannian theory investigates certain transformational relationships among the 12 major and 12 minor triads in ways that are algebraically elegant, musically suggestive, and readily visualized in various forms of a graph known as a Tonnetz (tone network), in which the harmonic path traced by a musical composition may be plotted (see the figure).

In this representation, the mod-12 numbers in the background graph (dotted lines) designate pitch classes (itches under the assumption of octave equivalence), from C = 0 through B = 11. They are arranged by musically important intervals: by perfect fifths (interval 7) diagonally (red), major thirds (interval 4) vertically (blue), and minor thirds (interval 3) horizontally (green). This two-dimensional graph is an unwrapped torus whose right edge is identified with the left and the top with the bottom. The graph in the foreground (solid arrows) depicts major and minor triads (labeled in upper and lower case, respectively), each positioned within the triangle formed by the corresponding pitch classes in the background; for example, the B-flat major triad comprises the notes B-flat (pitch class 10), D (pitch class 2), and F (pitch class 5). Each triad shares two of its notes with three different triads of the opposite mode, to which it is related by the transformations P (parallel, red arrows), R (relative, blue), and L [Leitton-



Harmonic plan of Beethoven, Violin Sonata, Op. 24 (“Spring”), second movement, measures 37–54



Mapping music. A harmonic path in the neo-Riemannian Tonnetz.

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wechsel (leading tone exchange), green], which generate a dihedral group of order 24, isomorphic to the group of rotations and reflections of a 12-sided polygon. Each transformation exhibits efficient voice leading, preserving two pitch classes and moving the third by a small interval; arrows always cross dotted lines of the same color. The Beethoven progression (highlighted in the graph and expanded at the bottom of the figure) unfolds a PL-cycle that circumnavigates the torus, starting and ending in B-flat major, and illustrates that the composite transformation PLPLPL is the identity element of the group.

The Tonnetz is only one of many possible geometric representations of musical spaces (11), and recent studies have extended neo-Riemannian methods to larger and more powerful transformation groups, to other chord types besides triads, and in various other directions (12). In addition to group theory and other algebraic techniques, ideas from graph

theory, combinatorics, geometry, and topology have found musical application. The work of Tymoczko *et al.* embraces all of these strategies in an innovative and wide-ranging investigation of chordal space. One of the great attractions of this work is its generality: It aims to describe what is in effect a “space of all chords” wherein the Tonnetz and many other familiar depictions of musical relationships appear as subspaces, projections, and cross sections. The spaces appearing here are of a type known as orbifolds, as they possess singularities—points where the geometry is not locally Euclidean. (The appeal to the recent topological concept of orbifolds is notable in a field that relies mainly on mathematics of a more classical vintage.) Other valuable contributions include a fresh perspective on the elusive notions of consonance and dissonance, connections between symmetries of the spaces and various musical practices, and many impli-

cations for the efficient chord-to-chord voice leading that has long been considered a hallmark of successful composition.

References

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DEVELOPMENTAL BIOLOGY

Morphing into Shape

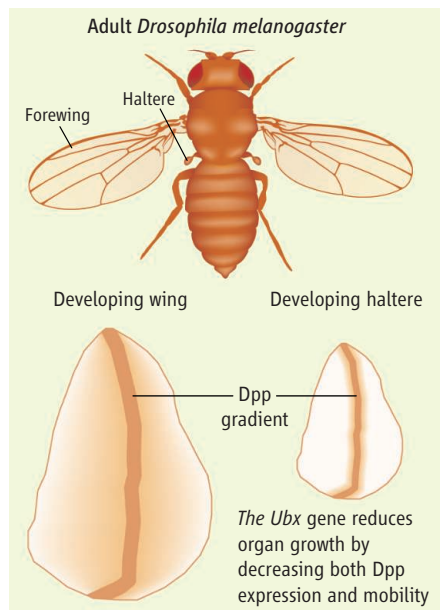
David L. Stern

In 1917, British polymath D’Arcy Thompson proposed that the shapes of different organisms—say, a human and a chimpanzee—could be imagined as simple alterations of the same underlying pattern (1). Thompson famously demonstrated this idea by overlaying transformed Cartesian coordinates on drawings of related animals. This holistic view of organism shape inspired the British biologist Julian Huxley to point out that changes in shape can be thought of most simply as differences in the relative sizes of different body parts, thus reducing shape change to a more manageable problem (2). On page 63 of this issue, Crickmore and Mann (3) present a detailed analysis of the mechanisms controlling one striking difference in the relative size of two organs and uncover what may be a general mechanism of shape evolution.

In segmented organisms, such as flies and humans, similar structures that differ mainly in size and shape are produced in several locations along the main body axis. For example, humans produce arms and legs, largely using many of the same developmental mechanisms to pattern both organs. In fruit flies, two flying appendages, the wings and halteres (see the figure), also are built largely by shared developmental mechanisms. Halteres are delicate club-shaped organs that work like gyroscopes during flight. They evolved about

225 million years ago from more traditional-looking wings—such as the hind wings of butterflies—and have undergone a drastic reduction in size.

All of the differences between the wing and the haltere are determined by expression of a single “selector” gene called *Ultrabithorax* (*Ubx*), which is expressed in all cells of the developing haltere. When *Ubx* is experimentally removed from these cells, a fully formed wing grows instead of a haltere (4), revealing some of the



Two appendages of the fly, the haltere and the wing, grow to very different sizes. Limited expression and mobility of a growth morphogen is partly responsible for this difference.

underlying similarities between the two flight organs. *Ubx* somehow instructs other genes to alter the growth and development of haltere cells. In 1998, Weatherbee *et al.* (5) showed that *Ubx* regulates a battery of genes in the haltere, but until now we have not known precisely which genes are regulated to cause the greatest difference between the wing and the haltere: their five-fold difference in cell number in the adult.

Crickmore and Mann focused their attention on how *Ubx* influences the activity of *decapentaplegic* (*dpp*), a gene that is one of the key regulators of wing growth. Dpp protein is produced by cells that lie in a line that is several cells wide along the middle of both the wing and the haltere. The protein is then secreted from these cells and diffuses to neighboring cells. When the Dpp protein binds to its receptor, Thickveins (Tkv), two things happen. First, a signal is triggered within the cell and this signal is interpreted as “grow more.” Second, the Dpp protein is captured by the cell and eventually destroyed. Thus, Dpp protein diffuses away from the central cells and forms a gradient whose extent and steepness is controlled, at least in part, by the receptor Tkv.

Crickmore and Mann first noted that the width of the stripe of cells producing Dpp was narrower in the haltere than in the wing, and the level of expression per cell was also lower in the

How is organ size controlled? The *Ubx* gene is expressed in haltere cells, restricting the growth effect of the morphogen Dpp during development.

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