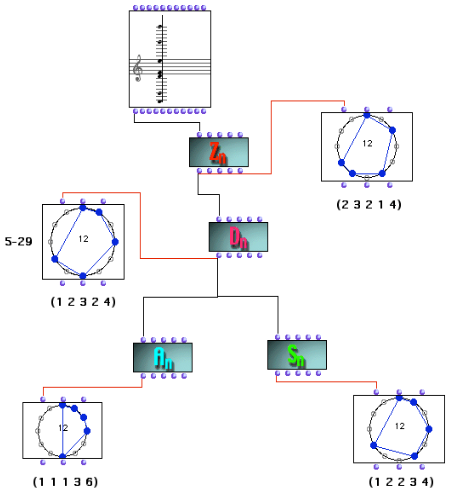
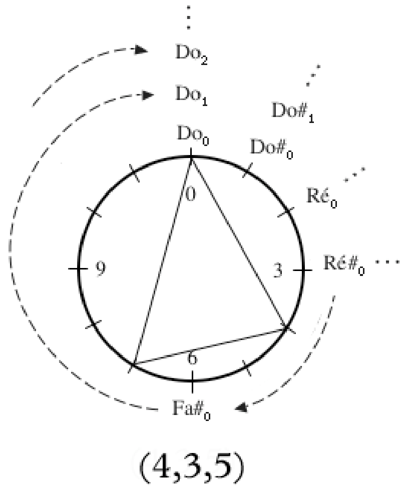


Pisa - Domus Galileiana



Introduzione alla formalizzazione algebrica delle strutture musicali

Moreno Andreatta
Equipe Représentations Musicales
IRCAM/CNRS
Moreno.Andreatta@ircam.fr

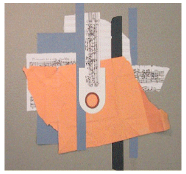
ircam Centre Pompidou

CNRS CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Sommario

Algebra combinatoria nella teoria, analisi e composizione musicali:

- Classificazione e enumerazione delle strutture musicali:
  - Serie dodecafoniche:
    - ♬ Otterström, Eimert, Morris&Starr, Jedrzejewski, …
    - ∑ Riotte, Reiner, Fripertinger, Jedrzejewski,
  - Accordi:
    - ♬ Costère, Zalewski, Vieru, Forte, Babbitt, Carter, Morris, Estrada, [Xenakis]…
    - ∑ Halsey&Hewitt, Mazzola, Reiner, Fripertinger, Read, Broué …
  - Canoni ritmici "a mosaico":
    - ∑ Minkowski/ Hajos, Fuglede, Vuza, Agon, Andreatta, Fripertinger, Noll, Amiot, Jedrzejewski, Hall, Gilbert, …
    - ♬ Messiaen, Levy, Johnson, Bloch, Wild, …
- Teoria/analisi trasformazionale:
  - Sistemi d'intervalli generalizzati e *K-nets*:
    - ♬ Lewin, Klumpenhouwer, Perle, …
    - ∑ Vuza, Mazzola/Andreatta, …



OM=♬+∑

Ramificazioni percettivo-cognitive e filosofiche dell'approccio algebrico in musica

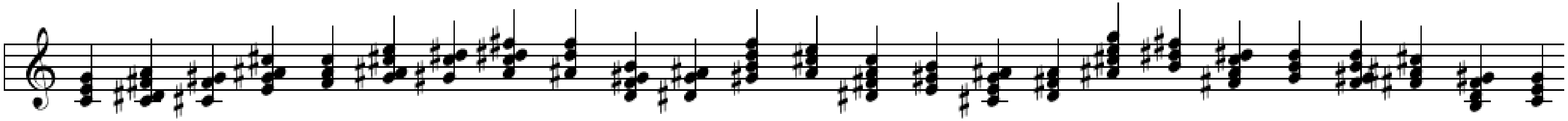
- Cassirer, Lewin, Balzano, Leyton, Granger, Piaget, …

Matematica/Musica…una storia recente!

- 1999: 4e Forum Diderot (Paris, Vienne, Lisbonne),*Mathematics and Music* (Assayag et al., Springer, 2001)
- 2000-2001: Séminaire *MaMuPhi, Penser la musique avec les mathématiques ?* (Assayag, Mazzola, Nicolas ed., Coll. « Musique/Sciences », Ircam/Delatour, 2006)
- 2000-2003: International Seminar on *MaMuTh* (*Perspectives in Mathematical and Computational Music Theory* (Mazzola, Noll, Luis-Puebla, epOs, 2004) `http://www.epos.uos.de/music/`
- 2003: *The Topos of Music* (G. Mazzola et al.)
- 2003: *Music and Mathematics. From Pythagoras to Fractals* (J. Fauvel et al.)
- 2001 - 2006: Séminaire *MaMuX* de l'Ircam `http://recherche.ircam.fr/equipes/repmus/mamux/`
- 2004 - 2006 : Séminaire « Musique et Mathématique » (Ens/Ircam) `http://www.entretemps.asso.fr/maths`
- 2006 : *Mathematical Theory of Music* (Franck Jedrzejewski), Ircam/Delatour
- 2007 : *Music. A Mathematical Offering* (Dave Benson), Cambridge University Press
- 2007 : *Les mathématiques naturelles* (Marc Chemillier), Odile Jacob
- 2007 : Journal of Mathematics and Music (Taylor & Francis) `http://www.tandf.co.uk/journals/titles/17459737.asp`

L’emergenza delle strutture algebriche in musica

Thorvald Ötterstrom, *A Theory of Modulation*, Chicago UP, 1935



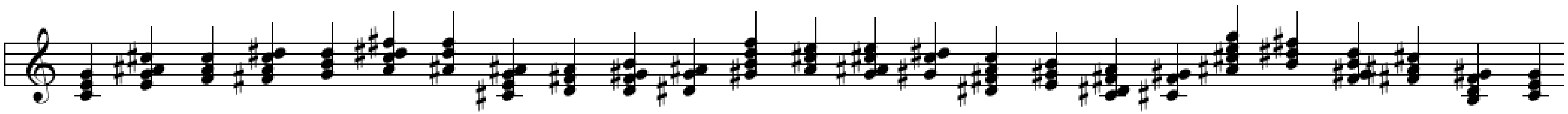
C C# F G# A# D# A E D B G F# (C)

0 1 5 8 10 3 9 4 2 11 7 6 (0)

α = 1 4 3 2 5 6 7 10 9 8 11 (6)

←   ←

11 8 9 10 7 6 5 2 3 4 1 (6)



E(α)= 5 2 3 4 1 6 11 8 9 10 7 (6)

=> *OpenMusic*

key-form; otherwise it is called an *acentral* key-form. By Theorem 11, when b is even, there always exists a central key-form. If b is greater than 4, then by the Corollary we need to find only one-fourth of all the key-forms in order to have them all, since each one *generates* three more. To find a complete set of generating key-forms we may proceed as follows:

1. Find all the central key-forms with the element 1 in the first half of A or in its middle.
2. Find all the acentral key-forms with B longer than A and with the element 1 in the first half of A or in its middle.
3. Find all the acentral key-forms with B longer than A and with the element 1 in the first half of B or in its middle.

Complete sets of generating key-forms for $b=2, 4, 6, 8, 10$ are as follows:

$b=2$:

1

$b=4$:

123

$b=6$:

14325

$b=8$:

CENTRAL

1234567
1634527
5174632

ACENTRAL

1643752
3241576
7245136
6542137

$b=10$:

CENTRAL

$k_1=1$: 126357489
124753689
124759863
174258639
138654279
183654729
176852439
176859342

$k_2=1$: 312654897
318456297
813456792

ACENTRAL

$k_4=5$: 129574836
162548793
184597263

216534789

498512367
367512498
867512493
926513847
427513968
472518963

347521689
849571326
342571689

386524179
638592147

$k_3=5$: 125978436
135869427
175948326
185369427

765123849
985132674
485132679
265134879
935162784
435162789

475218693
865714293
365714298
425718693

975241683
925741683

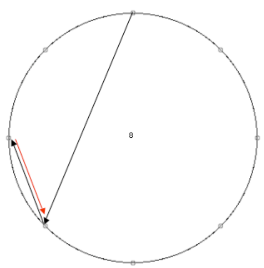
$k_2=5$: 153869247
158324967
157632498

351267849
851327694
851396724
751623489
251678439
851762349
351872694
351896274

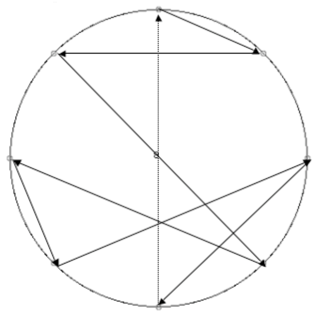
657132948
256143879
756148329
657149238
952168743
453186927

653814297
259714386
759214836
953416872
452618793

452781693
957241863



(5,1,7,4,6,3,2)



(1,6,4,3,7,5,2)

Base b	Generating Key-Forms				Total Key-Forms
	Central		Acentral	Total	
	$k_1=1$	Total			
2	1	1	0	1	1
4	1	1	0	1	2
6	1	1	0	1	4
8	2	3	4	7	28
10	8	11	55	66	264

The problem of the number of key-forms for the general base b is a problem in partitions and probably admits of no formula.

The problem of the number of key-forms for the general base *b* is a problem of partition and probably admits of no formula

Enumerazione delle serie "omni-intervallari" (*all-interval rows*)

H. Fripertinger: «Enumeration in Musical Theory», *Beiträge zur Elektr. Musik*,1,1992

G_1= groupe de Klein → 1.

G_2= <I,R,E> → 2.

G_2= <I,R,E,Q> → 3.

Theorem 25 (Number of Patterns of All-Interval-Rows) *For $i = 1, 2, 3$ the number of patterns of all-interval-rows in regard to G_i is*

1. $\frac{1}{4}(\chi(\mathrm{id}) + \chi(\varphi_I \circ \varphi_R))$ *for* $i = 1$.
2. $\frac{1}{8}(\chi(\mathrm{id}) + \chi(\varphi_I \circ \varphi_R) + \chi(\varphi_I \circ V))$ *for* $i = 2$.
3. *For $i = 3$ we calculate*

$$\frac{1}{16}(\chi(\mathrm{id}) + \chi(\varphi_I \circ \varphi_R) + \chi(\varphi_I \circ V) + \chi(\varphi_Q \circ \varphi_R \circ V)) =$$

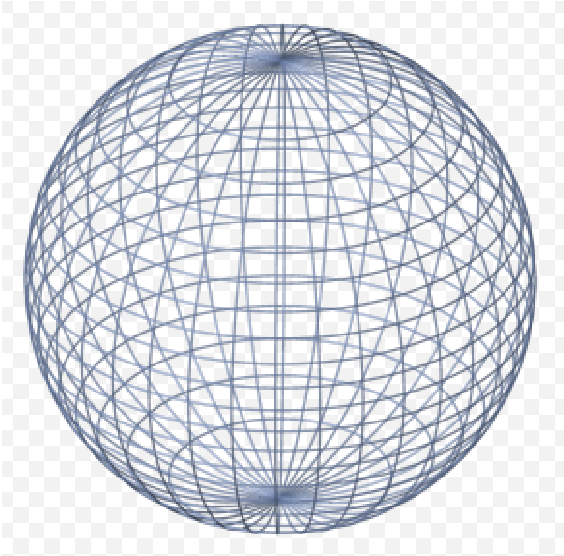
$$= \frac{1}{16}(3\,856 + 176 + 120 + 120) = 267.$$

• D. Halsey & E. Hewitt: « Eine gruppentheoretische Methode in der Musik-theorie », *Jahresber. Der Dt. Math.-Vereinigung*, 80, 1978.
• D. Reiner: «Enumeration in Music Theory», *Amer. Math. Month*. 92:51-54, 1985
• **H. Fripertinger: «Enumeration in Musical Theory», *Beiträge zur Elektr. Musik*,1,1992**
• R.C. Read: « Combinatorial problems in the theory of music », *Discrete Math.*, 1997
• H. Fripertinger: « Enumeration of mosaics », *Discrete Math.*, 1999
• H. Fripertinger: « Enumeration of non-isomorphic canons », *Tatra Mt. Math. Publ.*, 2001
• M. Broué : « Les tonalités musicales vues par un mathématicien », *Le temps des savoirs, Revue de l'Institut Universitaire de France,* 2002
• David J. Hunter & Paul T. von Hippel : « How Rare Is Symmetry in Musical 12-Tone Rows? », *The American Mathematical Monthly*, Vol. 110, No. 2., Feb., 2003
• H. Fripertinger: « Tiling problems in music theory », in *Perspectives in Mathematical and Computational Music Theory* (Mazzola, Noll, Puebla ed., Epos, 2004)
• Rachel W. Hall & P. Klingsberg: « Asymmetric Rhythms, Tiling Canons, and Burnside's Lemma », *Bridge Proceedings*, 2004
• …

L'approccio assiomatico in matematica

David Hilbert: i fondamenti assiomatici della geometria e il ruolo dell'intuizione

In order to be constructed in a right way, geometry […] only needs few simple principles. These principles are called the ***axioms*** *of the geometry. […] This study (of the axioms) goes back to the* ***logical analysis of our spatial intuition*** (*Grundlage der Geometrie*, 1899).



At the moment there are two tendencies in mathematics. From one side, the tendency toward abstraction aims at 'cristallizing' the logical relations inside of a study object and at organizing this material in a systematic way. But there is also a tendency towards the ***intuitive understanding*** *which aims at understanding the* ***concrete meaning of their relations*** (*Anschauliche Geometrie*, 1932)

Verso una definizione « moderna » della teoria musicale

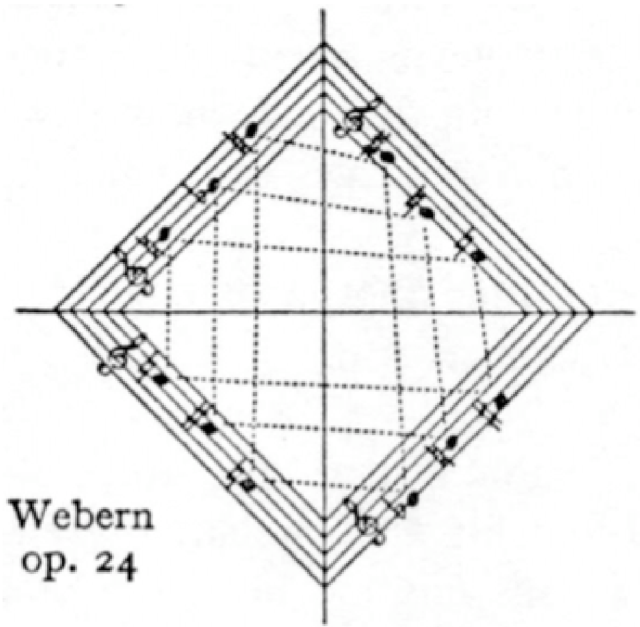
Ernst Krenek e l'approccio assiomatico in musica

- *The Relativity of Scientific Systems*
- *The Significance of Axioms*
- *Axioms in music*
- *Musical Theory and Musical Practice*

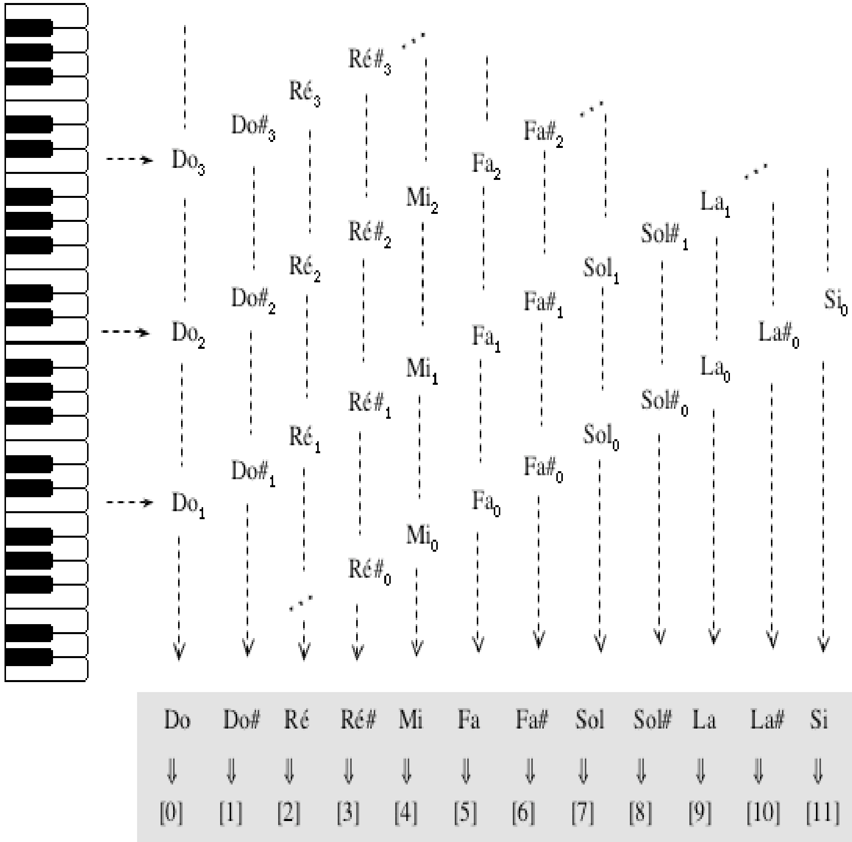
Ernst Krenek : *Über Neue Musik*, 1937 (Engl. Transl. *Music here and now,* 1939).

Physicists and mathematicians are far in advance of musicians in realizing that their respective sciences do not serve to establish a concept of the universe conforming to an objectively existent nature

As the study of axioms eliminates the idea that axioms are something absolute, conceiving them instead as ***free propositions of the human mind****, just so would this* ***musical theory*** *free us from the concept of major/minor tonality […] as an irrevocable law of nature.*



Funzione e struttura del sistema dodecafonico



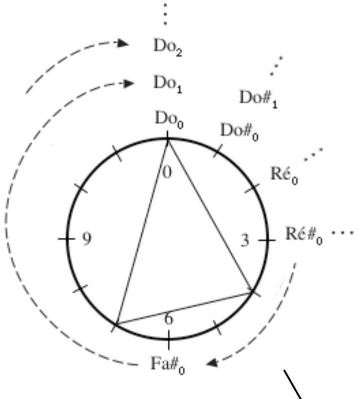
La relazione di congruenza mod 12

Camille Durutte:

- *Technie, ou lois générales du système harmonique* (1855)
- *Résumé élémentaire de la Technie harmonique, et complément de cette Technie* (1876)

Il sistema dodecafonico è "*un insieme d'elementi,* ***relazioni*** *fra gli elementi e* ***operazioni*** *sugli elementi*. […] *Uneffettiva matematizzazione avrebbe bisogno di una formulazione e di una presentazione dettata dal fatto che il sistema dodecafonico è un* ***gruppo di permutazioni*** *determinato* [shaped] *dalla struttura di questo modello matematico*"
(M. Babbitt: *The function of Set Structure in the Twelve-Tone System,* 1946)

Sistema dodecafonico e teoria dei gruppi



(4,3,5)

	S	I	R	RI
S	S	I	R	RI
I	I	S	RI	R
R	R	RI	S	I
RI	RI	R	I	S

S: $(a,b) \to (a,b)$

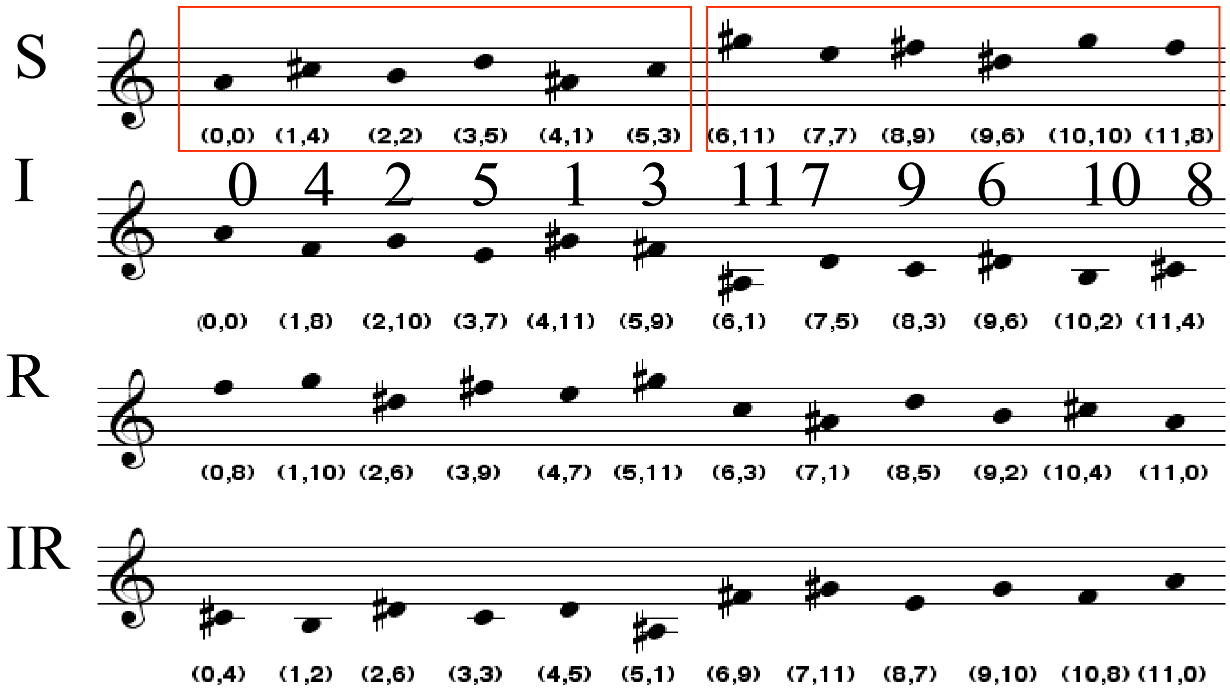
I: $(a,b) \to (a,\ \mathbf{12\text{-}}\boldsymbol{b}\ \mathbf{mod.12})$

R: $(a,b) \to (11\text{-}a,b)$.

RI: $(a,b) \to (a,\ 12\text{-}b \text{ mod.}12)$

$\downarrow$

$(11\text{-}a,\ 12\text{-}b \text{ mod.}12)$



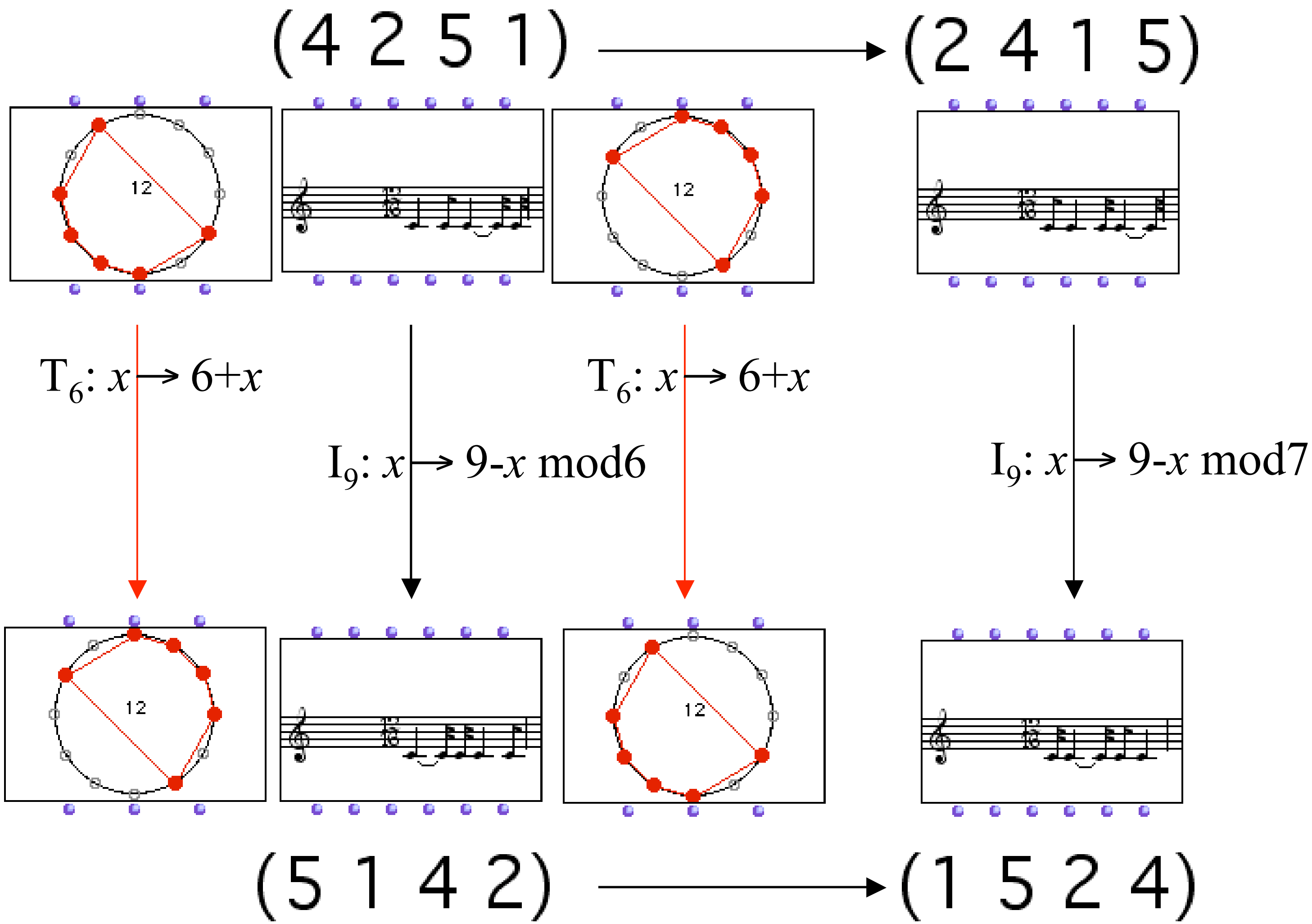
I: $x \to 12\text{-}x$

T_k: $x \to k+x$

$T_{11}I$: $x \to 11\text{-}x$

Le origini algebriche del serialismo integrale

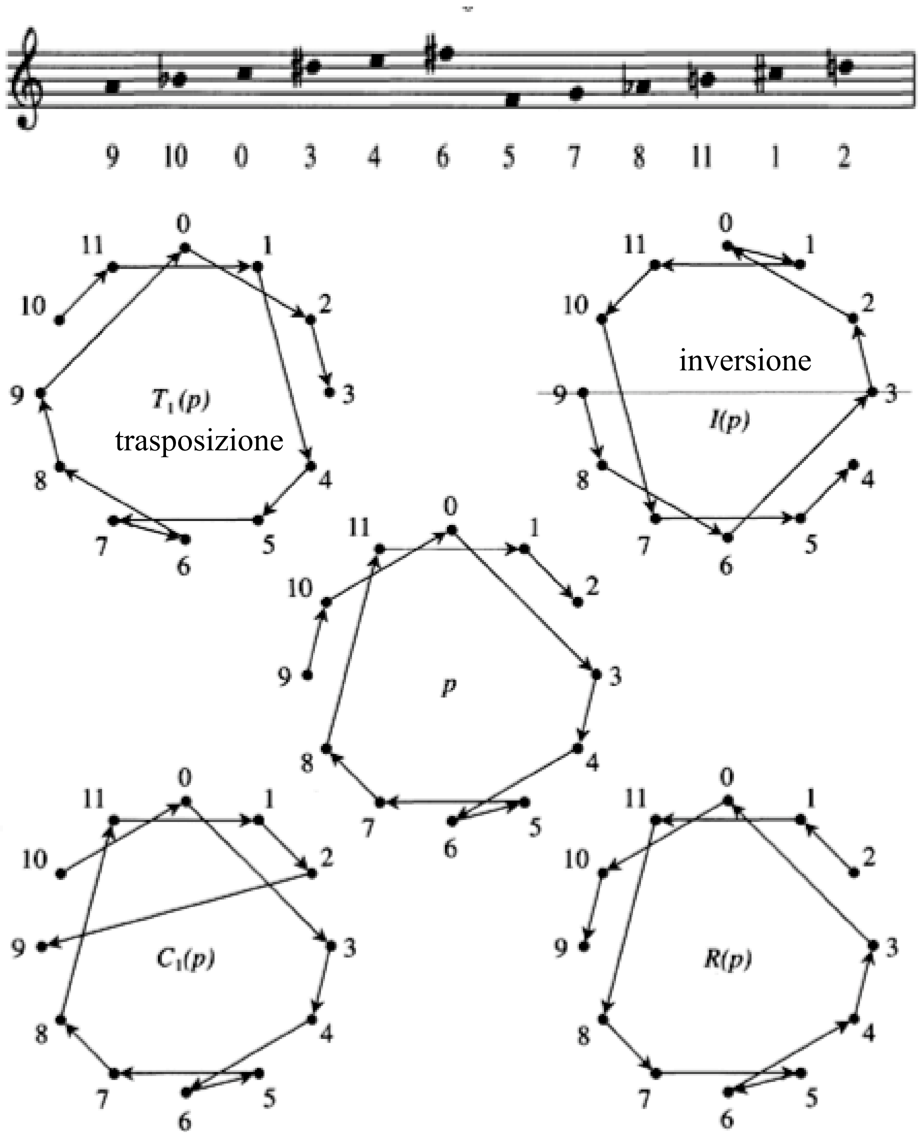
- Milton Babbitt: *Three compositions for piano* (1948)



mm.14-16

Rappresentazioni geometriche delle trasformazioni dodecafoniche

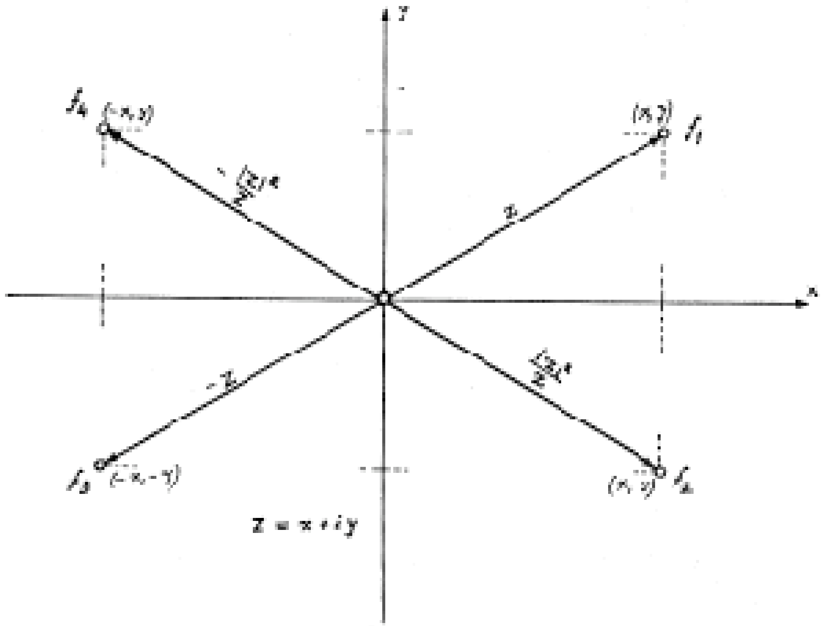
R.C. Read: « Combinatorial problems in the theory of music », *Discrete Math.*, 1997



« rotazione »
(permutation circulaire)

retrogradazione

I. Xenakis: *Formalized Music*, 1971 (1992)



$Z = x + yi$

$f_1 = Z = x + yi = Z = f_1(Z)$ = original form

$f_2 = x - yi = |Z|^2/Z = f_2(Z)$ = inversion

$f_3 = -x - yi = -Z = f_3(Z)$ = inverted retrogradation

$f_4 = -x + yi = -(|Z|^2/Z) = f_4(Z)$ = retrogradation

Funzione e struttura di una teoria della musica

«…rendere possibile da una parte lo studio della ***struttura*** *dei sistemi musicali […] e la formulazione dei vincoli di questi sistemiin una prospettiva compositiva […] ma anche, come condizione preliminare, una terminologia adequata […] per rendere possibile e stabilire un* ***modello*** *che autorizza delle proposizioni ben determinate e verificabili sulle opere musicali»*

M. Babbitt : « The Structure and Function of Music Theory », 1965

La *Set Theory*

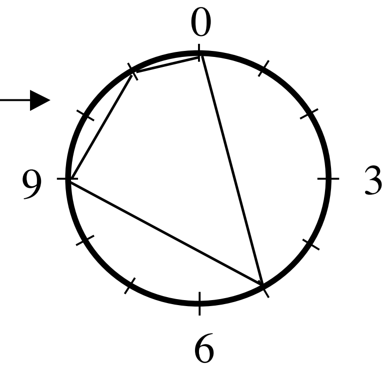
- A. Forte : *The Structure of Atonal Music*, 1973.
- D. Lewin : *Generalized Musical Intervals and Transformation*, 1987

- E. Carter : *Harmony Book*, 2002 (sketches 1960)

{do, fa, si, la}

{0, 5, 9, 11}

[1, 1, 1, 1, 1, 1]



4-Z29

23

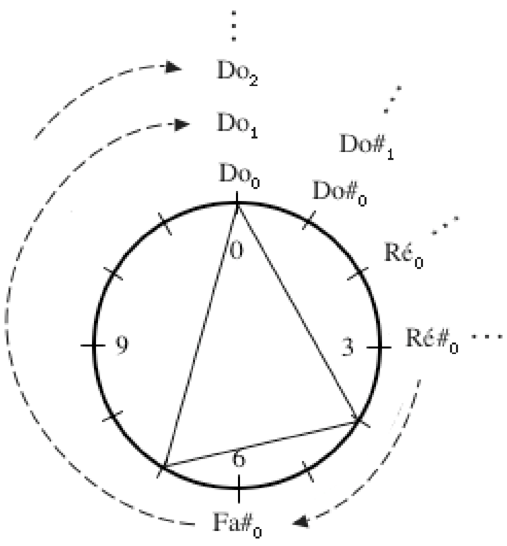
- A. Vieru : *The Book of modes*, 1993 (orig. 1980)

(5 4 2 1)

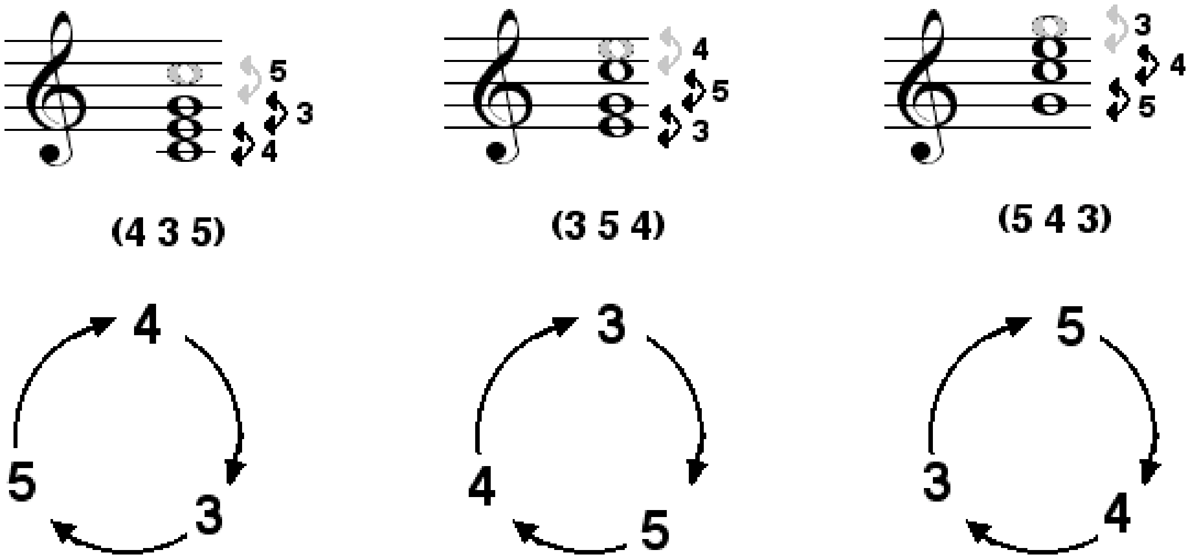
- A. Riotte, M. Mesnage : l'analyse formalisée

0-5421

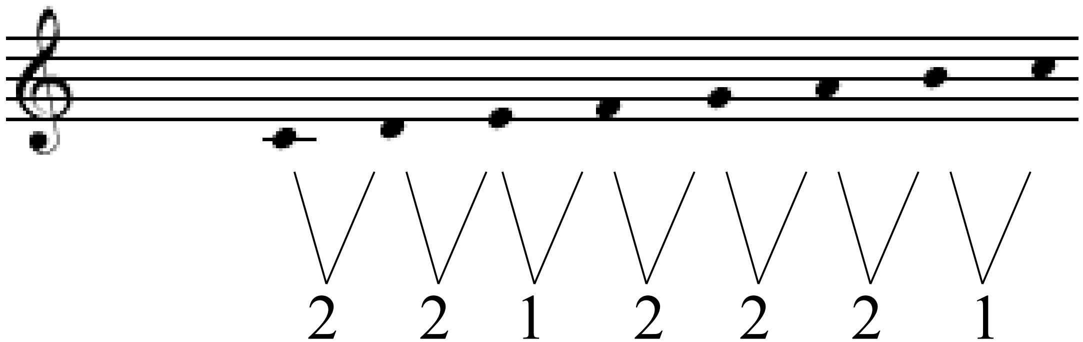
Rappresentazione circolare e struttura intervallare

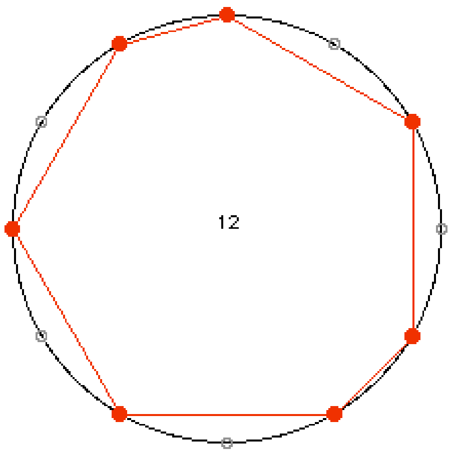


(4,3,5)



I rivolti di un accordo corrispondono alle permutazioni circolari della struttura intervallare



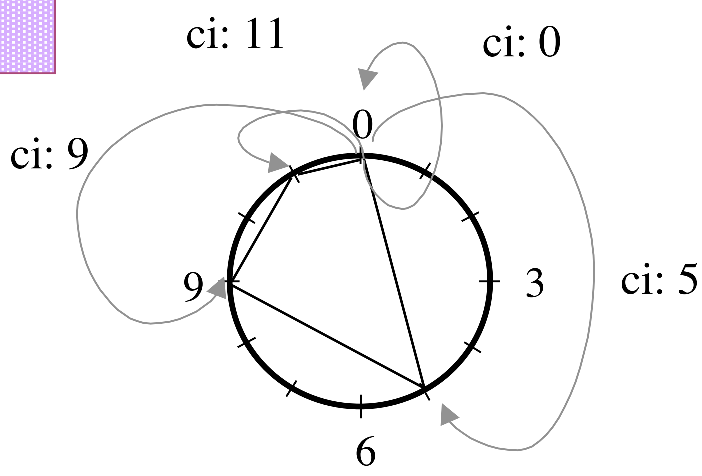


Vettore intervallare (IV) e funzione intervallare (IFUNC)

Il vettore intervallare (Forte) esprime la frequenza di apparizione di ogni intervallo modulo il suo complementare

IV(A) = [1 1 1 1 1 1]

ci 1 ci 2 ci 3 … ci 6.



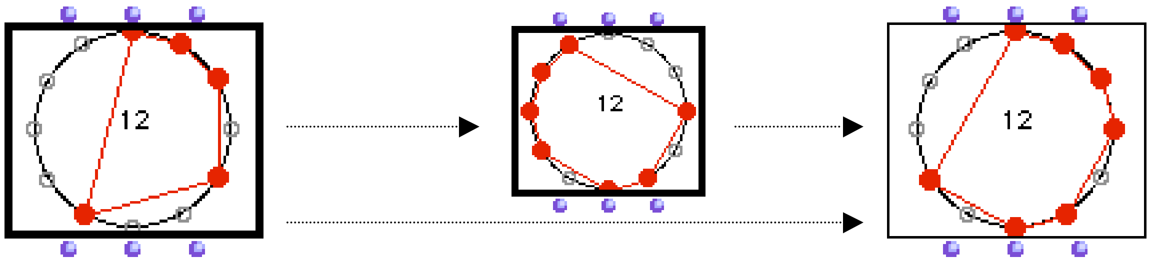
A = {0, 5, 9, 11}

La funzione intervallare IFUNC (Lewin) esprime la frequenza di apparizione di ogni intervallo (dall'unisono alla settima maggiore)

IFUNC(A, A) = [4 1 1 1 1 1 1 1 1 1 1 1]

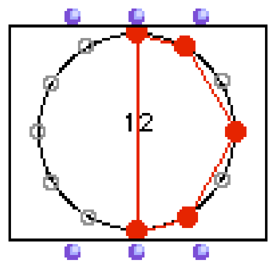
ci 0 ci 1 ci 2 … ci 11.

Il catalogo dei *pcs* d'Allen Forte (1973)



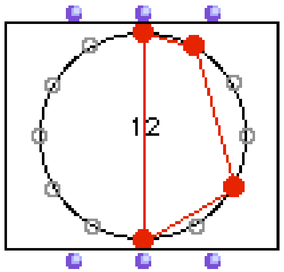
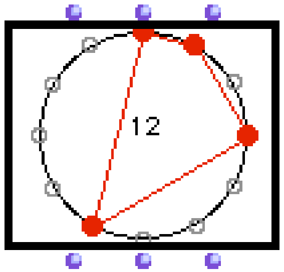
5-Z36	0,1,2,4,7	222121

7-Z36	0,1,2,3,5,6,8	444342



5-Z12

Elliott Carter's *Harmony Book* (2002)

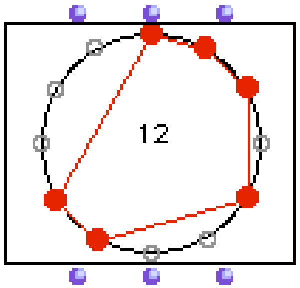
		pcset	*vecteur*	*Forte*
18		(0 1 4 6)	[111111]	4-Z15
23		(0 1 3 7)	[111111]	4-Z29

Utilizzazione (implicita) della *Z-relation*

- *Quartetto n°1* (1951)
- *Night Fantasies* (1980)
- *90+* (1994)
- …

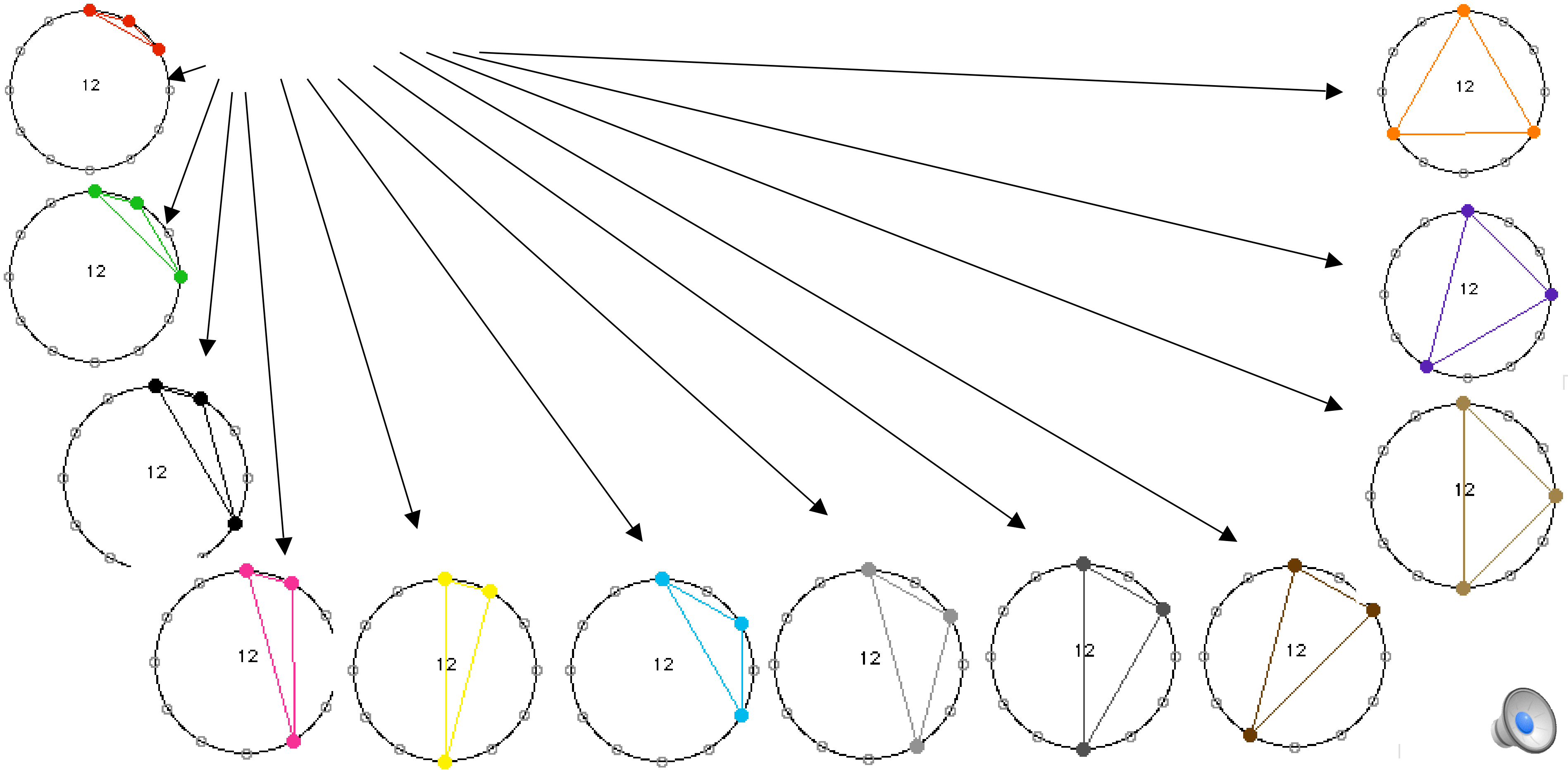
D. Ghisi, *Vettori intervallari : non degenerazione e Z-relation*, tesi di laurea, Milano, ottobre 2006

Elliott Carter: 90+ (1994)



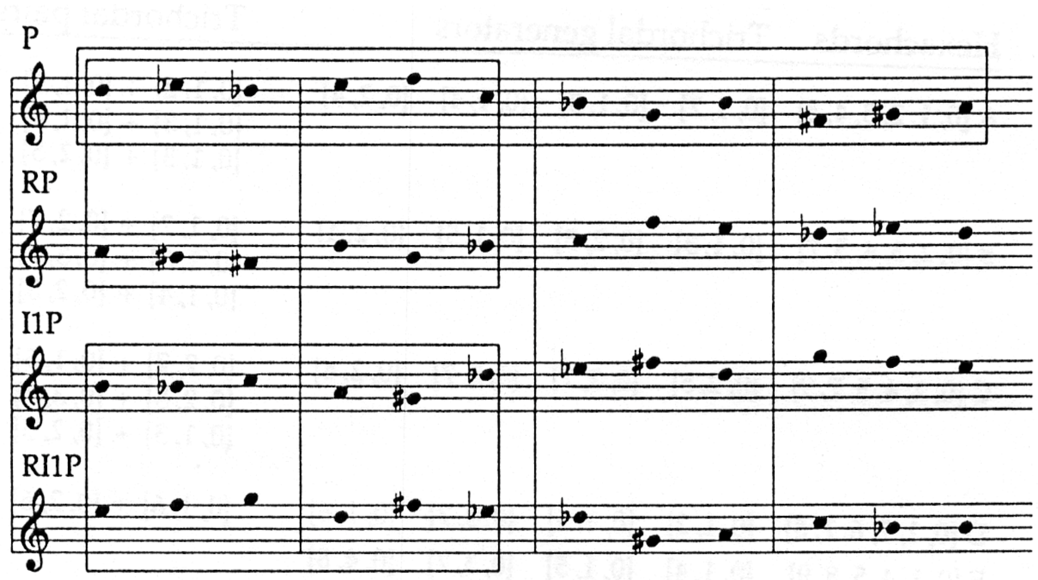
	pcset	vecteur	Forte
35	(0 1 2 4 7 8)	[322332]	6-Z17

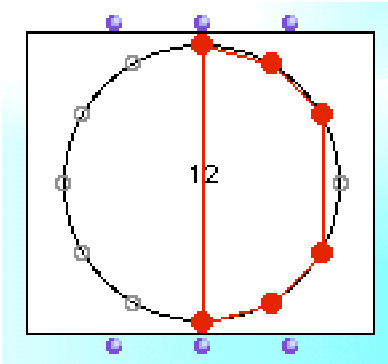
All-triad hexachord



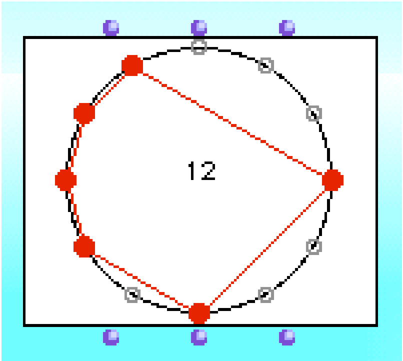
Teorema dell'esacordo (o teorema di Babbitt)

(Wilcox, Ralph Fox (?), Chemillier, Lewin, Mazzola, Schaub, …, Amiot [2006])





A



A'

IV(*A*) = [4, 3, 2, 3, 2, 1] = [4, 3, 2, 3, 2, 1] = IV(*A'*)

« Un esacordo e il suo complementare hanno lo stesso contenuto intervallare »

David Lewin e la trasformata di Fourier

Journal of Music Theory, 1958

- Il contenuto intervallare di due accordi *A* e B è uguale al prodotto di convoluzione delle loro funzioni caratteristiche

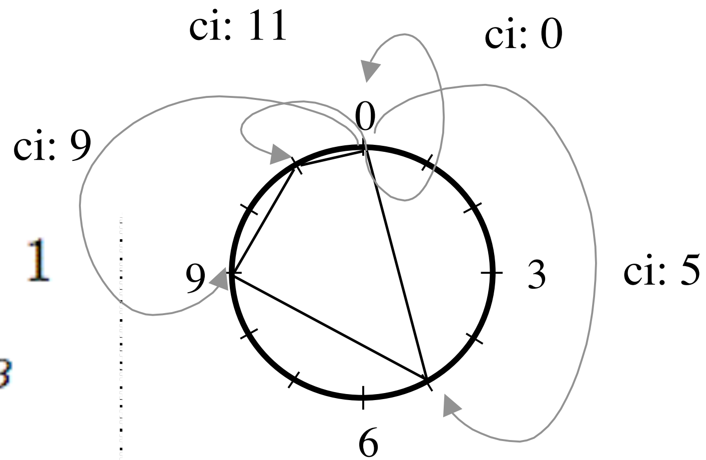
$$IC_A(k) = \text{Card}\{(x,y) \in A \times A \mid x + k = y\}$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

$$1_A * \tilde{1}_B(k) = \sum_i 1_A(i) \times 1_B(i-k) = \sum_{\substack{i \in A \\ i-k \in B}} 1$$

$$\mathcal{F}(1_A * \tilde{1}_B) = \mathcal{F}(1_A) \times \mathcal{F}(\tilde{1}_B)$$

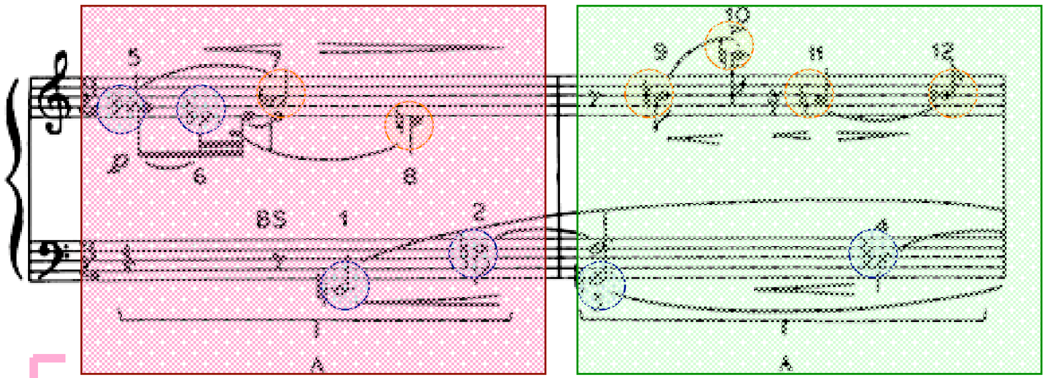
$$\forall k \ \mathcal{F}(IC_{\mathbb{Z}_c \setminus A})(k) = \mathcal{F}(IC_A)(k)$$



A = {0, 5, 9, 11}

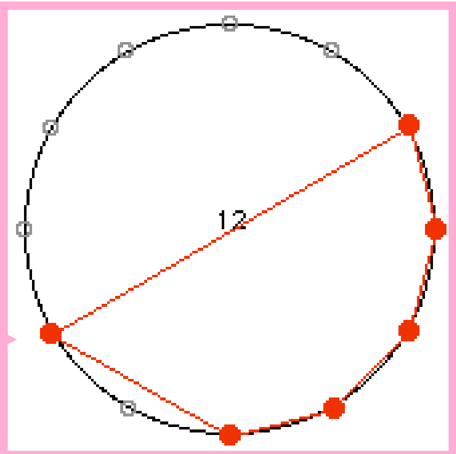
$IC_A(k) = 1 \ \forall \ k = 1 \ldots 11$

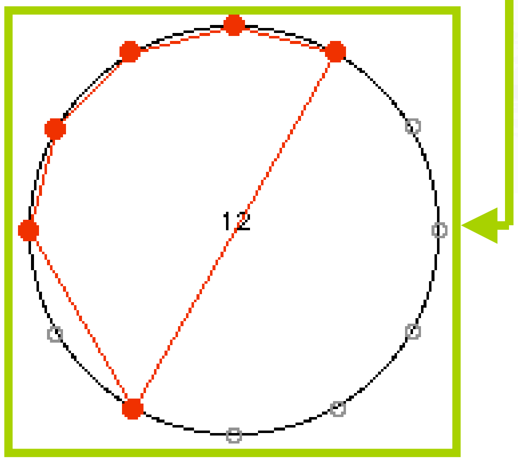
« Combinatorialità esacordale » (Schoenberg, Hauer, Krenek, Babbitt)

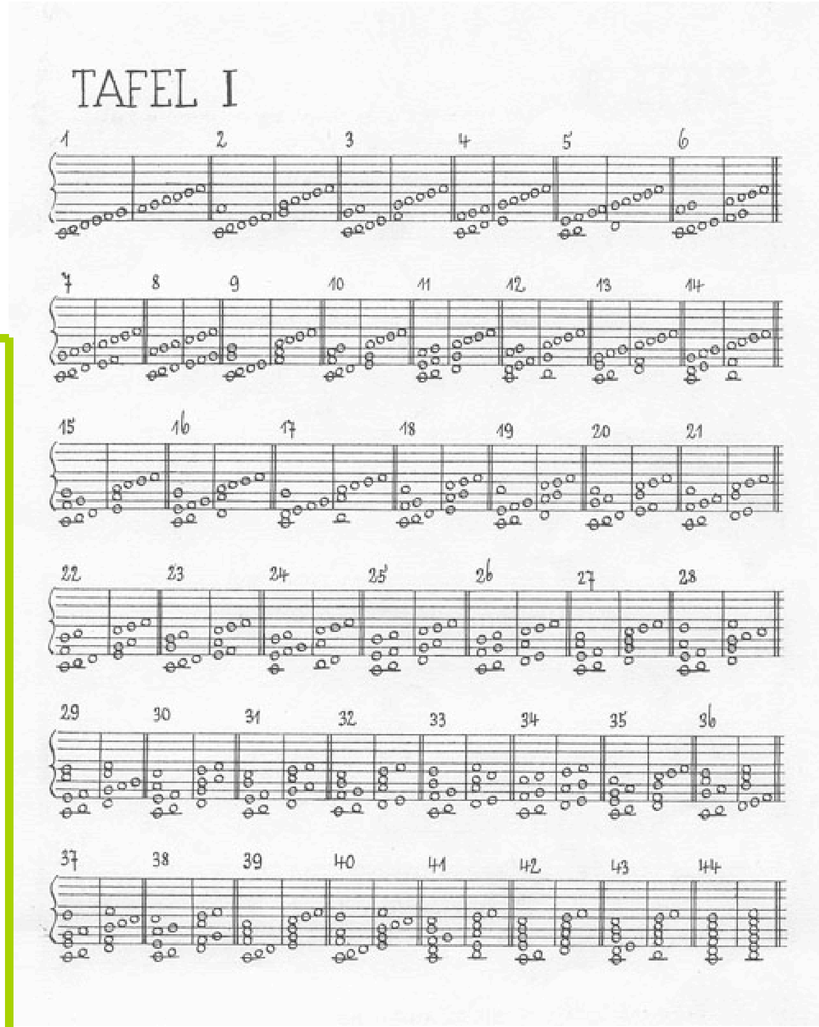


Basic Set: E F G D-flat G-flat E-flat (A) A-flat D B C A B-flat (A)

Example 1. Suite, Op. 25, Minuet



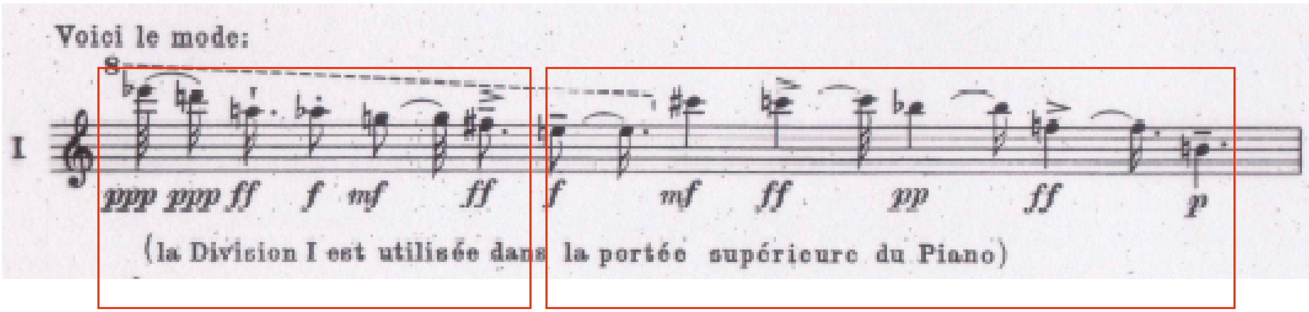


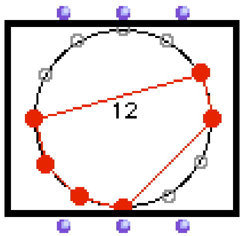


Joseph Mathias Hauer, *Zwölftontechnik: Die Lehre von den Tropen*, 1926

Combinatorialità esacordale in Messiaen

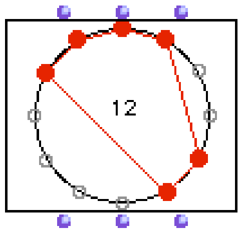
- Mode de valeurs et d’intensités (1950)





$\{3,2,9,8,7,6\} \longrightarrow \{4,5,10,11,0,1\}$

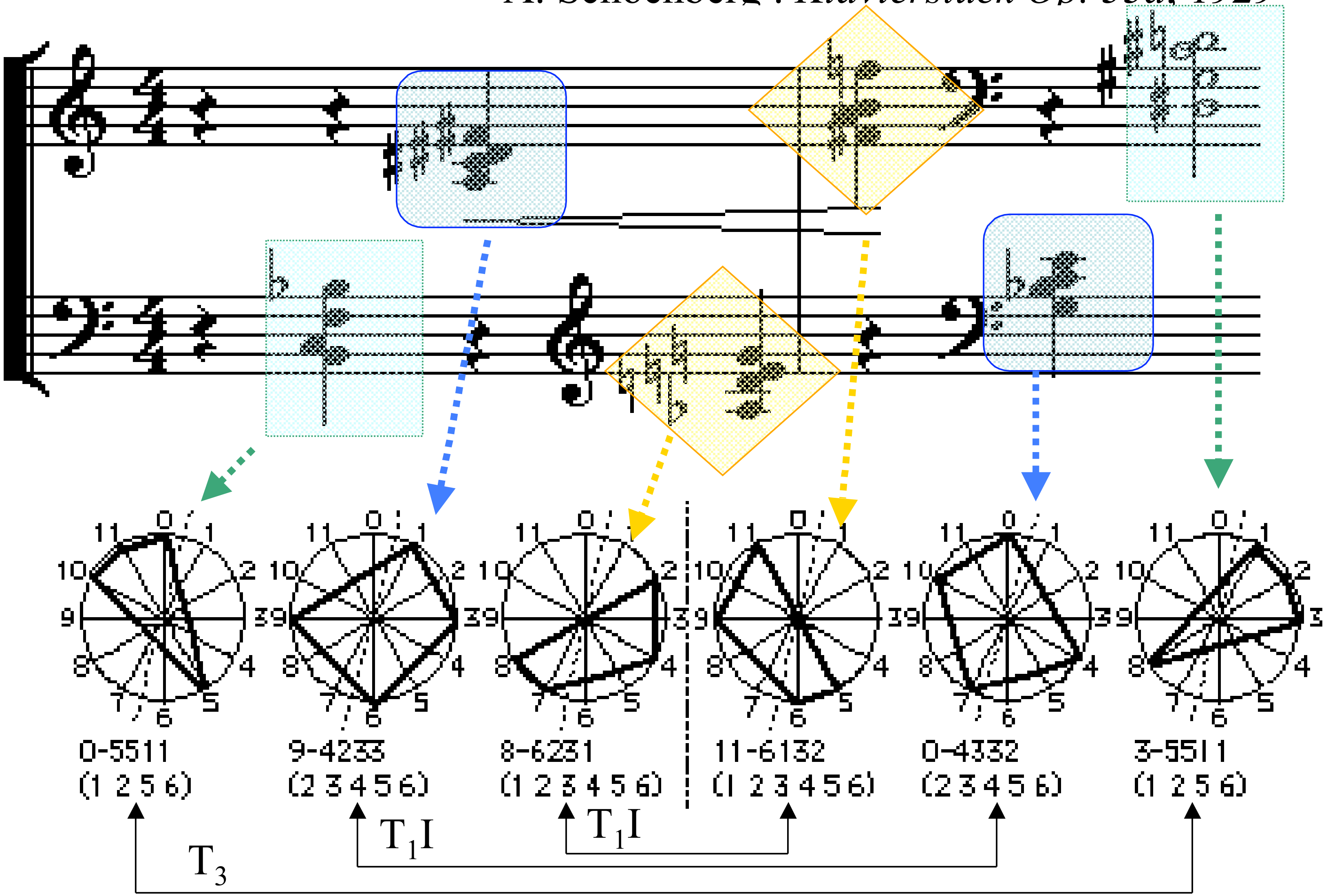
$T_7I : x \rightarrow 7-x$



L'analisi formalizzata o le « entités formelles » in musica

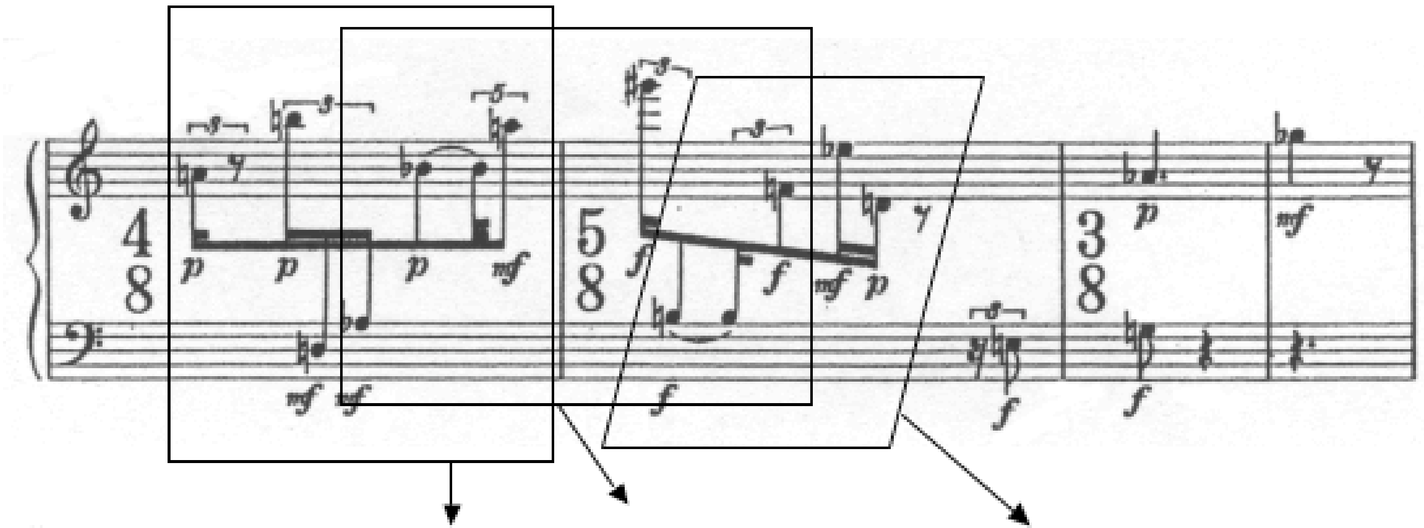
André Riotte e Marcel Mesnage

A. Schoenberg : *Klavierstück Op. 33a*, 1929

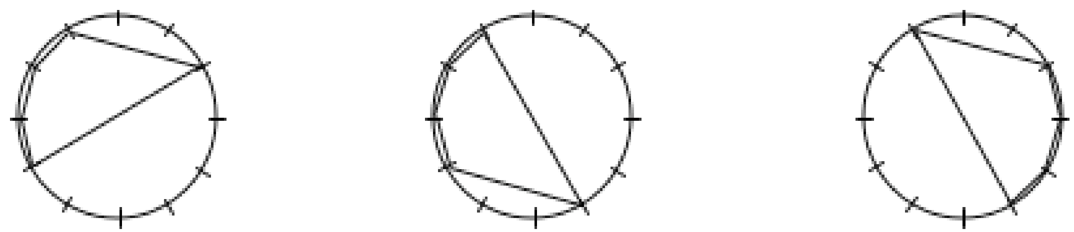


Un esempio di analisi trasformazionale

Stockhausen: *Klavierstück III* (Analisi di D. Lewin)



SI:	(1, 1, 1, 3, 6)	(6, 3, 1, 1, 1)	(6, 3, 1, 1, 1)
IFUNC:	[5 3 2 2 1 1 1 1 1 2 2 3]	[5 3 2 2 1 1 1 1 1 2 2 3]	[5 3 2 2 1 1 1 1 1 2 2 3]
VI:	[3 2 2 1 1 1]	[3 2 2 1 1 1]	[3 2 2 1 1 1]

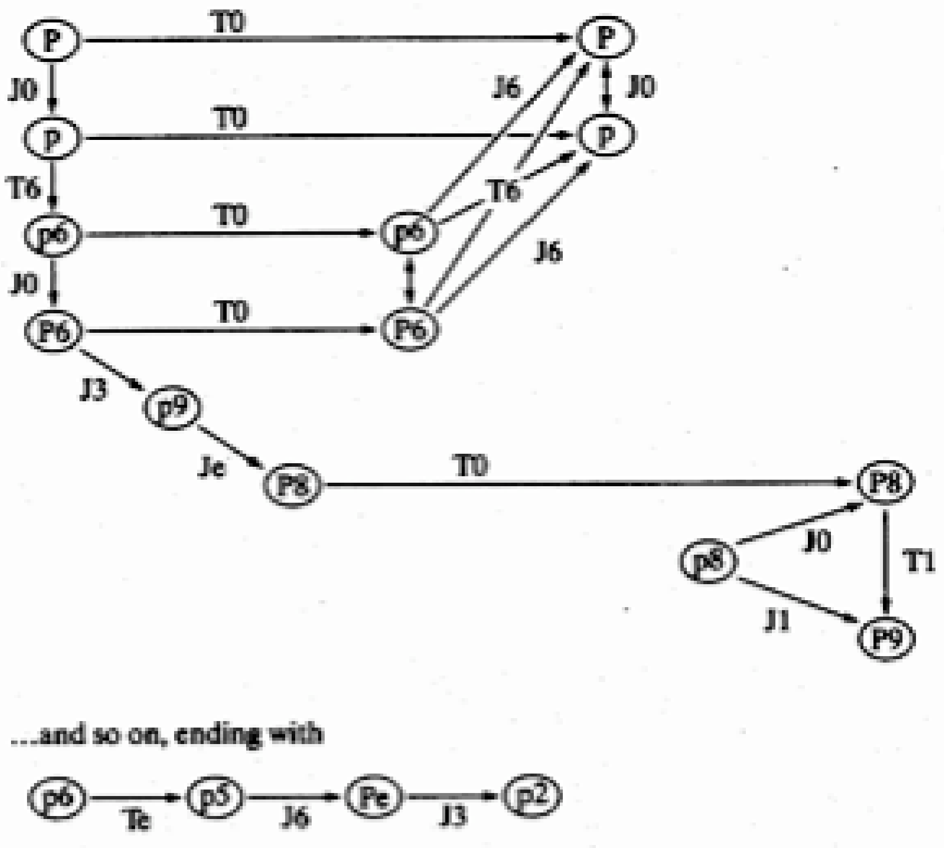


Henck

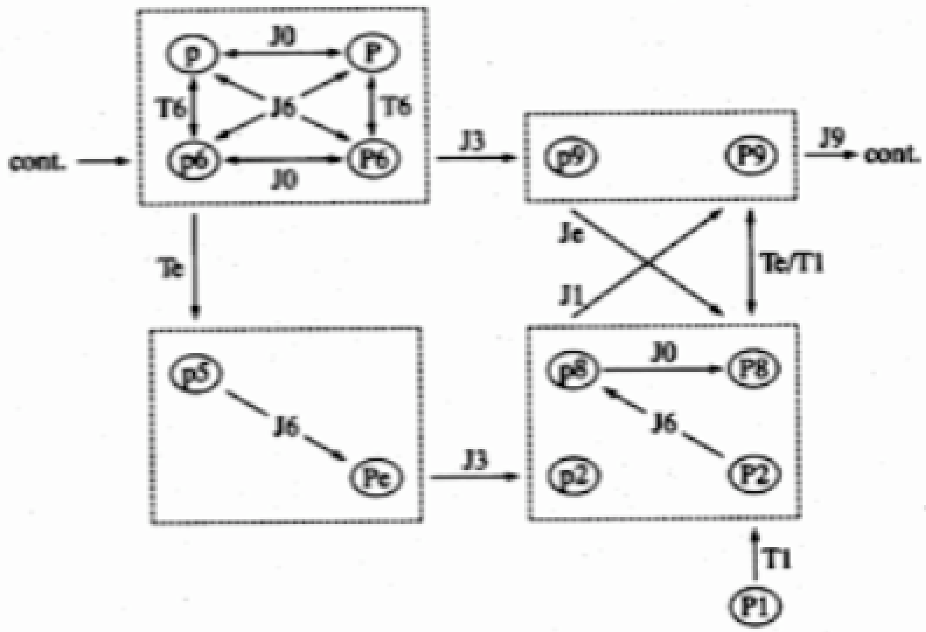
Kontarsky

Tudor

Progressione transformazionale *vs* reticolo transformazionale



Example 2.4. A network whose left-to-right layout reflects the chronological progress of the piece through P/p forms.

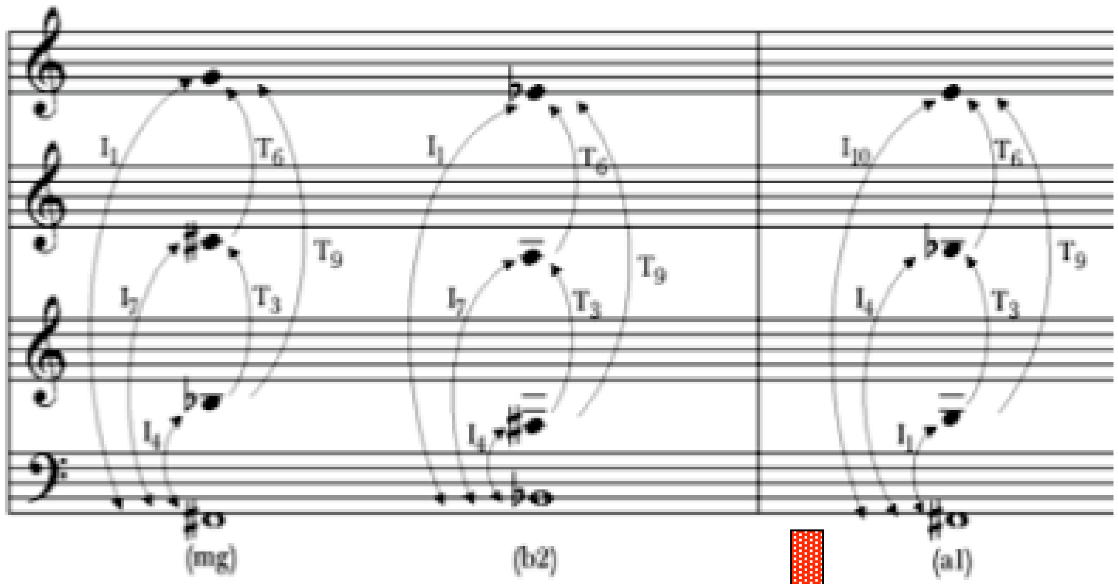


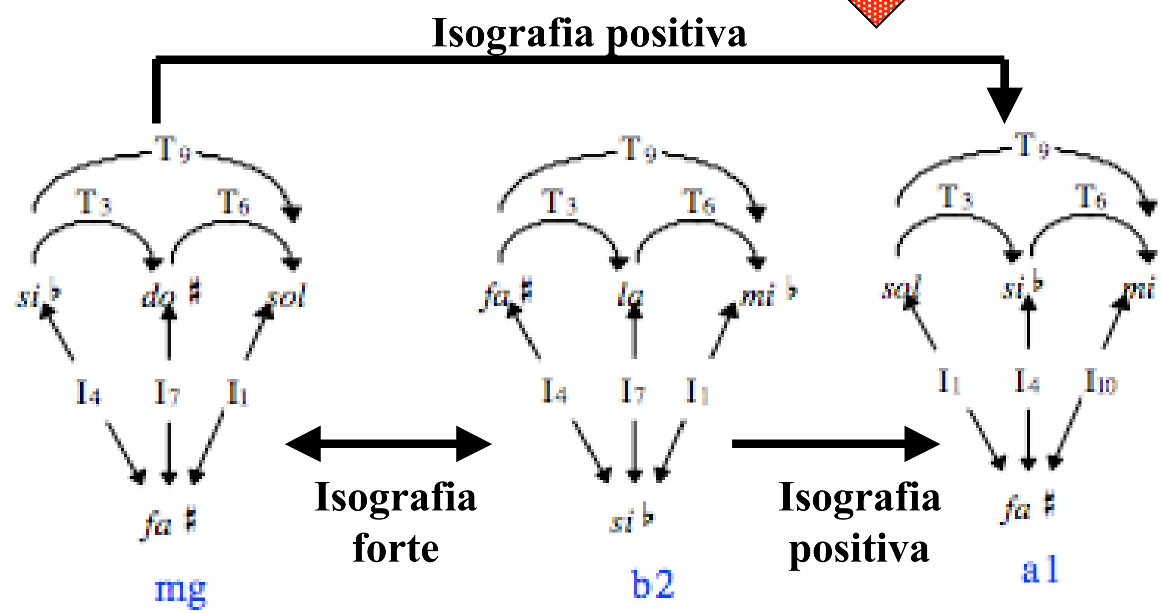
« Rather than asserting a network that follows pentachord relations one at a time, according to the chronology of the piece, I shall assert instead a **network that displays all the pentachord forms used and all their potentially functional interrelationships, in a very compactly organized little spatial configuration**. »

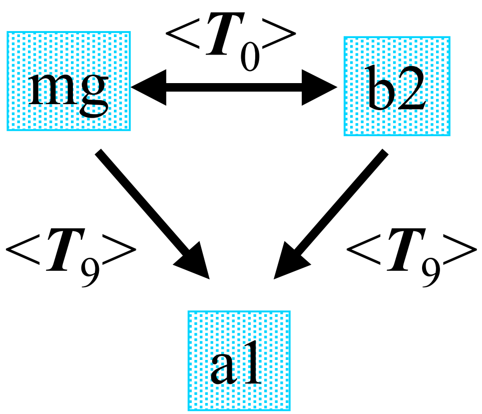
Klumpenhouwer Networks (K-nets)

Xavier Hascher: « Liszt et les sources de la notion d'agrégat », Analyse Musicale, 43, 2002

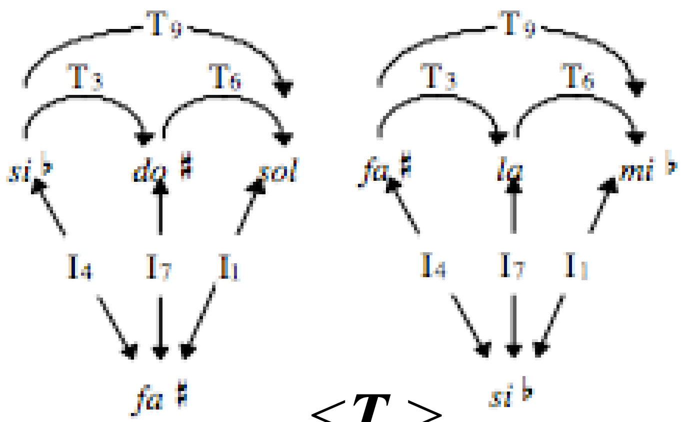
Ex. 1 - « Ladislaus Teleki » (*Historische ungarische Bildnisse* n° 4), mes. 1-7
Les agrégats dans la classification de Forte







Enumerazione dei K-nets in relazione d'isografia forte



mg $\xleftrightarrow{\langle T_0 \rangle}$ b2 $\longleftrightarrow$? …

Isografia forte — Isografia forte

$x \xrightarrow{T_3} x+3 \xrightarrow{T_6} x+9$

I_4, I_7, I_1

$4\text{-}x=7\text{-}(x+3)=1\text{-}(x+9)$

⇨ 12 soluzioni

re $\xrightarrow{T_4}$ *fa#*, *re* $\xrightarrow{M_5}$ *sib*, *fa#* $\xrightarrow{T_2I}$ *sol#*, *sib* $\xrightarrow{T_6I}$ *sol#*

⟷ **Isografia forte**

$x \xrightarrow{T_4} x+4$, $x \xrightarrow{M_5} 5x$, $x+4 \xrightarrow{T_2I}$, $5x \xrightarrow{T_6I}$ $6\text{-}5x=2\text{-}(x+4)$ ==> $8=4x$ ==> $x=2$

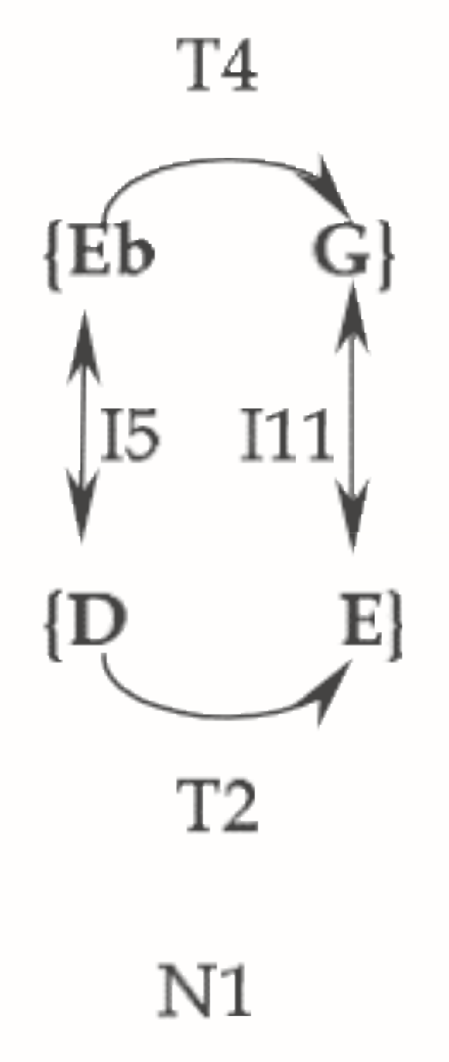
⇨ 1 soluzione

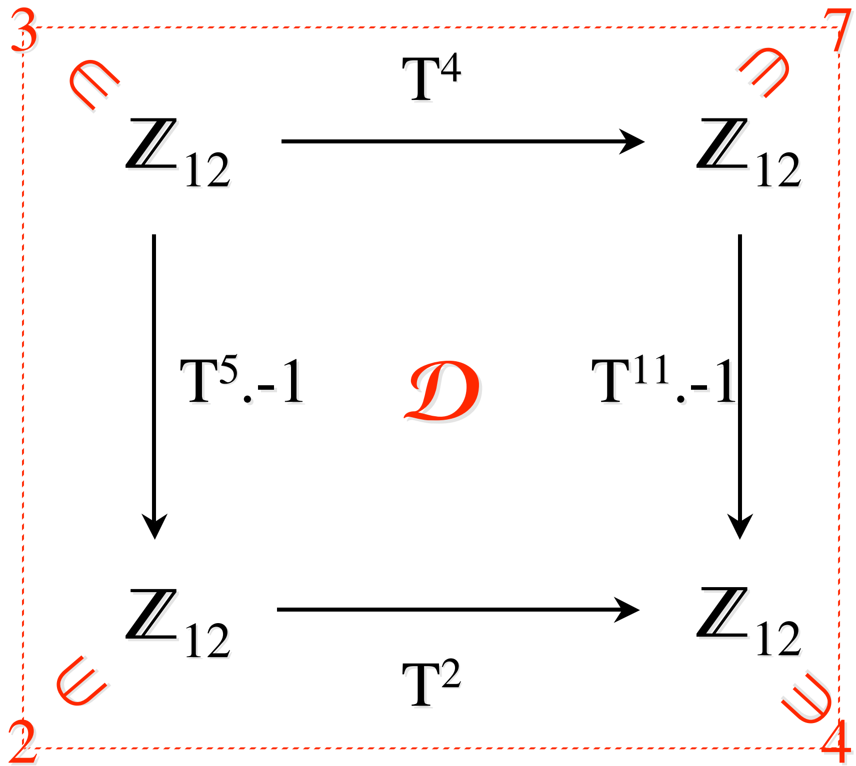
re# $\xrightarrow{M_3}$ *la*, *re#* $\xrightarrow{M_1}$ *re#*, *la* $\xrightarrow{M_7}$ *la*, *re#* $\xrightarrow{M_{11}}$ *la*

⟷ **Isografia forte**

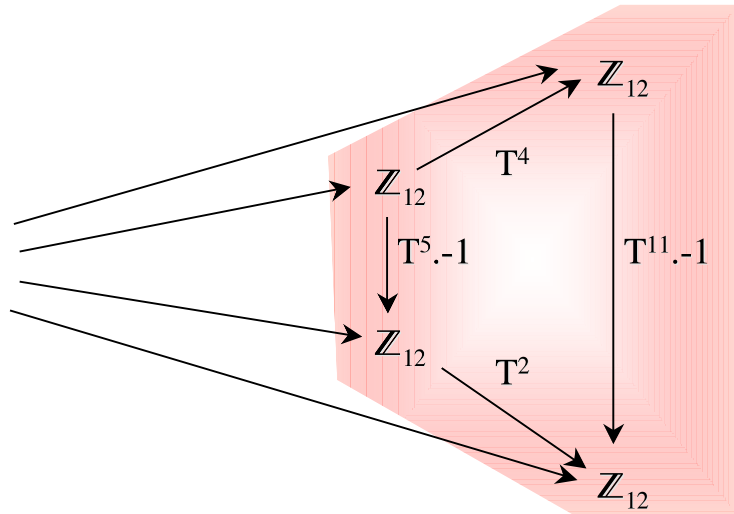
$x \xrightarrow{M_1} x$, $x \xrightarrow{M_1} x$, $x \xrightarrow{M_7}$, $x \xrightarrow{M_{11}}$ $11x=7x$ ==> $4x=0$ ==> $x=0, 3, 6, 9$

⇨ 4 soluzioni





$$(3, 7, 2, 4) \in \lim(\mathcal{D})$$



$\mathcal{Z}_i = \mathbb{Z}_{12}$

$f_{ij}{}^t \in \mathcal{Z}_i @ \mathcal{Z}_j$

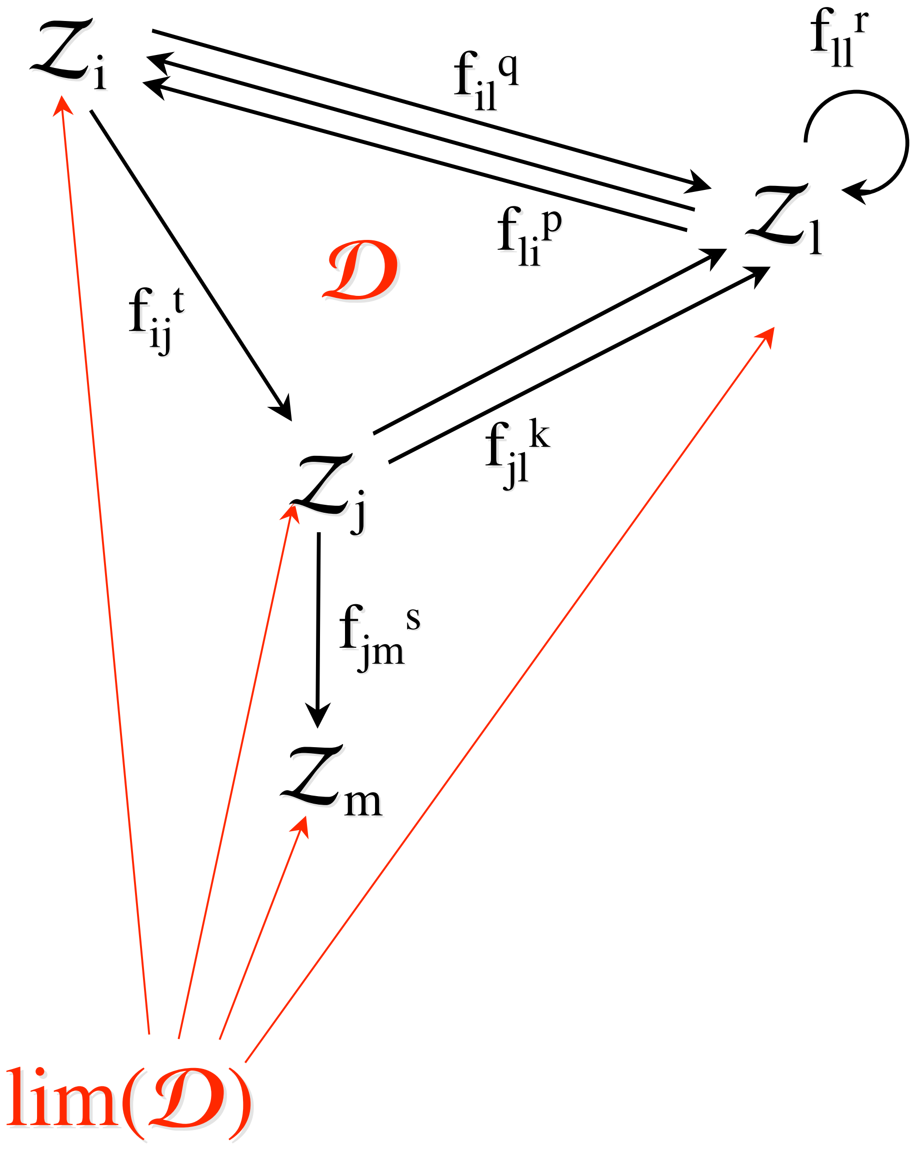
$\lim(\mathcal{D})$ = family of strongly-isographic networks

G. Mazzola & M. Andreatta: **From a Categorical Point of View: K-nets as Limit Denotators**, *PNM*, 2006

Fact:
$\lim(\mathcal{D}) \approx U$

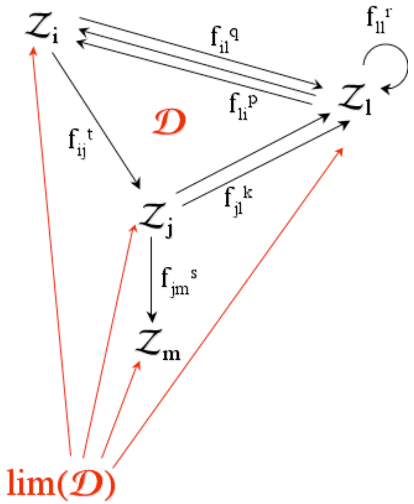
U = (empty or) subgroup of $(\mathbb{Z}_{12})^n$

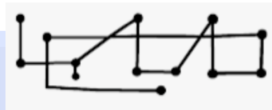
If $f_{**}{}^* =$ isomorphisms card (U) (= 0 or) divides 12



$\mathcal{Z}_i = \mathbb{Z}_{12}$

$f_{ij}^{t} \in \mathcal{Z}_i @ \mathcal{Z}_j$

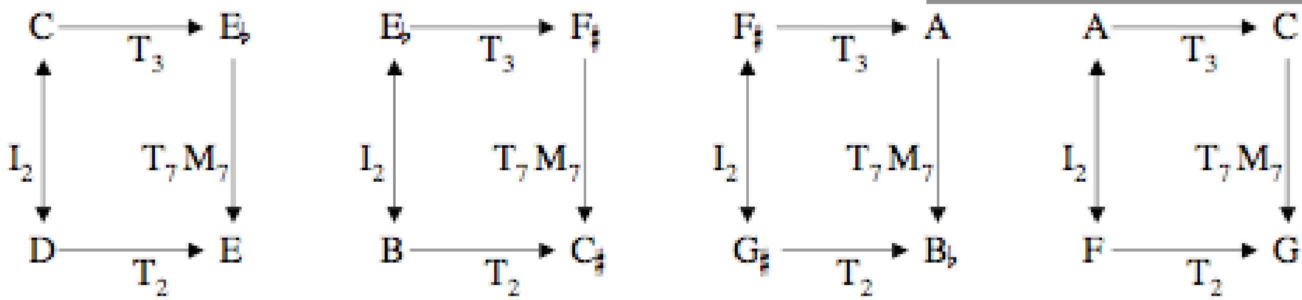




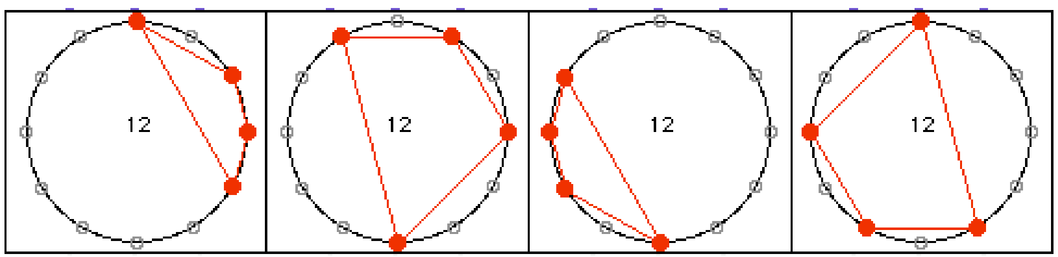
Fact:
lim($\mathcal{D}$) ≈ U

U = (empty or) subgroup of $(\mathbb{Z}_{12})^n$

If f_{**}^{*} = isomorphisms card (U) (= 0 or) divides 12

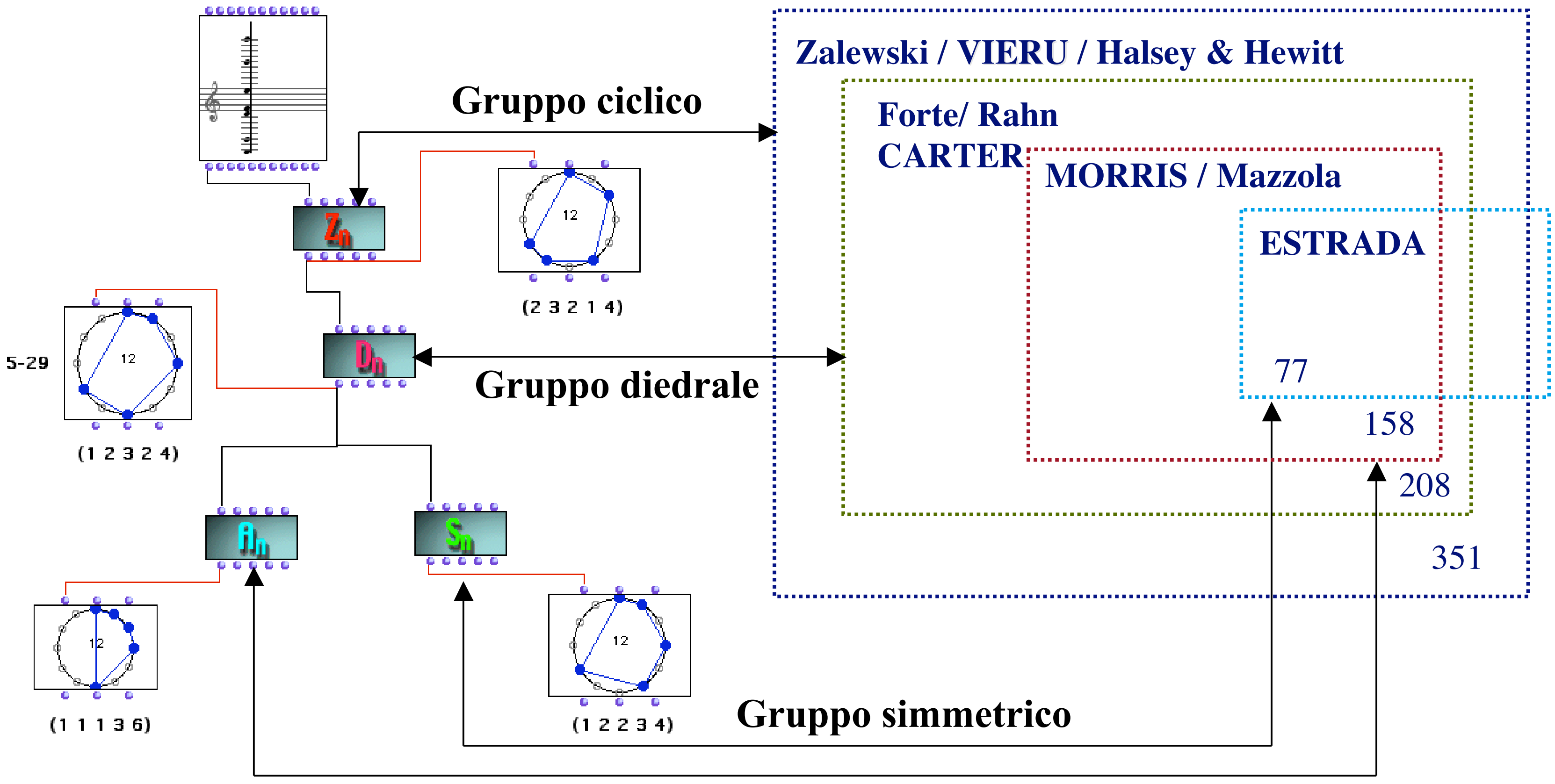


EXAMPLE 6: THE FOUR SOLUTIONS (STRONGLY ISOGRAPHIC K-NETS) OF THIS DIAGRAM ILLUSTRATE THAT THE CARDINALITY OF THE SOLUTION SET IS A DIVISOR OF 12. HERE, THE OPERATOR M_7 DENOTES THE MULTIPLICATION BY 7



Classificazione 'paradigmatica' delle strutture musicali

$G\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12
C_{12}	1	6	19	43	66	80	66	43	19	6	1	1
D_{12}	1	6	12	29	38	50	38	29	12	6	1	1
$\mathrm{Aff}_1(Z_{12})$	1	5	9	21	25	34	25	21	9	5	1	1



Architettura paradigmatica

Formule d'enumerazione d'accordi in un sistema temperato

(Reiner, 1985)

Z_n

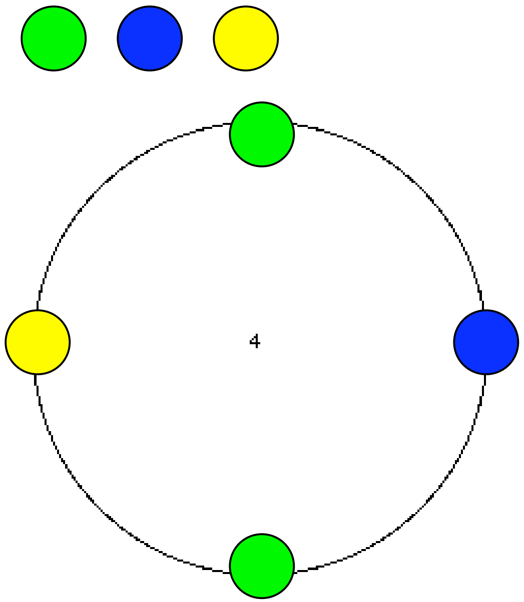
$$\#\text{ of } k\text{-chords} = \frac{1}{n}\sum_{j|(n,k)} \phi(j)\binom{n/j}{k/j} = \frac{1}{n}\Phi_n(k),$$

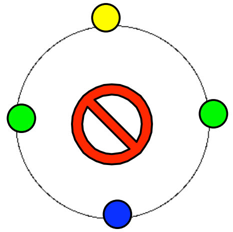
D_n

$$\#\text{ of } k\text{-chords} = \begin{cases} \frac{1}{2n}\left[\Phi_n(k) + n\binom{(n-1)/2}{[k/2]}\right], & \text{if } n \text{ is odd,} \\ \frac{1}{2n}\left[\Phi_n(k) + n\binom{n/2}{k/2}\right], & \text{if } n \text{ is even and } k \text{ is even,} \\ \frac{1}{2n}\left[\Phi_n(k) + n\binom{(n/2)-1}{[k/2]}\right], & \text{if } n \text{ is even and } k \text{ is odd.} \end{cases}$$

- D. Halsey & E. Hewitt: « Eine gruppentheoretische Methode in der Musik-theorie », *Jahresber. Der Dt. Math.-Vereinigung*, 80, 1978.
- **D. Reiner: «Enumeration in Music Theory», *Amer. Math. Month.* 92:51-54, 1985**
- H. Fripertinger: «Enumeration in Musical Theory», *Beiträge zur Elektr. Musik*,1,1992
- R.C. Read: « Combinatorial problems in the theory of music », *Discrete Math.*, 1997
- H. Fripertinger: « Enumeration of mosaics », *Discrete Math.*, 1999
- H. Fripertinger: « Enumeration of non-isomorphic canons », *Tatra Mt. Math. Publ.*, 2001
- M. Broué : « Les tonalités musicales vues par un mathématicien », *Le temps des savoirs, Revue de l'Institut Universitaire de France*, 2002
- David J. Hunter & Paul T. von Hippel : « How Rare Is Symmetry in Musical 12-Tone Rows? », *The American Mathematical Monthly*, Vol. 110, No. 2., Feb., 2003
- H. Fripertinger: « Tiling problems in music theory », in *Perspectives in Mathematical and Computational Music Theory* (Mazzola, Noll, Puebla ed., Epos, 2004)
- Rachel W. Hall & P. Klingsberg: « Asymmetric Rhythms, Tiling Canons, and Burnside's Lemma », *Bridge Proceedings*, 2004
- …

Enumerazione delle orbite sull'azione di un gruppo





Lemma di Burnside

$$n = \frac{1}{|G|}\sum_{g\in G}|X^g|.$$

$$X^g = \{x \in X : gx = x\}$$

Azione di **Z/4Z**

T_0 = identità
T_1 = rotazione di 90°
T_2 = rotazione di 180°
T_3 = rotazione di 270°

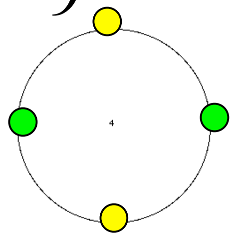
Configurazioni possibili = 3^4 = 81

T_0 fissa ogni configurazione => $|X^{T_0}|=81$

T_1 fissa ogni configurazione monocromatica => $|X^{T_1}|=3$

T_3 idem

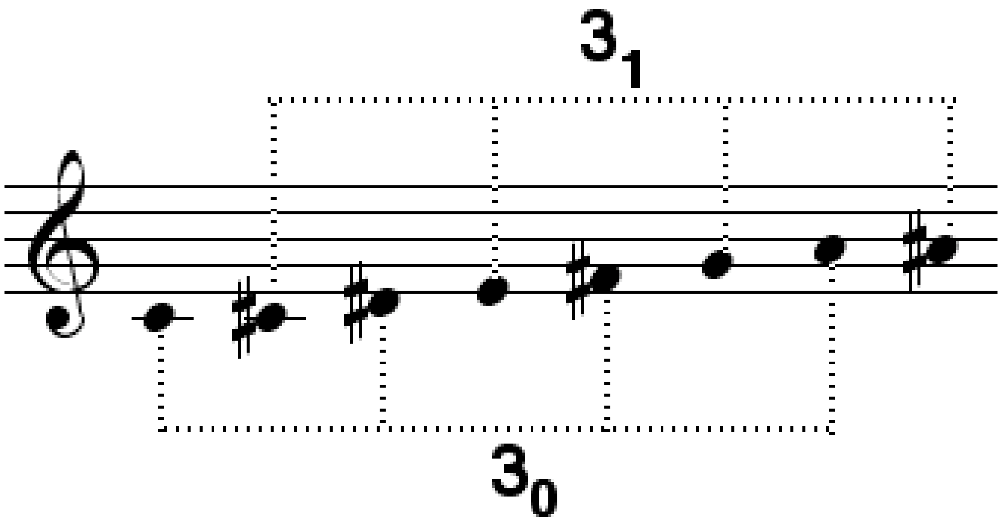
T_2 fissa ogni configurazione «a doppio-diametro » => $|X^{T_1}| = 3^2 = 9$

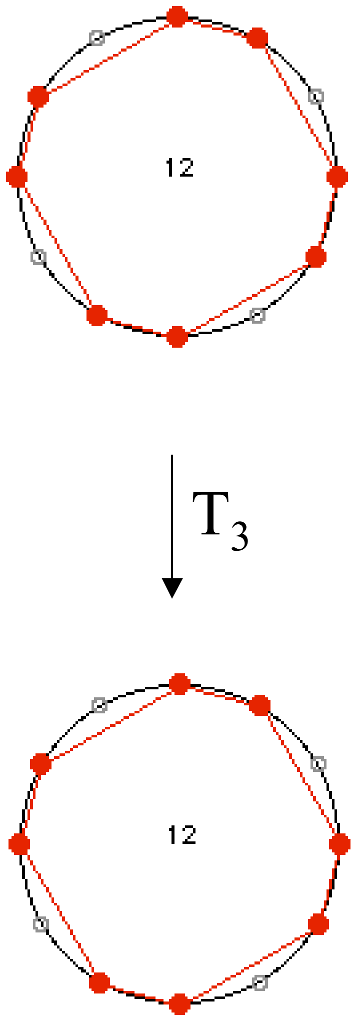


$n = 1/4\ (81+3+3+9) = 24$

Formalizzazione dei modi di Messiaen a trasposizione limitata

Quante e quali scale musicali hanno le stesse proprietà strutturali della scala ottatonica (semitono-tono)?





Trovare tutte le scale musicali che si ripetono esattamente ad una trasposizione T_k di k semitoni ($k \neq 0 \bmod 12$)

Enumerazione dei modi di Messiaen

R.C. Read: « Combinatorial problems in the theory of music », *Discrete Math.*, 1997

M. Broué : « Les tonalités musicales vues par un mathématicien », 2002

$$A_n = \sum_{k|n} \mu\left(\frac{n}{k}\right) 2^k$$

$$s_d(n) = \sum_{\{e\,;\,(e|(n/d))\}} \mu\left(\frac{n/d}{e}\right) 2^e$$

$\mu(k)=0$ se k è divisibile per un quadrato

$\mu(k)=(-1)^m$ se k è il prodotto di m numeri primi distinti

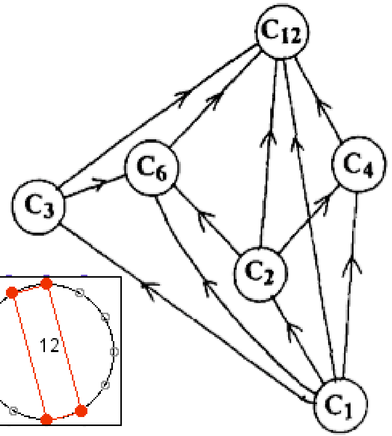
$$A_6=\mu(6)2+\mu(3)2^2+\mu(2)2^3+\mu(1)2^6=$$
$$= (-1)^2 2 + (-1)2^2 + (-1)\, 2^3 + 2^2 =$$
$$= 2 - 4 - 8 + 64 =$$
$$= 54$$

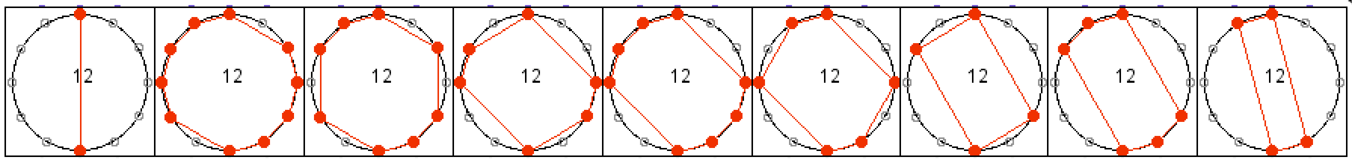
54/6 = 9

12/6 = 2

Table 1

Number of notes / Symmetry	0	1	2	3	4	5	6	7	8	9	10	11	12
1		1	5	18	40	66	75	66	40	18	5	1	
2			1		2		3		2		1		
3				1			1			1			
4					1				1				
6							1						
12	1												1
All scales	1	1	6	19	43	66	80	66	43	19	6	1	1

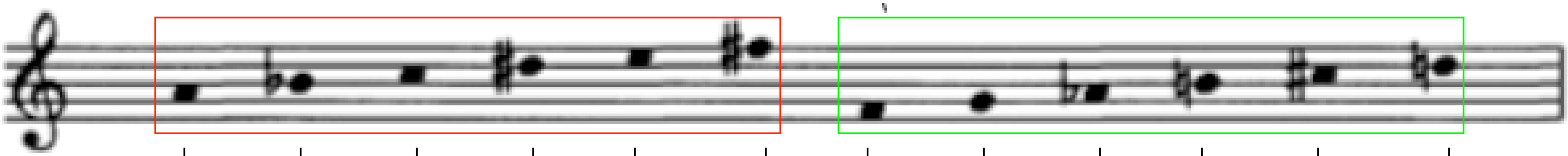




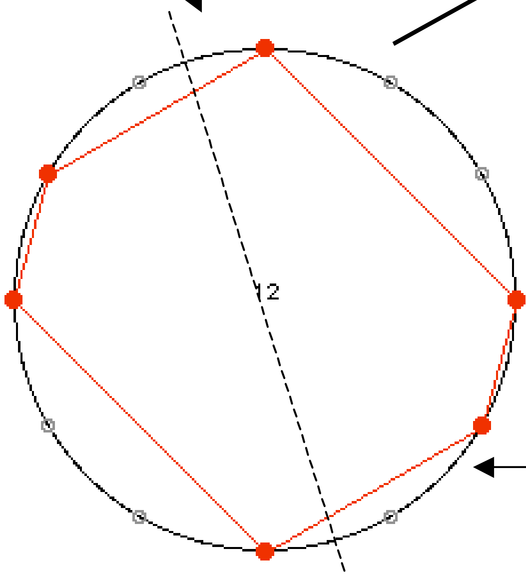
=> *OpenMusic*

Simmetria trasposizionale e "combinatorialità" esacordale

Schoenberg: Serenade Op.24, Mouvement 5



A = {9 , 10 , 0 , 3, 4 , 6} {5 , 7, 8, 11, 1, 2} =A'

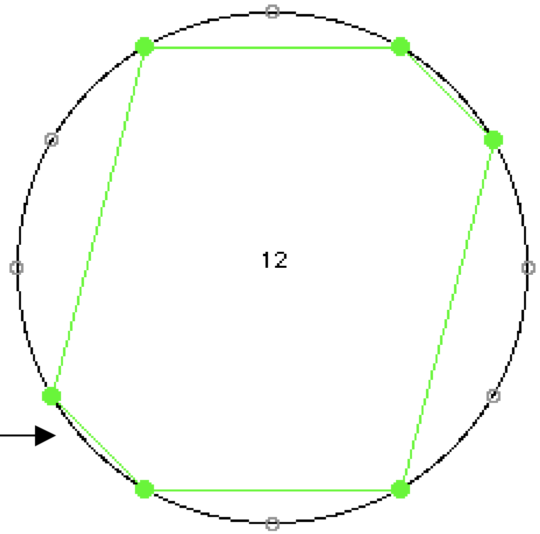


(3, 1, 2, 3, 1, 2)

T6{9,10,0,3,4,6}=
={6+9,6+10,6,6+3,6+4,6+6}=
={3,4,6,9,10,0}

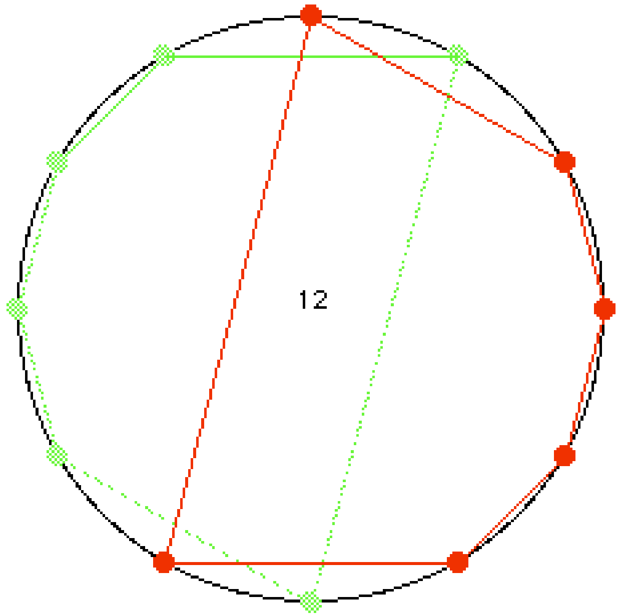
T6(A)=A

$$I_{11} = T_{11}\, I$$



(2, 1, 3, 2, 1, 3)

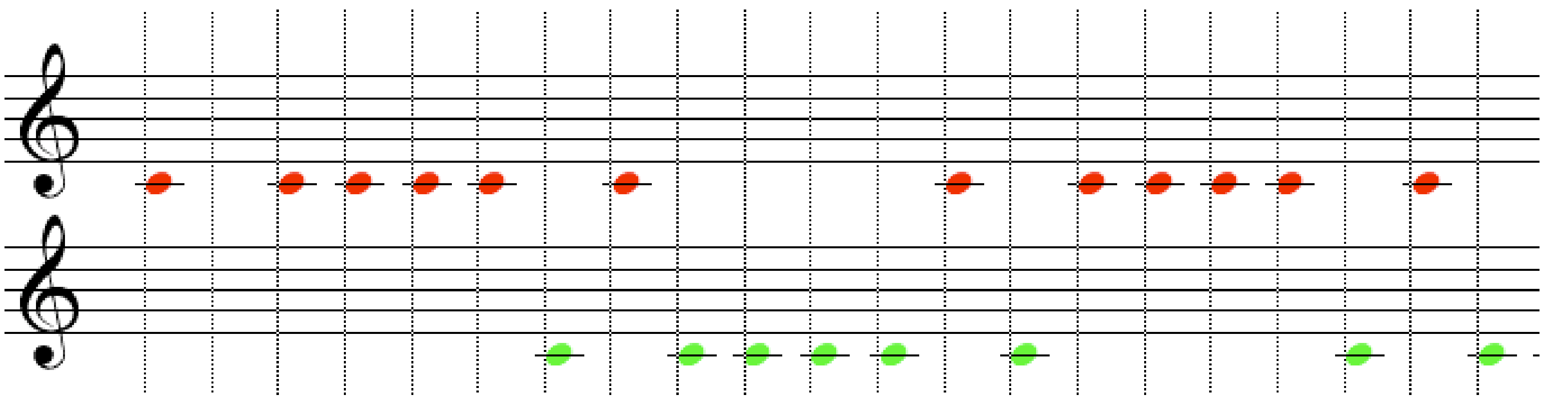
Combinatorialità e canoni ritmici "a mosaico"



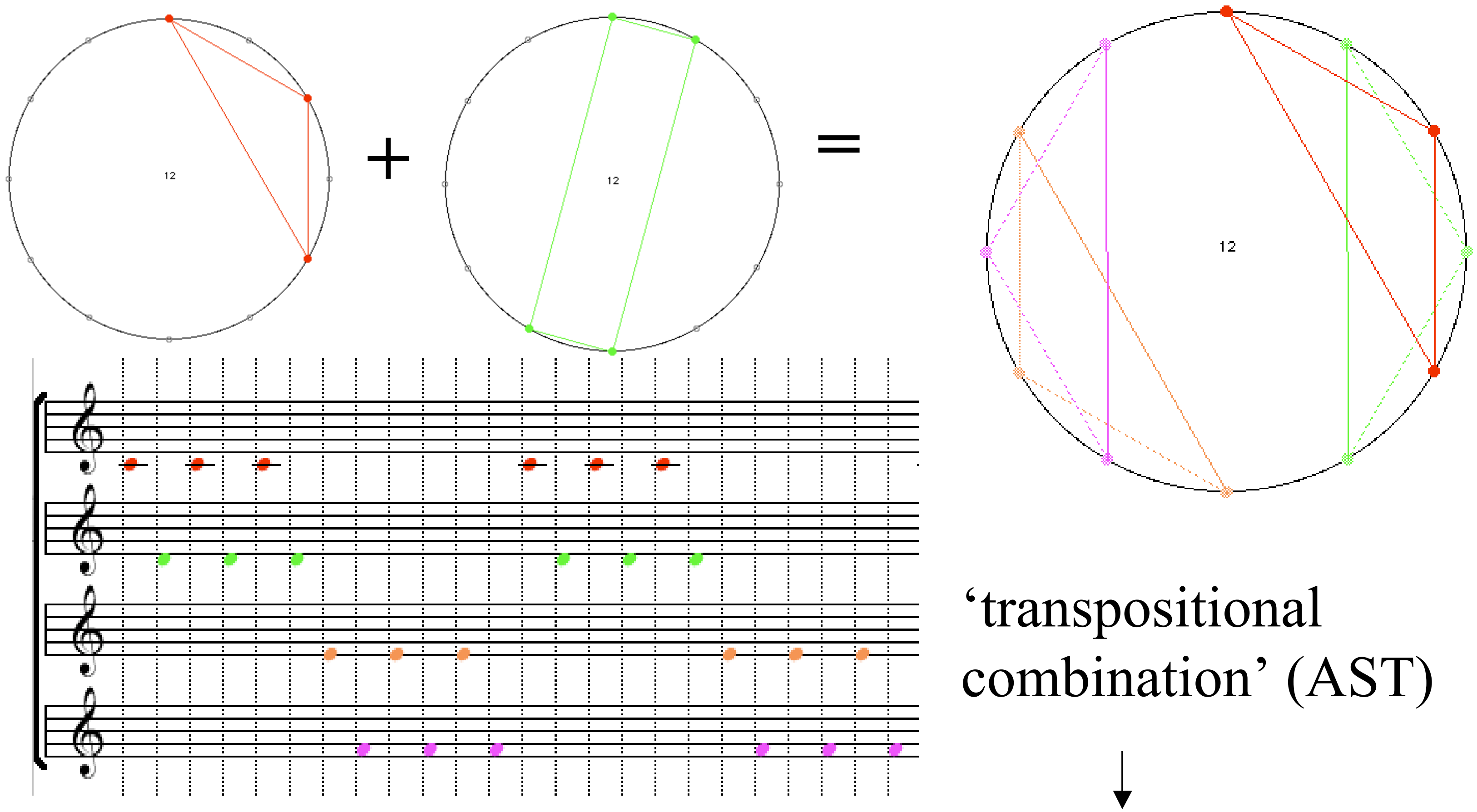
H1= {0, 2, 3, 4, 5, 7}

T6

H2= {1, 6, 8, 9, 10, 11}



Canoni ritmici e fattorizzazione di gruppi

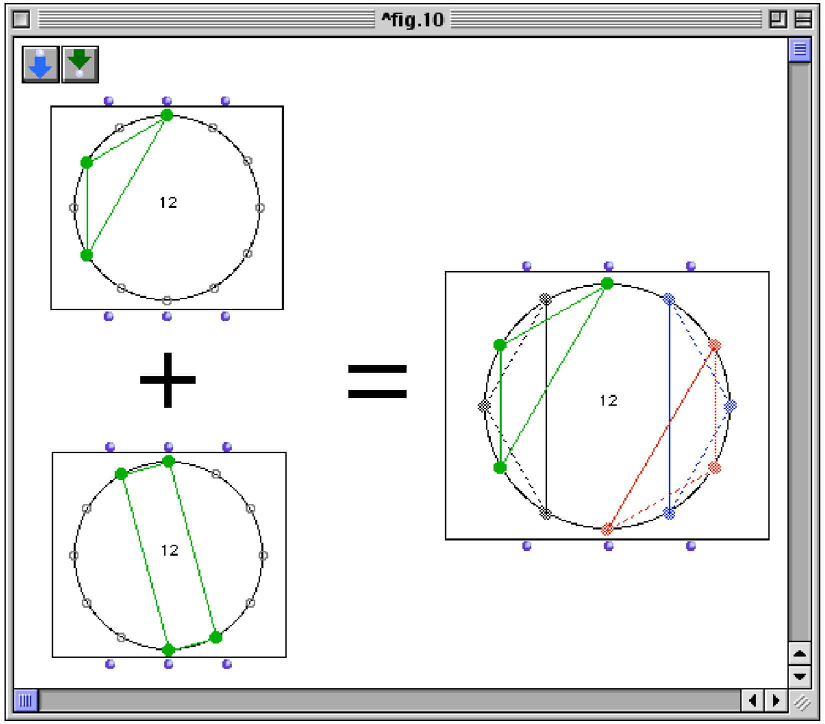


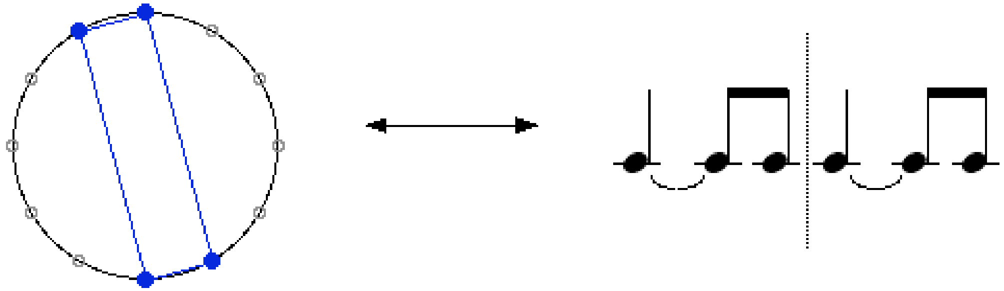
'transpositional combination' (AST)

$$\{0,2,4\} \oplus \{0,1,6,7\} = \mathbf{Z_{12}} = (2\ 2\ 8) \bullet (1\ 5\ 1\ 5)$$

Uno dei due fattori è un modo di Messiaen a trasposizione limitata

Ridondanza ritmica della proprietà di Messiaen



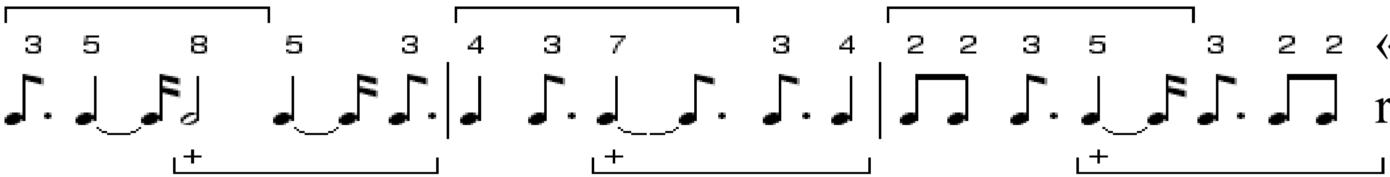


$$(2\ 8\ 2) \bullet (5\ 1\ 5\ 1) = \mathbf{Z/12Z}$$

Modo a trasposizione limitata

Un esempio *ante litteram* di canone ritmico a mosaico

O. Messiaen: *Harawi* (1945)



« Pedale rythmique »

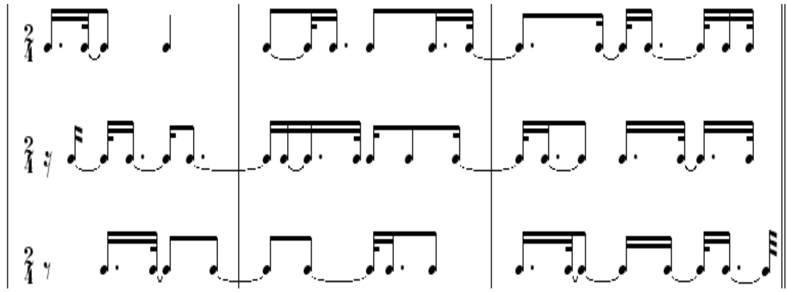
« Remarquons […] que les trois **rythmes non rétrogradables** divisent les durée en 5+5+7 durées, alors que les termes des trois ostinatos harmoniques contiennent toujours six sonorités pour le supérieur, et trois sonorités pour les deux autres. Ajoutons que les durées sont très inégales »

O. Messiaen : *Traité de Rythme, de Couleur et d'Ornithologie,* tome 2, Alphonse Leduc, Editions Musicales, Paris, 1992.

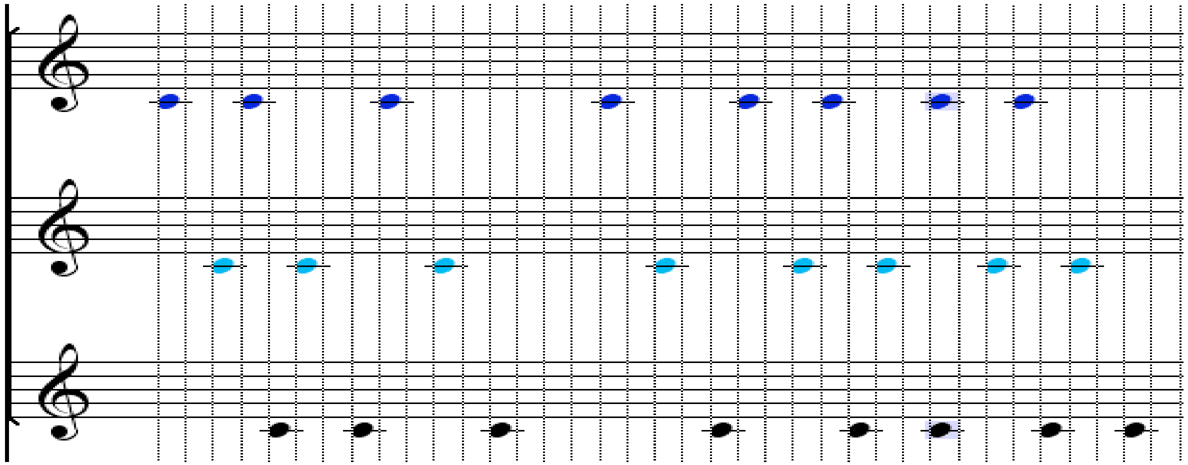
=> *OpenMusic*

Varie utilizzazioni dello stesso modello formale

Harawi (1945)



Visions de l'Amen (1943)



Modello ritmico

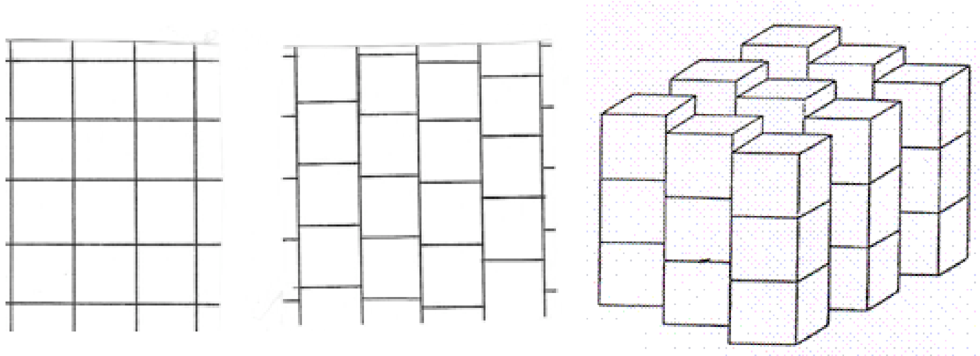
« …il résulte de tout cela que les différentes sonorités se mélangent ou s'opposent de manières très diverses, **jamais au même moment ni au même endroit […]. C'est du désordre organisé** »

O. Messiaen : *Traité de Rythme, de Couleur et d'Ornithologie,* tome 2, Alphonse Leduc, Editions Musicales, Paris, 1992.

Canoni ritmici a mosaico: le due storie

- Olivier Messiaen’s ‘formalization’ of rhythmic canons (Cf. Traité)
- Dan Tudor Vuza’s model of Regular Complementary Canons of Maximal Category (PNM, 1991-)
- The computer-aided model of RCMC Canons and first catalogues of solutions (Agon&Andreatta, 1999)
- Compositional applications of the model (by Fabien Levy, Georges Bloch)
- Enumeration and classifications of RCMC canons (Fripertinger, Amiot, Noll, Andreatta, Tangian, Jedrzejewski)
- Thomas Noll’s generalized model of augmented tiling canons
- Emmanuel Amiot’s model of cyclotomic tiling canons
- The *MathTools* environment in *OpenMusic* (Agon&Andreatta)

- Minkowski’s Conjecture (1896/1907)
- Hajos algebraic solution (1942)
- The classification of Hajos groups (Hajos, de Bruijn, Sands, …)
- The Tiling of the line problem and Fuglede’s Conjecture (Tijdeman,Lagarias, Laba, Coven-Meyerowitz)
- Fuglede’s Conjecture and Vuza’s Canons (Amiot, 2005)

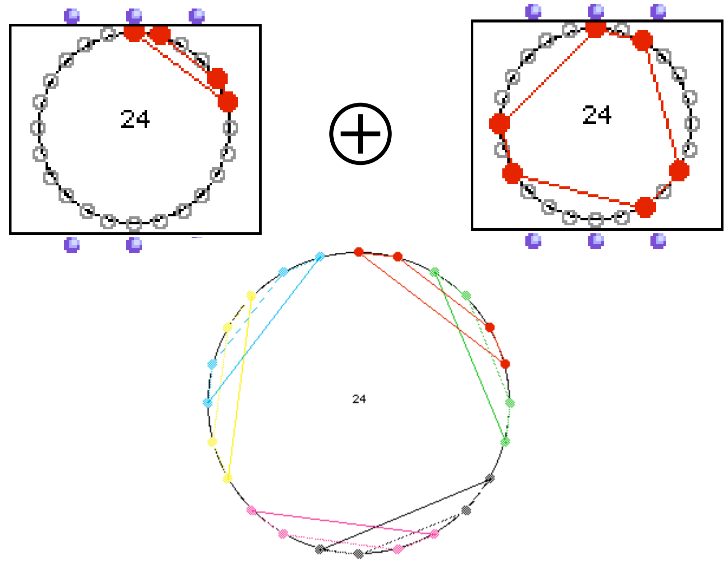


*In a simple lattice tiling of the n-dimensional space by unit cubes, at least one couple of cubes share a complete n-*1 *dimensional face*
(Cf. S. Stein, S. Szabó : *Algebra and Tiling,* 1994)

Gruppi di Hajos e periodicità dei fattori

A group G is an "Hajós group" if for all factorisation of G into a direct sum of subsets $A_1, A_2, \ldots, A_k$, at least one of the factors is periodic.

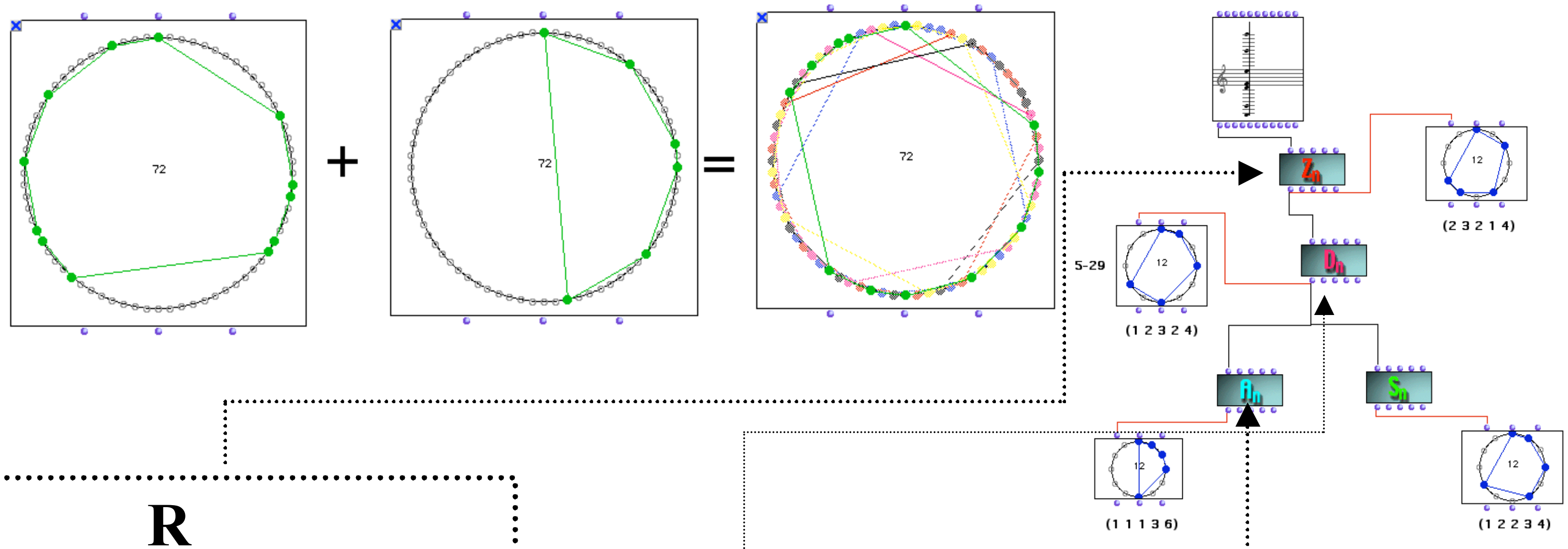
Rédei 1947	(p, p)
Hajós 1950	**Z**
	Z/nZ avec n= p^α
De Brujin 1953	(p^α, q)
	(p, q, r)
Sands 1957	(p^2, q^2)
	(p^2, q, r)
	(p, q, r, s)
Sands 1959	$(2^2, 2^2)$
	$(3^2, 3)$
	$(2^n, 2)$
Sands 1962	$(p, 3, 3)$
	$(p, 2^2, 2)$
	$(p, 2, 2, 2, 2)$
	$(p^2, 2, 2, 2)$
	$(p^3, 2, 2)$
	$(p, q, 2, 2)$
Sands 1964	**Q**
	Z+Z/pZ
	Q+Z/pZ



Groupes non-Hajós (bad groups)

72
108 120 144 168 180
200 216 240 252 264 270 280 288
300 312 324 336 360 378 392 396
400 408 432 440 450 456 468 480
500 504 520 528 540 552 560 576 588 594
600 612 616 624 648 672 675 680 684 696
700 702 720 728 744 750 756 760 784 792
800 810 816 828 864 880 882 888…

Classificazione (paradigmatica) dei canoni RCCM



=> *OpenMusic*

R

(1 3 3 6 11 4 9 6 5 1 3 20)
(20 3 1 5 6 9 4 11 6 3 3 1)
(1 4 1 19 4 1 6 6 7 4 13 6)
(6 13 4 7 6 6 1 4 19 1 4 1)
(1 5 15 4 5 6 6 3 4 17 3 3)
(3 3 17 4 3 6 6 5 4 15 5 1)

(8 8 2 8 8 38)
(16 2 14 2 16 22)
(14 8 10 8 14 18)

S

R

(1 3 3 6 11 4 9 6 5 1 3 20)
(1 4 1 19 4 1 6 6 7 4 13 6)
(1 5 15 4 5 6 6 3 4 17 3 3)

(8 8 2 8 8 38)
(16 2 14 2 16 22)
(14 8 10 8 14 18)

S

R

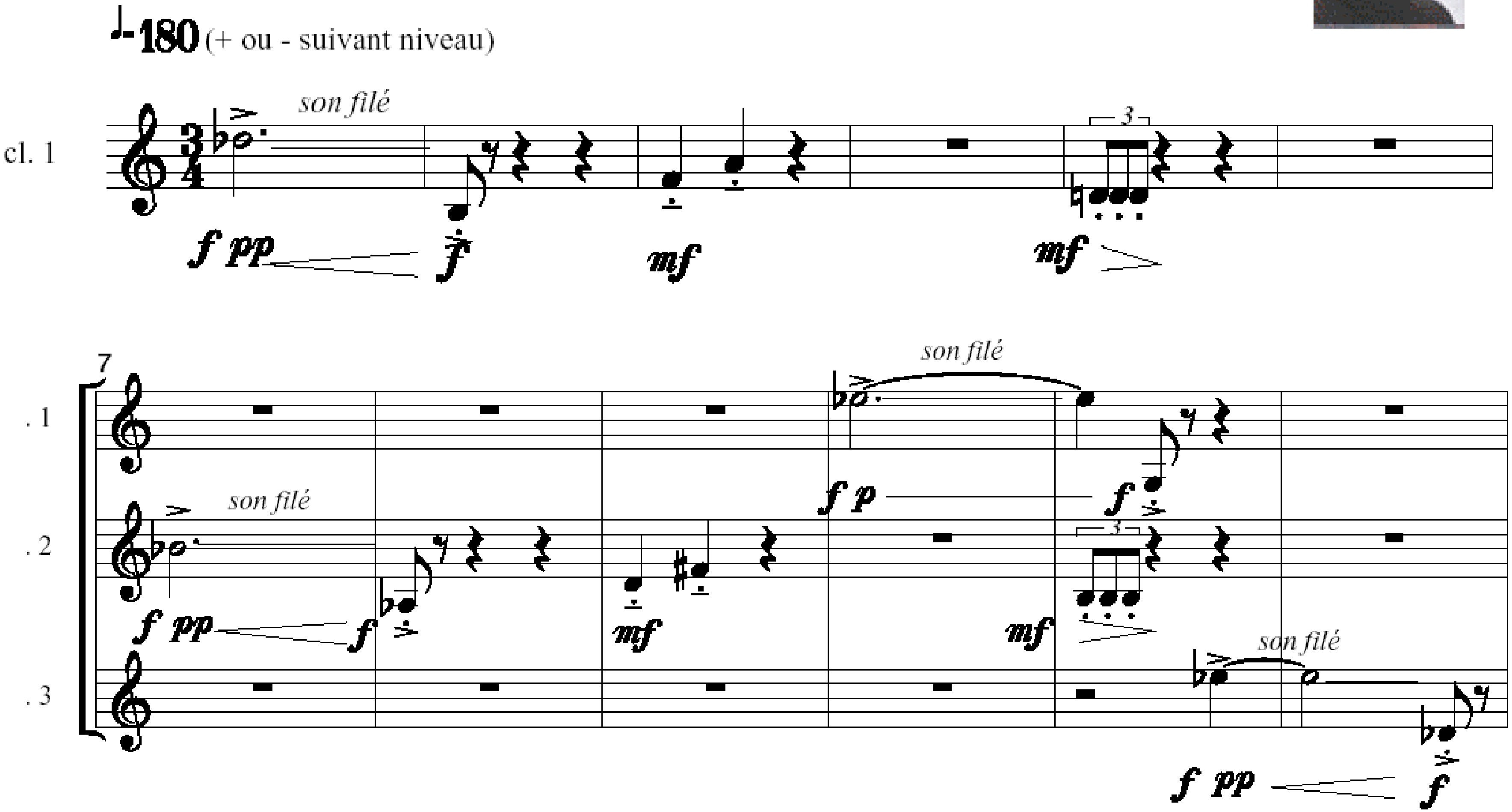
(1 3 3 6 11 4 9 6 5 1 3 20)

(14 8 10 8 14 18)

S

Fabien Levy

Strategie pedagogiche a partire da un modello formale



• *Où niche l'Hibou* (1999-2006)

Georges Bloch (2001-2004)

Prospettive teorico/compositive a partire dal modello formale

- Organizzazione metrica di un canone a mosaico
- Riduzione di un canone ritmico in *self-similar canons*
- Modulazione metrica fra canoni
- Trasformazione di un canone in *texture*

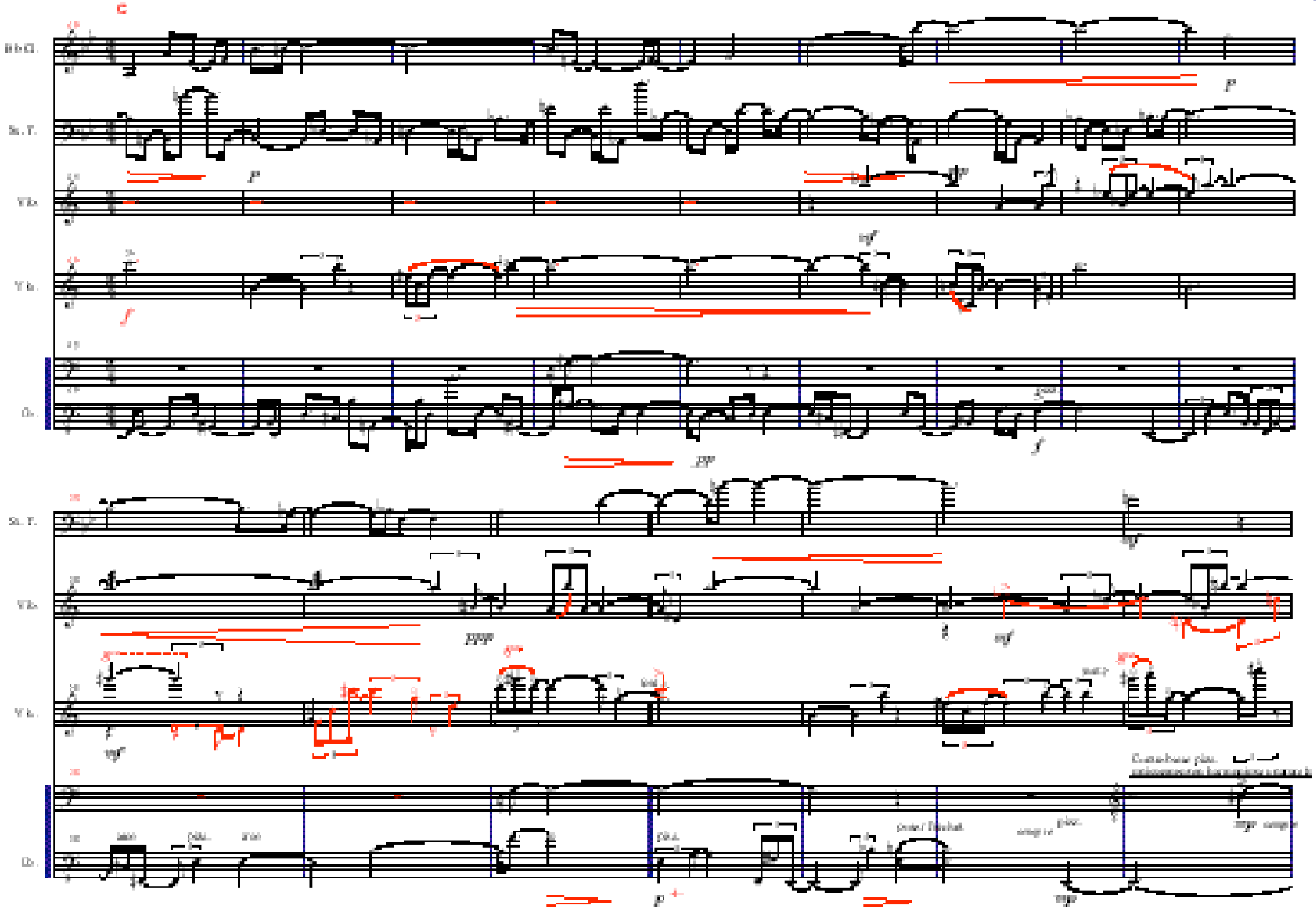
- ***Projet Beyeler*** (2001)
- *Projet Hitchcock*
- *Visite des tours de la cathédrale de Reims*
- *Noël des Chasseurs*
- *Canons à marcher*
- *Canon à eau*
- *Harawun* (2004)
- *L'Homme du champ* (2005)

- *A piece based on Monk* (2007)

Harawun: Utilizzazione di un canone di Vuza sulla griglia di *Harawi* di Messiaen

Georges Bloch:

Projet Beyeler (2001)



Canon2

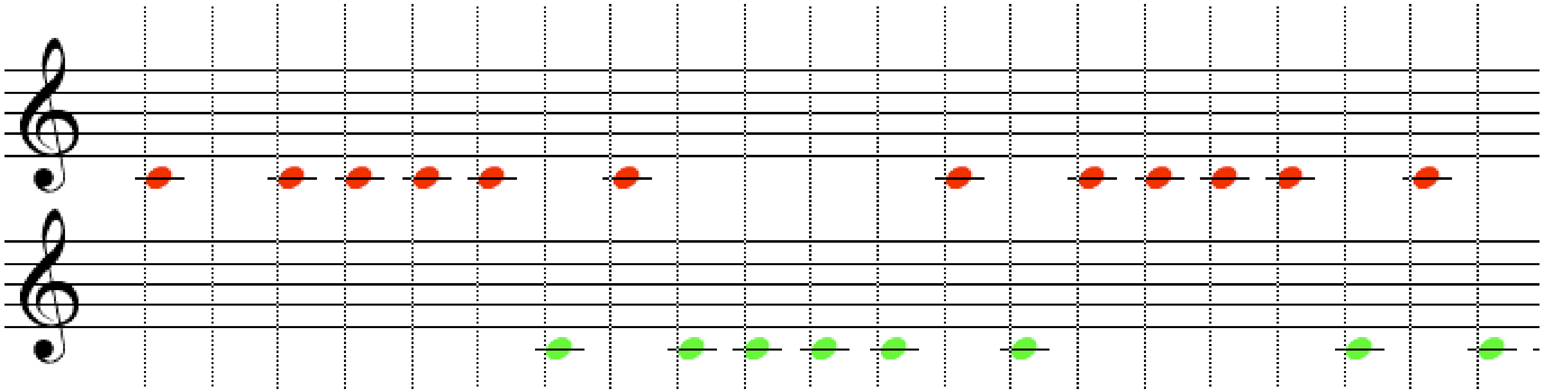
Canon2p

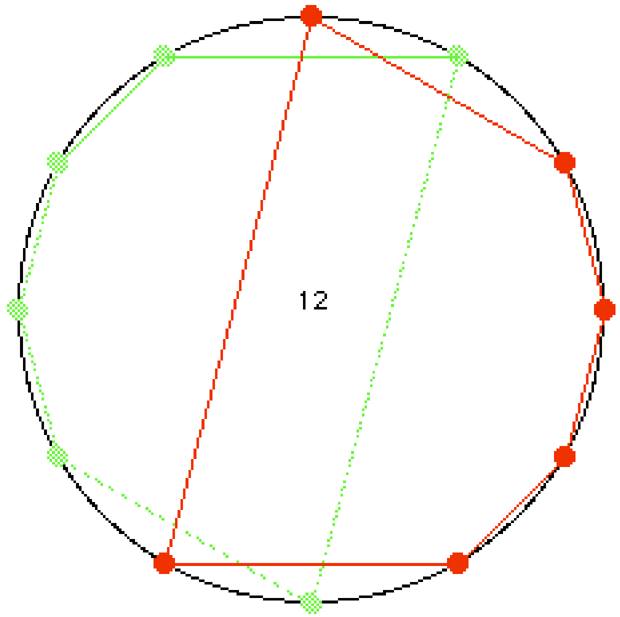
Canon final

Canon Final: da una struttura canonicaad una *texture*

Prime generalizzazioni: *Augmented Tiling Canons*

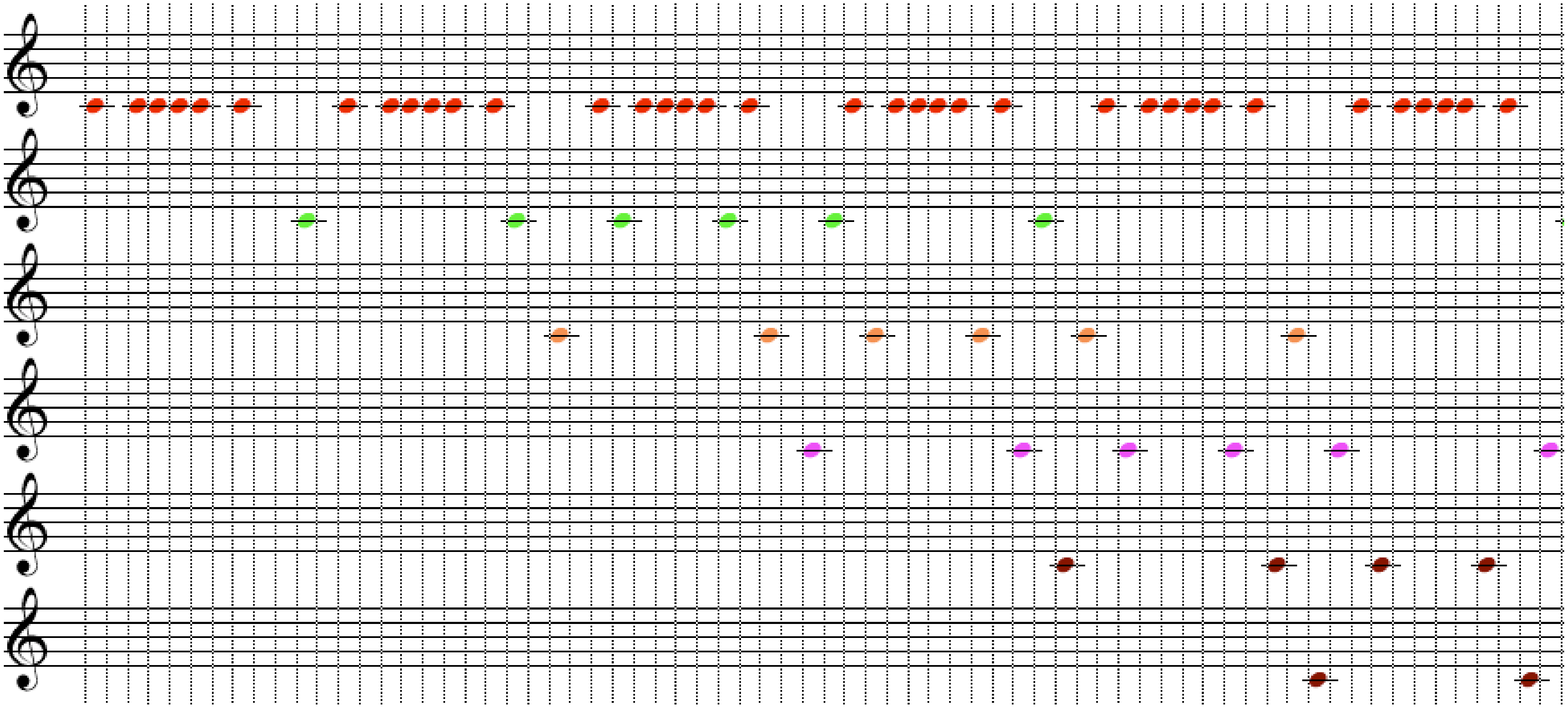
(in collaborazione con Thomas Noll)





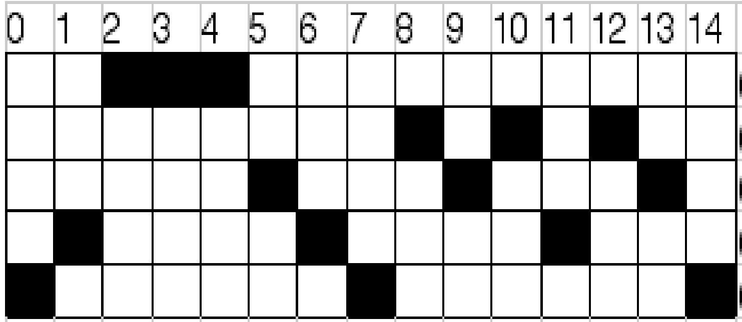

```
(((0 1 2 3 4 6) ((1 11)))
((0 1 2 3 4 5) ((1 11) (1 1)))
((0 1 2 3 5 7) ((1 11) (1 7)))
((0 1 3 4 7 8) ((1 5)))
((0 1 2 3 6 7) ((1 11)))
((0 1 3 4 6 9) ((1 11) (1 5)))
((0 1 3 6 7 9) ((1 11) (1 5)))
((0 1 2 6 7 8) ((1 11) (1 7) (1 5) (1 1)))
((0 1 4 5 8 9) ((1 11) (1 7) (1 5) (1 1)))
((0 1 2 5 6 7) ((1 7) (1 5)))
((0 2 3 4 5 7) ((1 11) (1 7) (1 5) (1 1)))
((0 1 4 5 6 8) ((1 11) (1 7)))
((0 1 2 4 5 7) ((1 5)))
((0 1 3 4 5 8) ((1 5) (1 1)))
((0 1 2 4 5 8) ((1 11)))
((0 1 2 4 6 8) ((1 11) (1 7)))
((0 2 3 4 6 8) ((1 11)))
((0 2 4 6 8 10) ((1 11) (1 7) (1 5) (1 1))))
```

Augmented Tiling Canons o l'azione del gruppo affine



• T. Noll, M. Andreatta, C. Agon, G. Assayag: « The Geometrical Groove: rhythmic canons between Theory, Implementation and Musical Experiment », *JIM*, Bourges, 2001, pp. 93-98.

Tom Johnson:
Perfect Tiling Canons



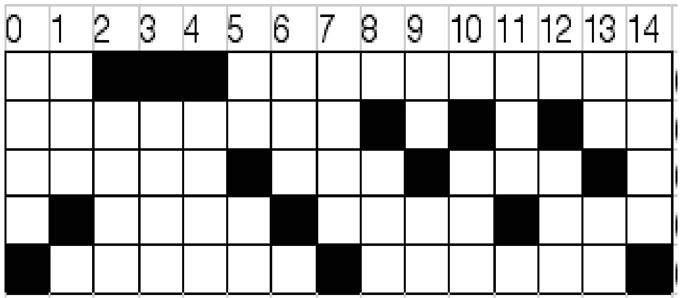
Tilework for Piano

perfect triplet tilings, 5th order

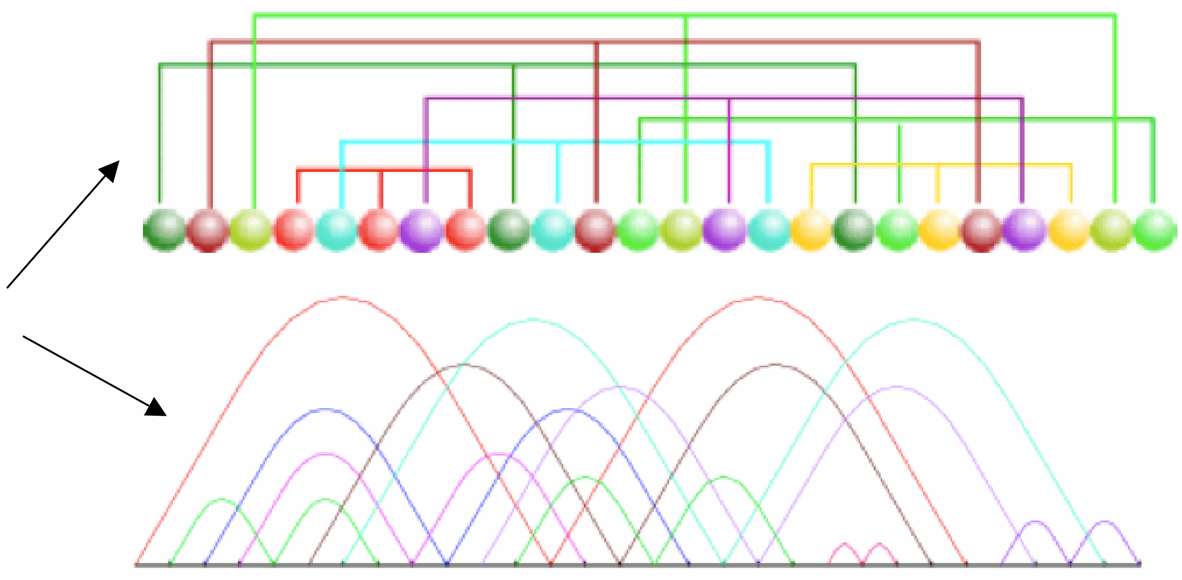
with thanks to Jon Wild and Erich Neuwirth

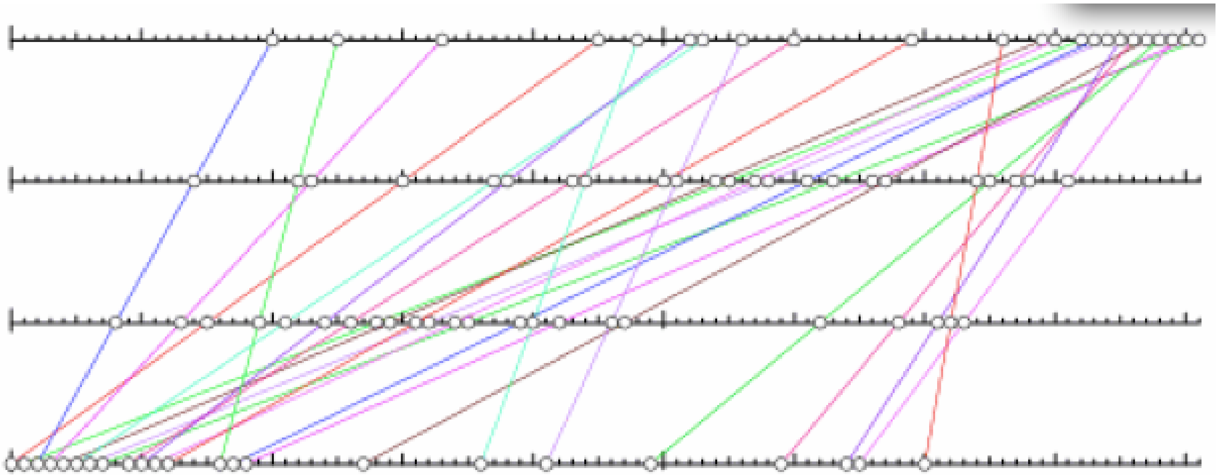
Perfect Tilings e rappresentazioni geometriche

(Jean-Paul Davalan)



Canone perfetto d'ordine 3





Canone perfetto d'ordine 4

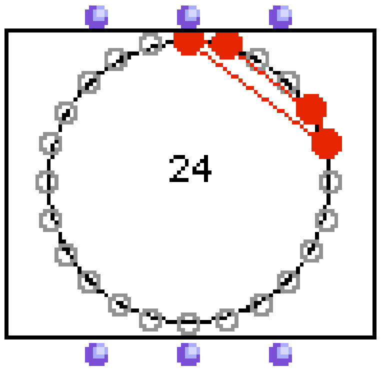
Problema aperto

Esiste un canone perfetto d'ordine 5?

Canoni a mosaico e polinomi

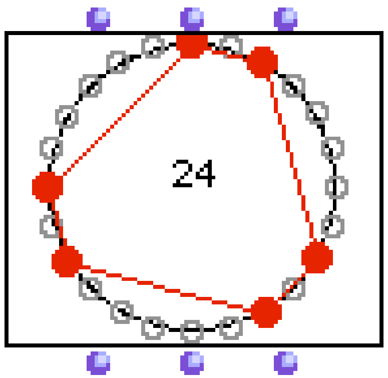
$$A \oplus B = \mathbb{Z}/n\mathbb{Z} \longrightarrow A(x) = \sum_{k \in A} x^k$$

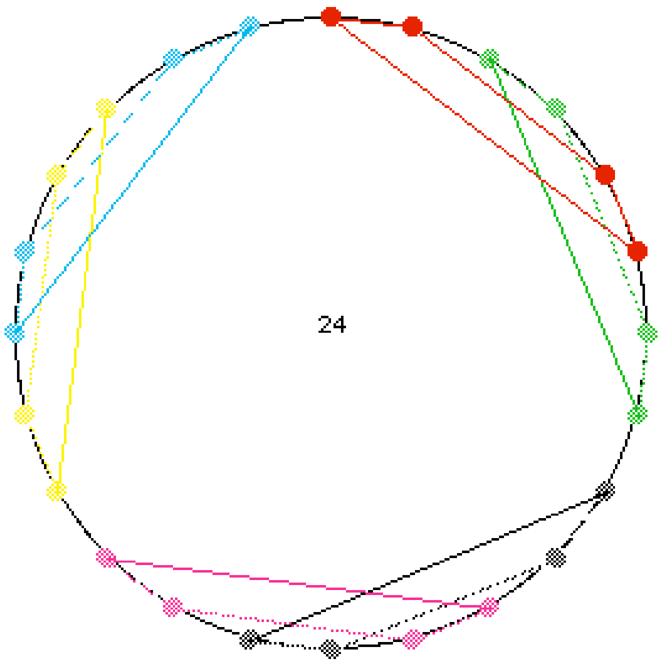
$$A(x) \times B(x) = (A \oplus B)(x) \equiv 1 + x + \dots x^{n-1} \pmod{X^n - 1}$$



$A(X) = 1+X+X^4+X^5$

$B(X) = 1+X^2+X^8+X^{10}+X^{16}+X^{18}$





Buone e cattive fattorizzazioni

$$\Delta_n = 1 + X + X^2 + \ldots + X^{n-1} = \prod_{\substack{d \mid n \\ d \neq 1}} \Phi_d(X)$$

$$\Phi_2(X) = 1 + X$$
$$\Phi_3(X) = 1 + X + X^2$$
$$\Phi_4(X) = 1 + X^2$$
$$\Phi_6(X) = 1 - X + X^2$$

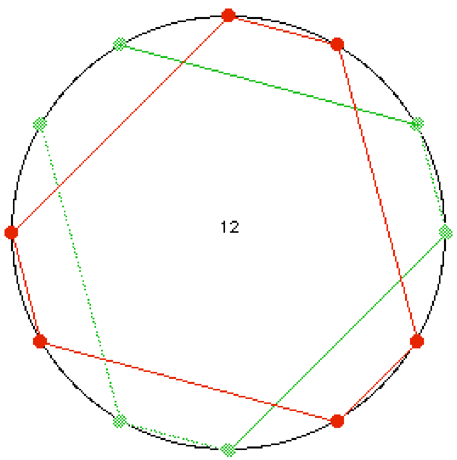
$$\Delta_{12} = 1 + X + \ldots + X^{11} = \Phi_2 \times \Phi_3 \times \Phi_4 \times \Phi_6 \times \Phi_{12}$$

$$A(X) = \Phi_2 \times \Phi_3 \times \Phi_6 \times \Phi_{12} = 1 + X + X^4 + X^5 + X^8 + X^9$$

$$B(X) = \Phi_4 = 1 + X^2$$

$$S = \{0, 2\}$$

$$R = \{0, 1, 4, 5, 8, 9\}$$



$$A^*(X) = \Phi_2 \times \Phi_3 \times \Phi_{12}$$

$$B^*(X) = \Phi_4 \times \Phi_6$$

Questa decomposizione non funziona!

Le condizioni di Coven-Meyerowitz

• E. Coven & A. Meyerowitz : "Tiling the integers with translates of one finite set", *J. Algebra*, 212, pp.161-174, 1999

There is no loss of generality in restricting attention to translates of a finite set A of *nonnegative* integers. Then $A(x) = \sum_{a \in A} x^a$ is a polynomial such that $\#A = A(1)$. Let S_A be the set of prime powers s such that the s-th cyclotomic polynomial $\Phi_s(x)$ divides $A(x)$. Consider the following conditions on $A(x)$.

(T1) $A(1) = \prod_{s \in S_A} \Phi_s(1)$.

(T2) If $s_1, \ldots, s_m \in S_A$ are powers of distinct primes, then $\Phi_{s_1 \cdots s_m}(x)$ divides $A(x)$.

Theorem A. *If $A(x)$ satisfies (T1) and (T2), then A tiles the integers.*

Theorem B1. *If A tiles the integers, then $A(x)$ satisfies (T1).*

Theorem B2. *If A tiles the integers and $\#A$ has at most two prime factors, then $A(x)$ satisfies (T2).*

Corollary. *If $\#A$ has at most two prime factors, then A tiles the integers if and only if $A(x)$ satisfies (T1) and (T2).*

Le condizioni di CM e canoni a mosaico

(T1) $A(1) = \prod_{s \in S_A} \Phi_s(1)$.
(T2) If $s_1, \ldots, s_m \in S_A$ are powers of distinct primes, then $\Phi_{s_1 \cdots s_m}(x)$ divides $A(x)$.

Theorem A. *If $A(x)$ satisfies (T1) and (T2), then A tiles the integers.*

$$A(X) = \Phi_2 \times \Phi_3 \times \Phi_6 \times \Phi_{12} = 1 + X + X^4 + X^5 + X^8 + X^9$$

$$\Phi_2(X) = 1 + X$$

$$\Phi_3(X) = 1 + X + X^2$$

(T1) $A(1) = 6 = \Phi_2(1) \times \Phi_3(1) = 2 \times 3$

(T2) $\Phi_2 \mid A(X)$ et $\Phi_3 \mid A(X) \Rightarrow \Phi_{2\times 3} \mid A(X)$

Theorem B1. *If A tiles the integers, then $A(x)$ satisfies (T1).*

$$A^*(X) = \Phi_2 \times \Phi_3 \times \Phi_{12} = 1 + 2X + 2X^2 - X^3 - X^4 + X^5 + 2X^6 + X^7$$

$$A^*(1) = 7 \neq \Phi_2 \times (1) \times \Phi_3(1) = 6$$

WOLFRAM RESEARCH

mathworld.wolfram.com

Fuglede's Conjecture

CONTRIBUTE TO THIS ENTRY

Portions of this entry contributed by Emmanuel Amiot

Fuglede (1974) conjectured that a domain Ω admits an operator spectrum iff it is possible to tile $\mathbb{R}^d$ by a family of translates of Ω. Fuglede proved the conjecture in the special case that the tiling set or the spectrum are lattice subsets of $\mathbb{R}^d$ and Iosevich *et al.* (1999) proved that no smooth symmetric convex body Ω with at least one point of nonvanishing Gaussian curvature can admit an orthogonal basis of exponentials.

Using complex Hadamard matrices of orders 6 and 12, Tao (2003) constructed counterexamples to the conjecture in some small Abelian groups, and lifted these to counterexamples in $\mathbb{R}^5$ or $\mathbb{R}^{11}$.

However, the conjecture has been proved in a great number of special cases (e.g., all convex bodies) and remains an open problem in small dimensions. For example, it has been shown in dimension 1 that a nice algebraic characterization of finite sets tiling $\mathbb{Z}$ indeed implies one side of Fuglede's conjecture (Coven-Meyerowitz 1998). Furthermore, it is sufficient to prove these conditions when the tiling gives a factorization of a non-Hajós cyclic group (Amiot).

Canoni a mosaico, condizioni di CM e congettura spettrale

• E. Coven & A. Meyerowitz : “Tiling the integers with translates of one finite set”, *J. Algebra*, 212, pp.161-174, 1999

T1 + T2 => mosaico

Mosaico => T1

• I. Laba : “The spectral set conjecture and multiplicative properties of roots of polynomials”, *J. London Math society*, 65, pp. 661-671, 2002

T1 + T2 => spettrale

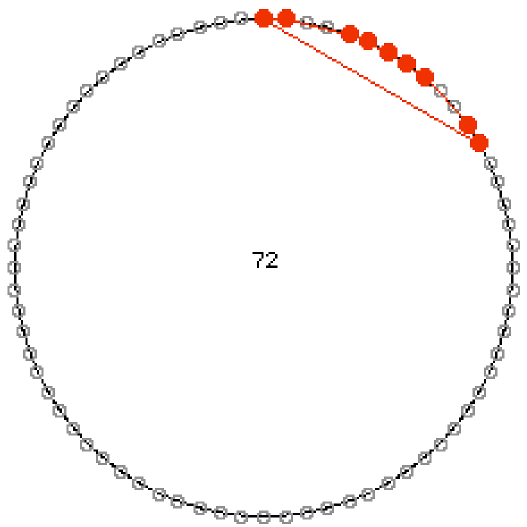
T2 => spettrale

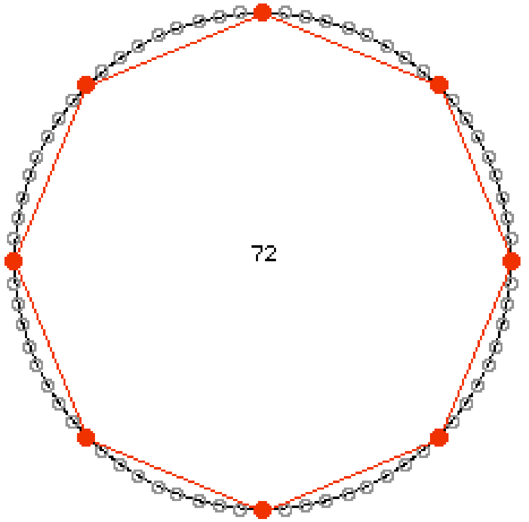
spettrale => T1

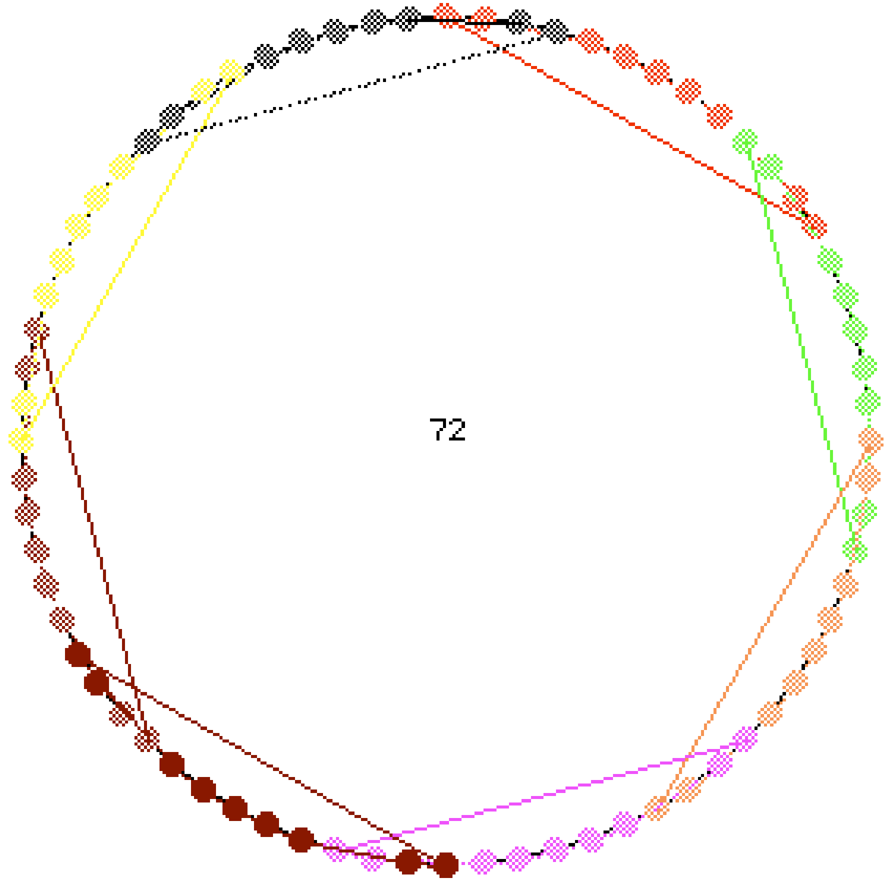
• E. Amiot : “A propos des canons rythmiques”, *Gazette des Mathématiciens,* n°106, Octobre 2005

Se A realizza un mosaico ma A non è spettrale => A è il pattern ritmico di un canone di Vuza

La classe dei “canons cyclotomiques”







- E. Amiot, M. Andreatta, C. Agon: « Tiling the (musical) line with polynomial: some theoretical and implementational aspects », *ICMC*, Barcelona, 2005, pp. 227-230.

Orizzonte filosofico di un approccio strutturale in musica

*G.-G. Granger e la dualità dell'*oggettuale *e dell'*operatorio

- « Pygmalion. Réflexions sur la pensée formelle », 1947
- *Pour la connaissance philosophique*, 1988
- *Formes, opérations, objets*, 1994

« [C'est la notion de groupe qui] donne un sens précis à l'idée de structure d'un ensemble [et] permet de déterminer les éléments efficaces des transformations en réduisant en quelque sorte à son schéma opératoire le domaine envisagé. [...]L'objet véritable de la science est le ***système des relations*** *et non pas les termes supposés qu'il relie. [...] Intégrer les résultats - symbolisés - d'une* ***expérience*** *nouvelle revient [...] à créer un canevas nouveau, un* ***groupe de transformations*** *plus complexe et plus compréhensif* »

G.-G. Granger : « Pygmalion. Réflexions sur la pensée formelle », 1947

Orizzonte filosofico di un approccio strutturale in musica

J. Piaget: teoria dei gruppi e delle categorie nell'epistemologia genetica

- *Le structuralisme*, 1968
- *Morphismes et Catégories. Comparer et transformer*
(avec G. Henriques, E. Ascher 1990)

*« ...**attitude relationnelle**, selon laquelle ce qui compte [sont] les relations entre les éléments, autrement dit les procédés ou processus de composition [...] La structure [de **groupe**] se referme sur elle-même, mais cette fermeture ne signifie en rien que la structure considérée ne peut pas entrer à titre de sous-structure dans une structure plus large »*

*« De même qu'en mathématique le structuralisme des Bourbaki est déjà doublé par un mouvement faisant appel à des **structures plus dynamiques** (les « catégories » [...]) de même toutes les formes actuelles du structuralisme [...] sont certainement grosses de développements multiples... »*

J. Piaget: *Le structuralisme*

Ramificazioni percettive e cognitive dei metodi algebrici in musica

La questione della **somiglianza percettiva** fra le diverse trasposizioni di uno stesso profilo melodico è legata *ad un problema molto più generale, un problema che interessa la matematica astratta.*

E. Cassirer : « The concept of group and the theory of perception », 1944

*Il carattere singolare dell'esperienza musicale è dovuto in parte alle strutture particolari di **gruppo** che la musica rende accessibile all'ascoltatore.*

G. Balzano : « The group-theoretic description of 12-fold and microtonal pitch systems », 1980

*Group Theory has emerged as a powerful tool for analyzing **cognitive structure**. The number of cognitive disciplines using group theory is now enormous. The power of group theory lies in its ability to identify organization, and to express organization in terms of generative actions that structure a space.*

Michael Leyton, The International Society for Group Theory in Cognitive Science