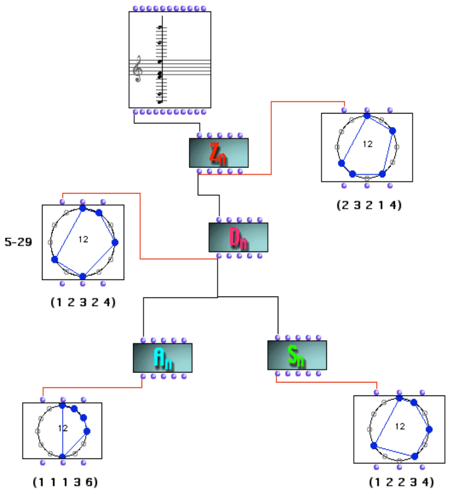
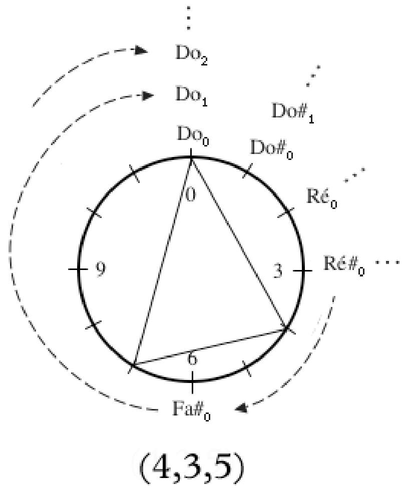



Elementi di Geometria Superiore 2




# *Matematica & Musica*

## Secondo trittico:
## tassellazioni e modello computazionale di *Nomos Alpha* di I. Xenakis


Moreno Andreatta
Equipe Représentations Musicales
IRCAM/CNRS
(In collaborazione con Carlos Agon e Emmanuel Amiot)

# Programma del corso

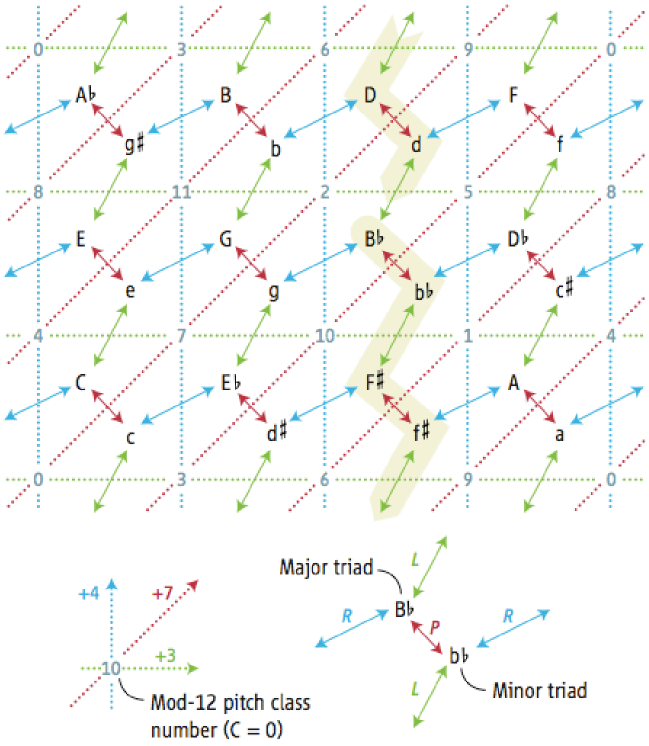
1.) Rappresentazione e formalizzazione delle strutture musicali
2.) Enumerazione e classificazione delle strutture musicali
3.) Teorie trasformazionali, diatoniche e neo-riemanniane
**4.) Tassellazioni musicali: la costruzione dei canoni ritmici a mosaico**
5.) Sequenze periodiche e calcolo delle differenze finite a valori in gruppi ciclici
6.) Ramificazioni filosofiche e cognitive dell'approccio algebrico in musica

# Alcune precisioni e complementi

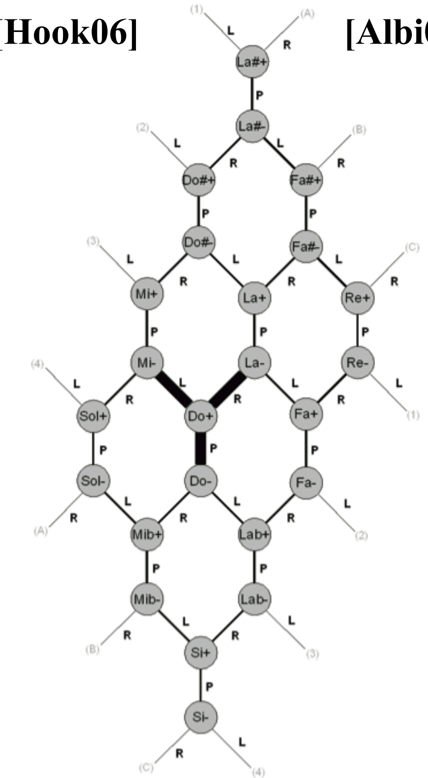
1.) Le due caratterizzazioni del gruppo diedrale $\mathbf{D_{24}}$
2.) Reticoli di Klumpenhouwer, isografie e gruppi di automorfismi
3.) Rappresentazioni topologiche di serie dodecafoniche attraverso la teoria dei nodi (applicazione alla musica di Elliott Carter)
4.) Teorema dell’esacordo e trasformata di Fourier discreta.

# Il *Tonnetz* di Oettingen/Riemann

[Hook06]

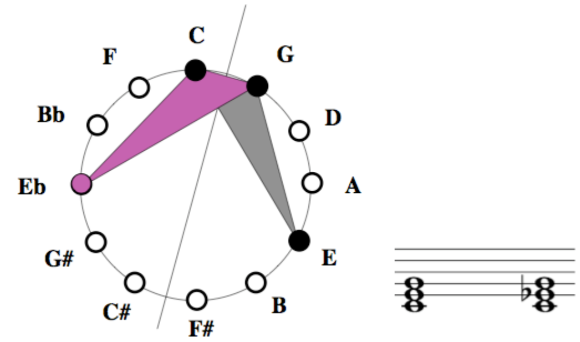



[Albi08]

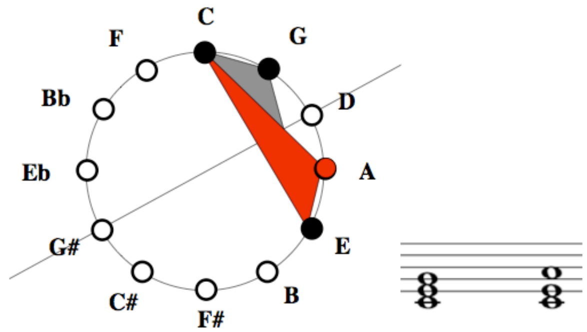


[Noll04]

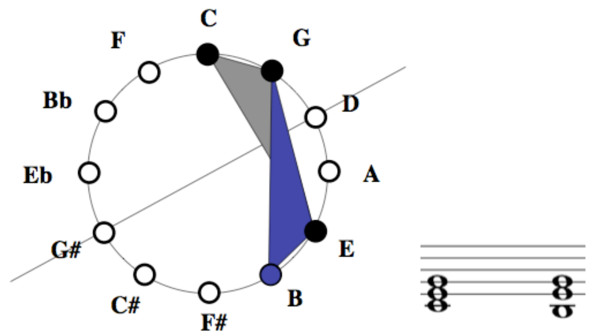
(Neo-)Riemannian Operation P = „Parallel“




(Neo-)Riemannian Operation R = „Relative“




(Neo-)Riemannian Operation L = „Leading-Tone“



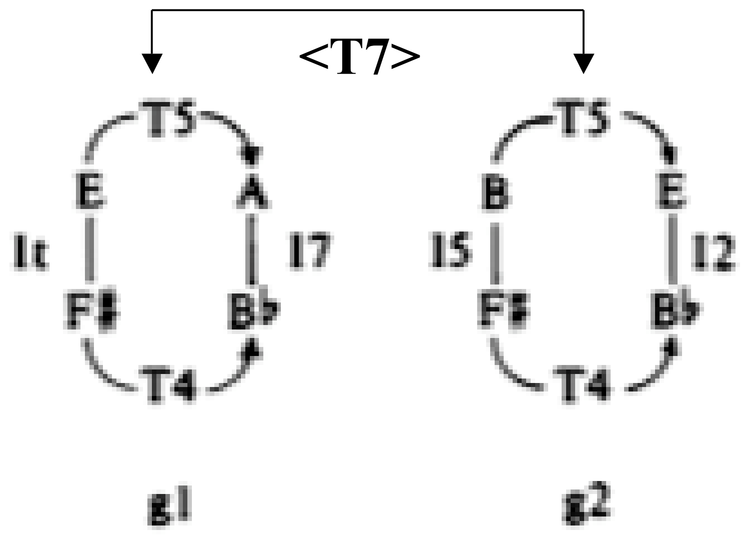

$$LPR = \langle L, R \mid L^2 = (LR)^{12} = 1 \ ; \ LRL = L(LR)^{-1} \rangle$$

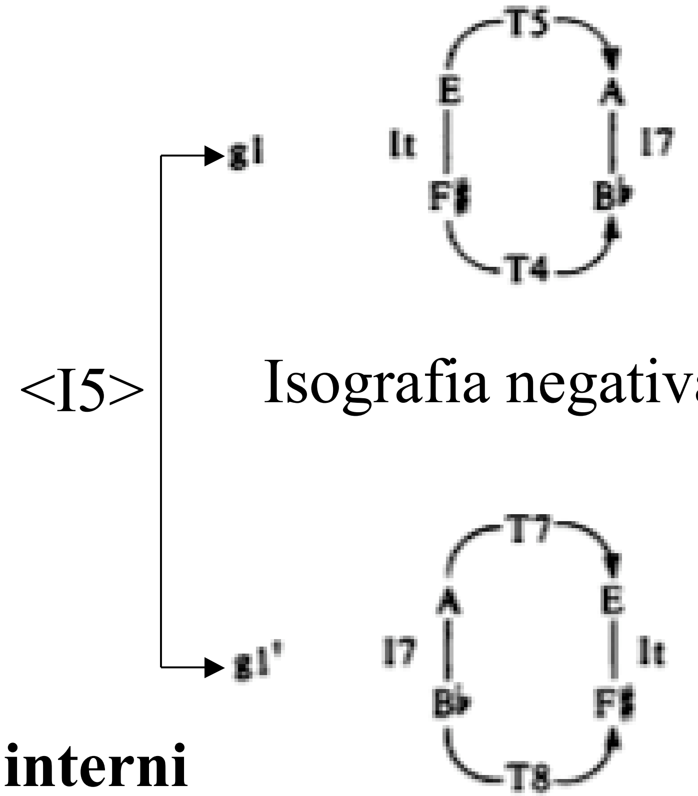
- LPR e $T_nI$ sono uno il *centralizzatore* dell'altro
- LPR e $T_nI$ condividono il sottogruppo abeliano massimale $\langle T_n \rangle = Z_n$
- LPR et $T_nI$ agiscono in maniera semplicemente transitiva sull'insieme delle 24 triadi consonanti

[Crans, Fiore & Satyendra, 2008]

## *Klumpenhouwer Networks* (K-nets), isografie e gruppi di automorfismi

### Isografia positiva




Isografia negativa

### Automorfismi esterni

$<T_k> : T_m \rightarrow T_m$
$\quad I_m \rightarrow I_{k+m}$

$<I_k> : T_m \rightarrow T_{-m}$
$\quad I_m \rightarrow I_{k-m}$

### Automorfismi interni

$[T_k] : T_m \rightarrow T_m$
$\quad I_m \rightarrow I_{2k+m}$

$[I_k] : T_m \rightarrow T_{-m}$
$\quad I_m \rightarrow I_{2k-m}$

# Teoria dei nodi e strutture musicali

MaMuX (11 mars 2006) : Théorie des noeuds et des tresses en mathématique et en musique.
http://recherche.ircam.fr/equipes/repmus/mamux/

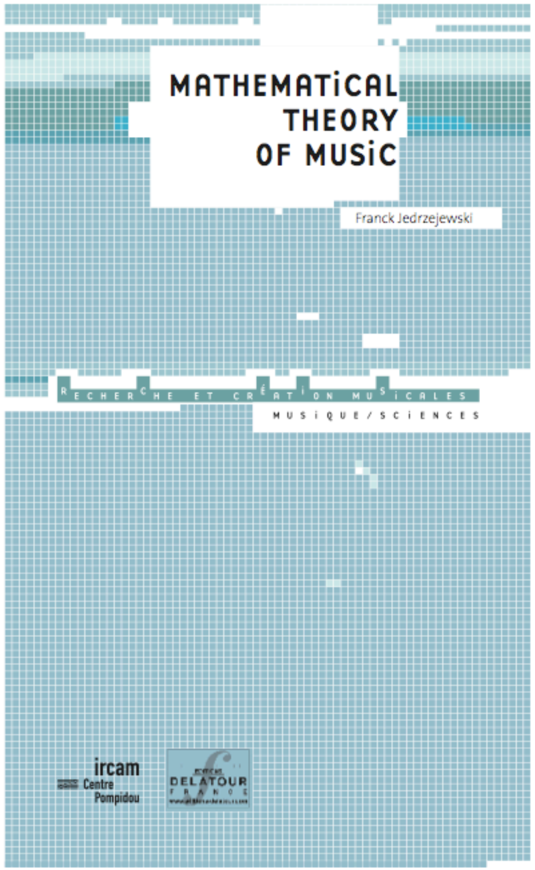
## Noeuds dodecaphoniques

**Comment construire un diagramme de Gauss pour une série de 12 sons ?**

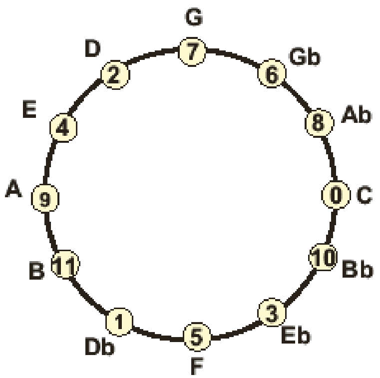
**1) Choisir une série :**

**C, Ab, Gb, G, D, E, A, B, Db, F, Eb, Bb**

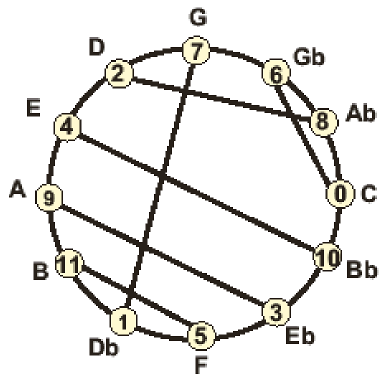
**0, 8, 6, 7, 2, 4, 9, 11, 1, 5, 3, 10**



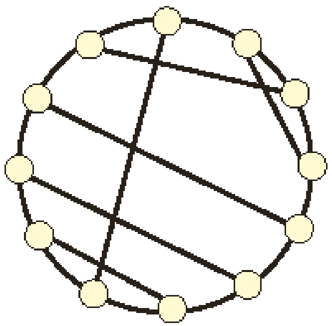

**2) Placer les notes sur un cercle**




**3) Joindre les tritons**




**4) Ne conserver que la structure**



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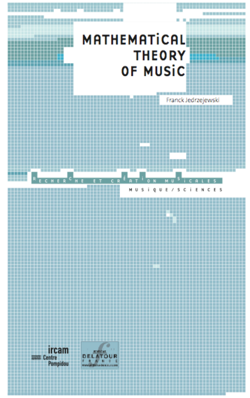
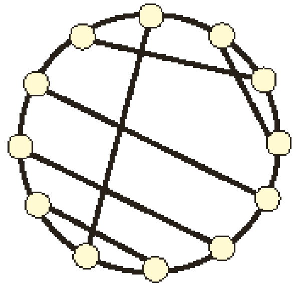


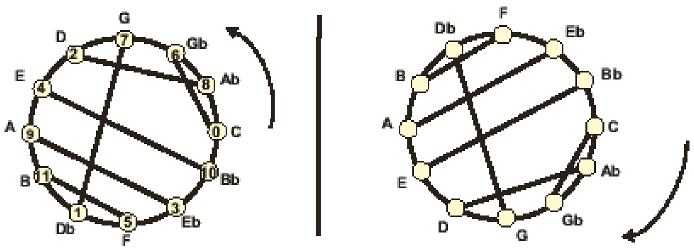

## Noeuds dodecaphoniques

**Un seul diagramme de Gauss représente les 48 formes dérivées de la série**



**1 - Transpositions : ont la même structure tritonique (Rotations)**

**2 - Retrogradation : symétrie miroir et rotation**




**3 - Renversement : même structure tritonique**


**4 - Retrogradation du renversement : Symétrie miroir**

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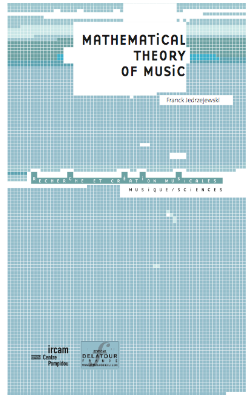
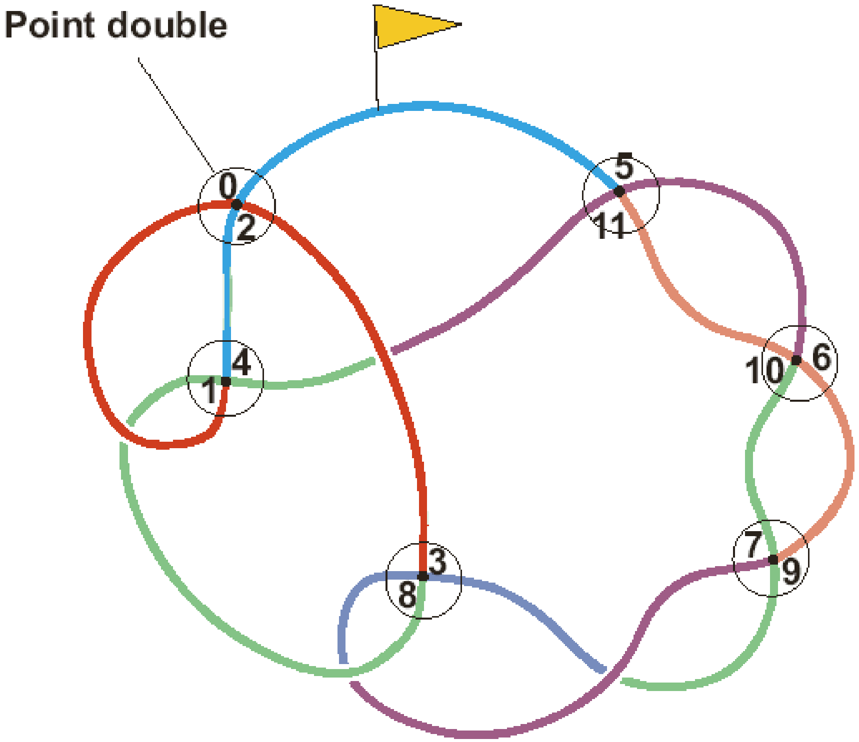
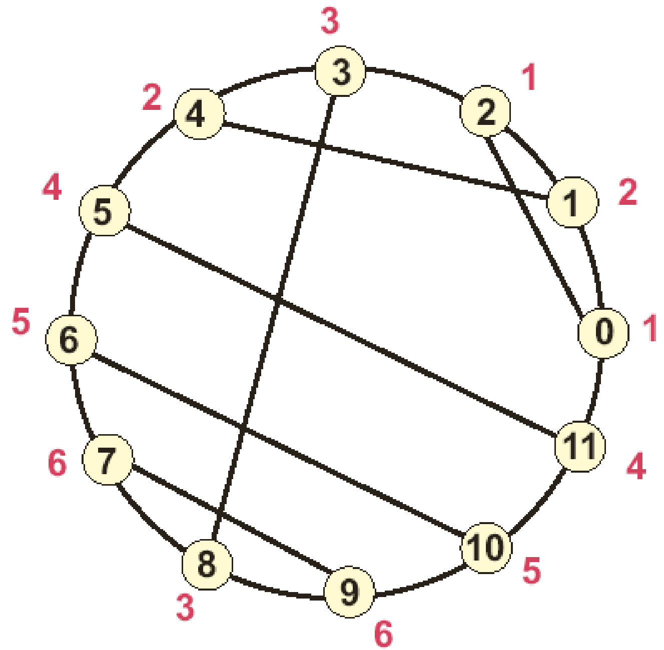



## Noeuds dodecaphoniques

**Un diagramme de Gauss représente un noeud de 6 points doubles**




**Noeud de 4 croisements et 6 points doubles**




**Mot de Gauss**

1, 2, 1, 3, 2, 4, 5, 6, 3, 6, 5, 4

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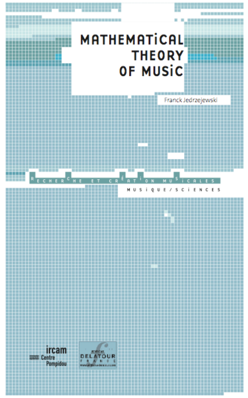
http://recherche.ircam.fr/equipes/repmus/mamux/




## Combinatoire des diagrammes de Gauss

**Peut-on calculer le nombre de diagramme de Gauss ?**

Il y a 12 ! series (environ 479 millions) en réalité seulement 9 985 920 séries.

Il existe 554 diagrammes de Gauss dans le tempérament égal

| $n$ | $c_n$ | $d_n$ | Temper. |
|---|---|---|---|
| 3 | 5 | 5 | 6-tet |
| 4 | 18 | 17 | 8-tet |
| 5 | 105 | 79 | 10-tet |
| 6 | 902 | **554** | **12-tet** |
| 7 | 9749 | 5283 | 14-tet |
| 8 | 127072 | 65346 | 16-tet |
| 9 | 1915951 | 966156 | 18-tet |
| 10 | 32743182 | 16411700 | 20-tet |
| 11 | 625002933 | 312702217 | 22-tet |

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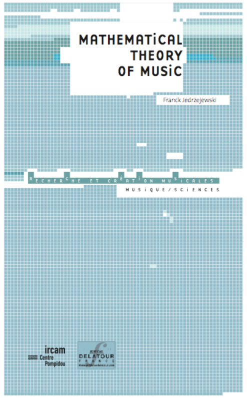
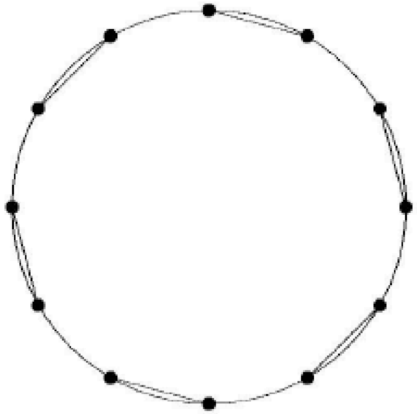


$D_1$ $\quad X = a^6$

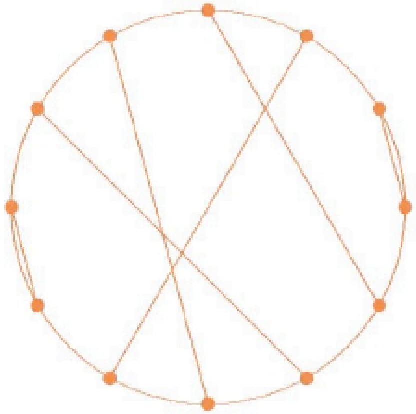
Mot de Gauss 112233445566

Vecteur Structural 600000

Permutation

(0 1) (2 3) (4 5) (6 7) (8 9) (10 11)

B.A. Zimmermann, Die Soldaten, Acte I



$D_{349}$ $\quad X = afd^{-1}e^2a$

Mot de Gauss 112345662453

Vecteur Structural 200121

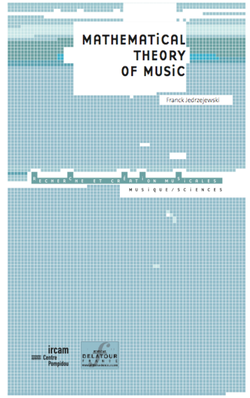
Permutation

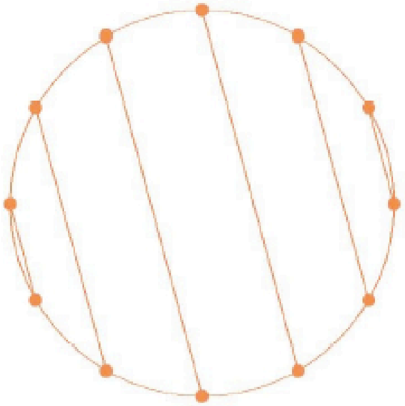
(0 1) (2 8) (3 11) (4 9) (5 10) (6 7)

Karel Goeyvaerts, Sonate pour deux pianos.

# Teoria dei nodi e strutture musicali

MaMuX (11 mars 2006) : Théorie des noeuds et des tresses en mathématique et en musique.
http://recherche.ircam.fr/equipes/repmus/mamux/



$D_{358}$ X $= ac^{-1}e^{-1}eca$

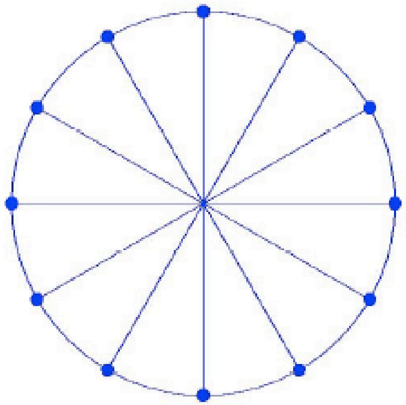
Mot de Gauss 112345665432

Vecteur Structural 202020

Permutation

(0 1) (2 11) (3 10) (4 9) (5 8) (6 7)

Anton Webern, Symphonie de chambre, opus 21



$D_{554}$ X $= f^6$

Mot de Gauss 123456123456

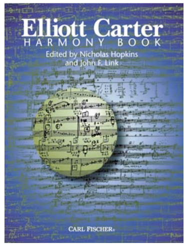
Vecteur Structural 000006

Permutation

(0 6) (1 7) (2 8) (3 9) (4 10) (5 11)

B.A. Zimmerman, Interludes (Die Soldaten)

# Elliott Carter : 90+ (1994)

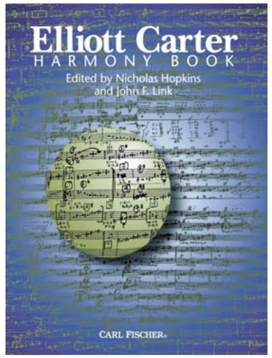


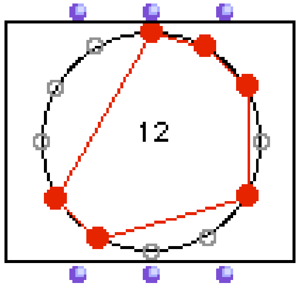

- **Combinatoire d'accords**
  - **Hexacordes**
  - **Tetracordes**
  - **Triades**
- **Séries tous–intervalles**
  - ***Link–chords***
- **Polyrythmie et modulations métriques**

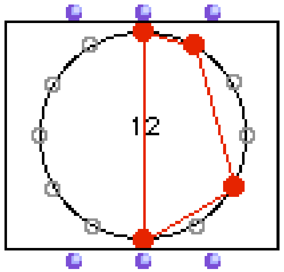
**(piano: John Snijders)**

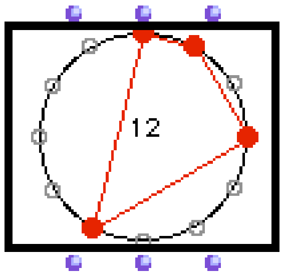
# Elliott Carter: 90+ (1994)

*« From about 1990, I have reduced my vocabulary of chords more and more to the six note chord n° 35 and the four note chords n° 18 and 23, which encompass all the intervals »* (*Harmony Book*, 2002, p. ix)



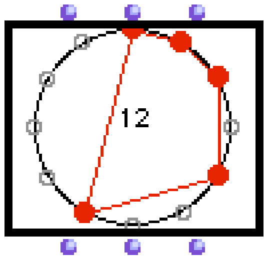




| | *pcset* | *vector* | *name* |
|---|---|---|---|
| 35 | (0 1 2 4 7 8) | [322332] | 6-Z17 |
| 18 | (0 1 4 6) | [111111] | 4-Z15 |
| 23 | (0 1 3 7) | [111111] | 4-Z29 |

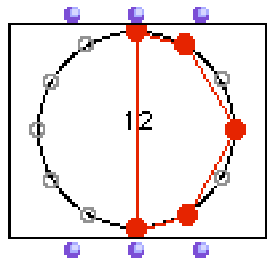
*All-triad hexachord*

# Vettore intervallare e relazione Z




**Due accordi *A* e *B* sono in relazione Z se hanno lo stesso contenuto intervallare, ovvero se i moduli delle trasformate di Fourier rispettive sono uguali.**

| | | |
|---|---|---|
| 5-Z36 | 0,1,2,4,7 | 222121 |

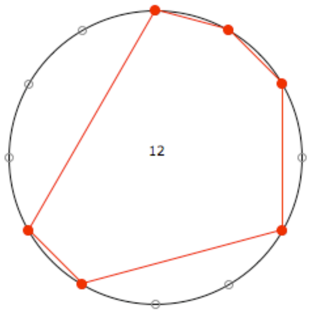



5-Z12

# Elliott Carter: 90+ (1994)

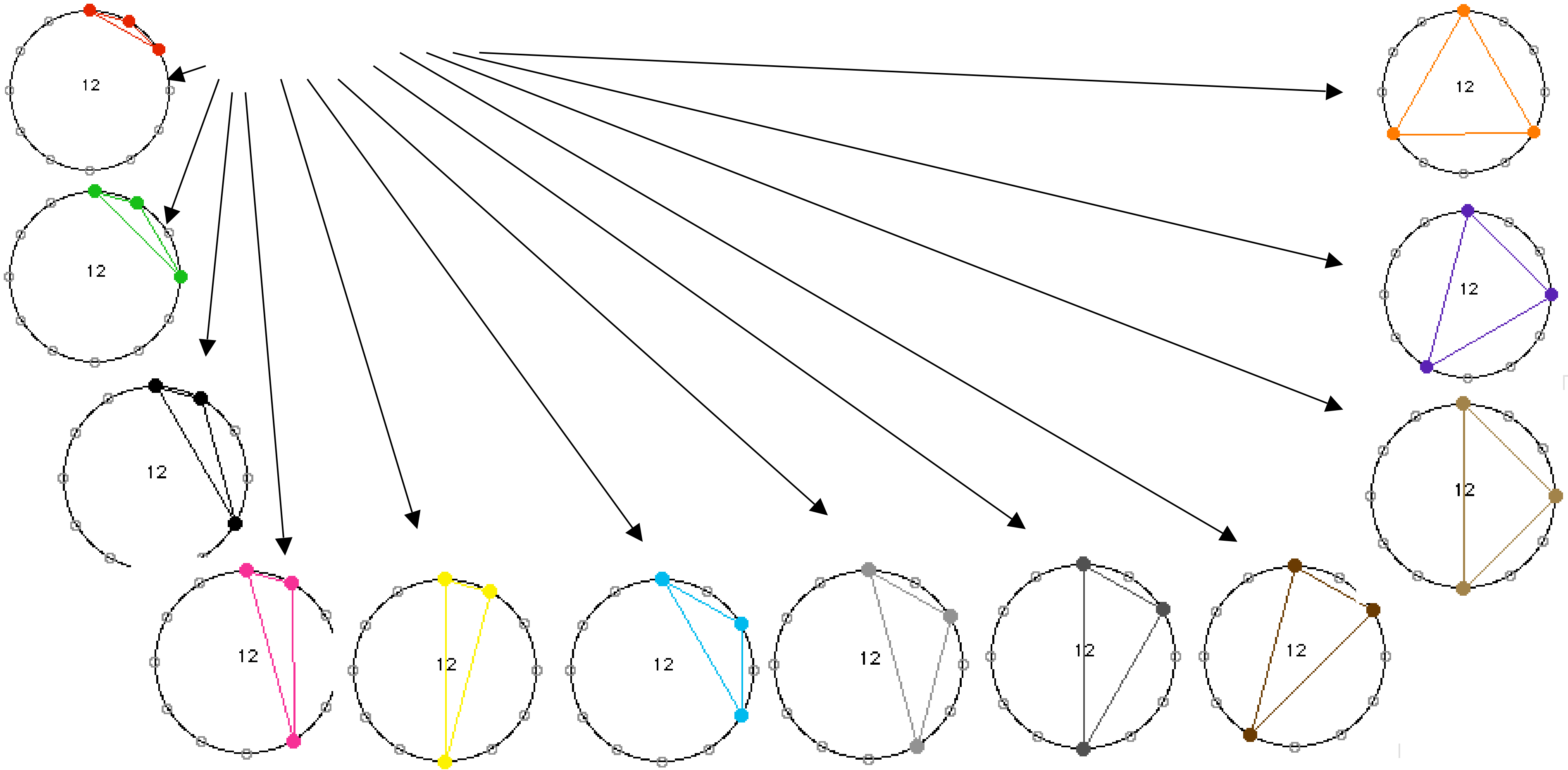
| $G \backslash k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_{12}$ | 1 | 6 | 19 | 43 | 66 | 80 | 66 | 43 | 19 | 6 | 1 | 1 |
| $D_{12}$ | 1 | 6 | 12 | 29 | 38 | 50 | 38 | 29 | 12 | 6 | 1 | 1 |
| $\mathrm{Aff}_1(Z_{12})$ | 1 | 5 | 9 | 21 | 25 | 34 | 25 | 21 | 9 | 5 | 1 | 1 |

35



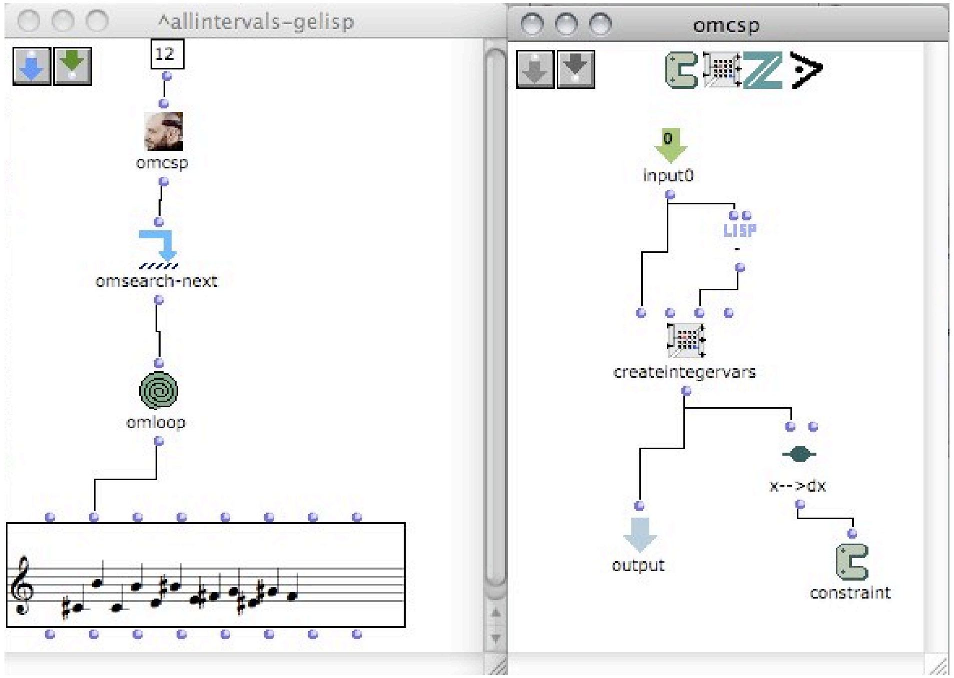

| *pcset* | *vector* | *name* |
|---|---|---|
| (0 1 2 4 7 8) | [322332] | 6-Z17 |

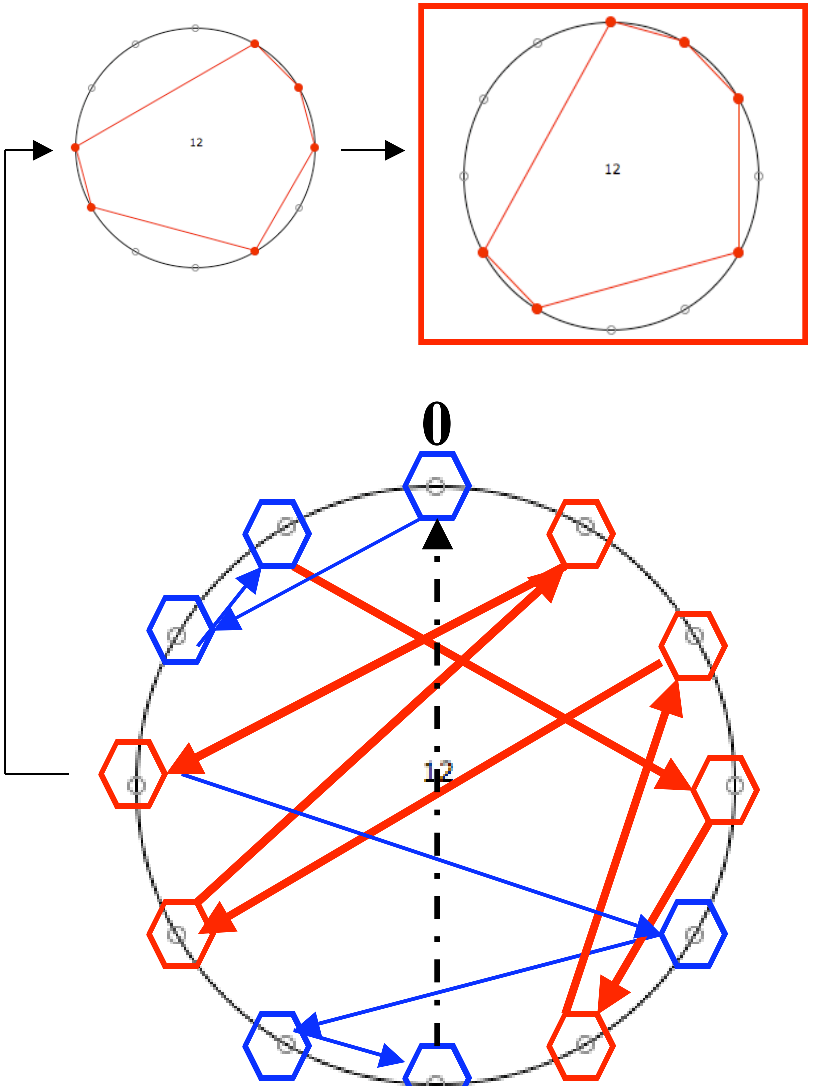
*All-triad hexachord*



# Elliott Carter : 90+ (1994) : Link–chords (mm. 49–68)

OM-> ((**0 10 11 3 5 2 8 1 9 4 7 6)** (1 2 3 5 8 9))
OM-> ((0 10 11 1 5 2 9 3 8 4 7 6) (1 2 3 5 8 9))
OM-> ((0 10 3 5 2 8 9 1 4 11 7 6) (1 2 3 5 8 9))
OM-> ((0 9 4 8 2 3 5 10 1 11 7 6) (0 2 3 4 8 9))
OM-> ((0 9 4 2 3 8 10 1 5 11 7 6) (0 2 3 4 8 9))
OM-> ((0 9 3 11 4 5 7 10 2 1 8 6) (3 4 5 7 10 11))
OM-> ((0 9 1 4 2 8 3 5 10 11 7 6) (3 5 6 7 10 11))
…

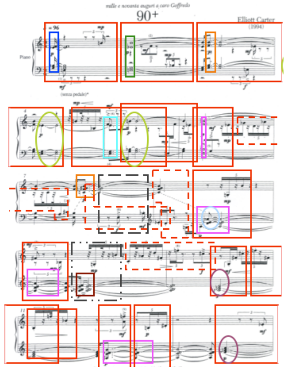


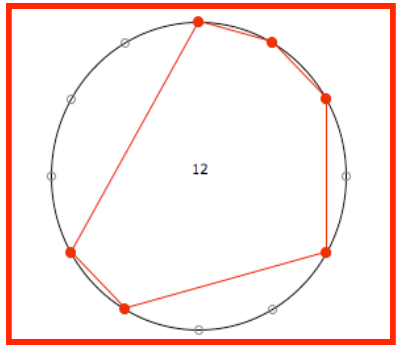



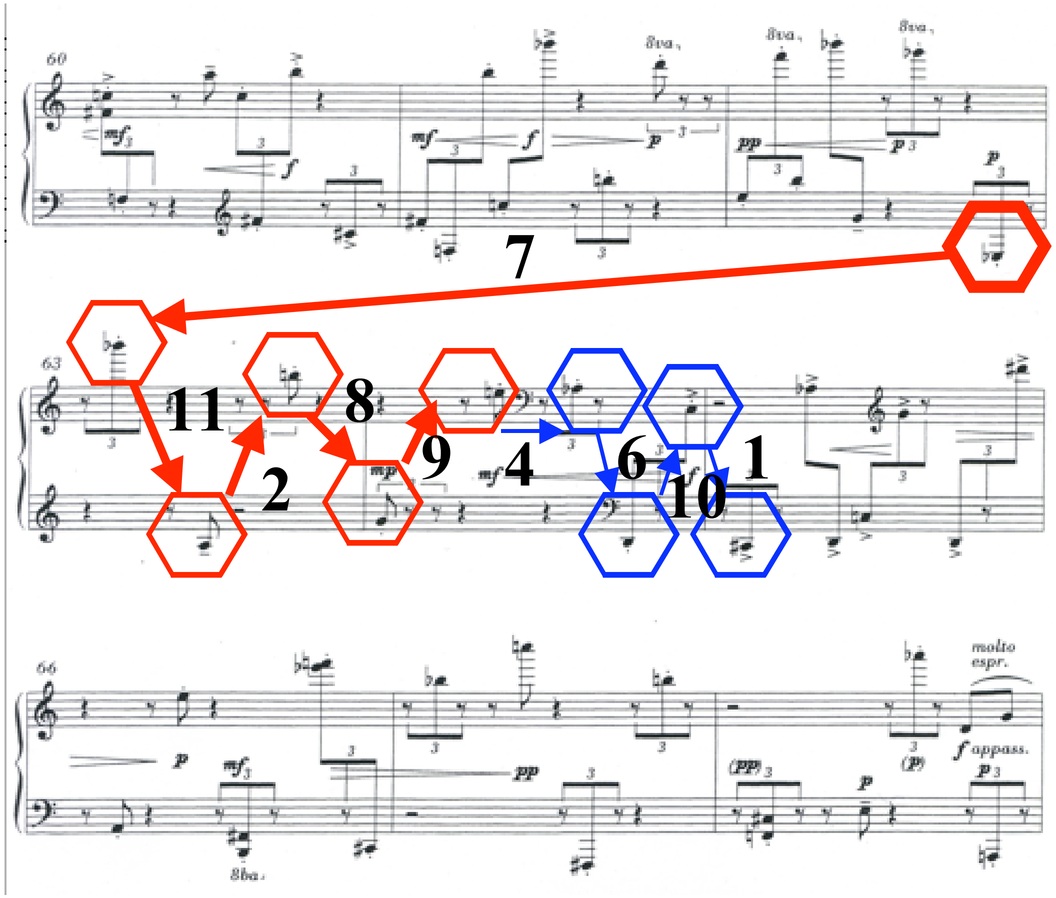

Mauricio Toro Universidad Javeriana, Colombia / IRCAM
http://gelisp.sourceforge.net/

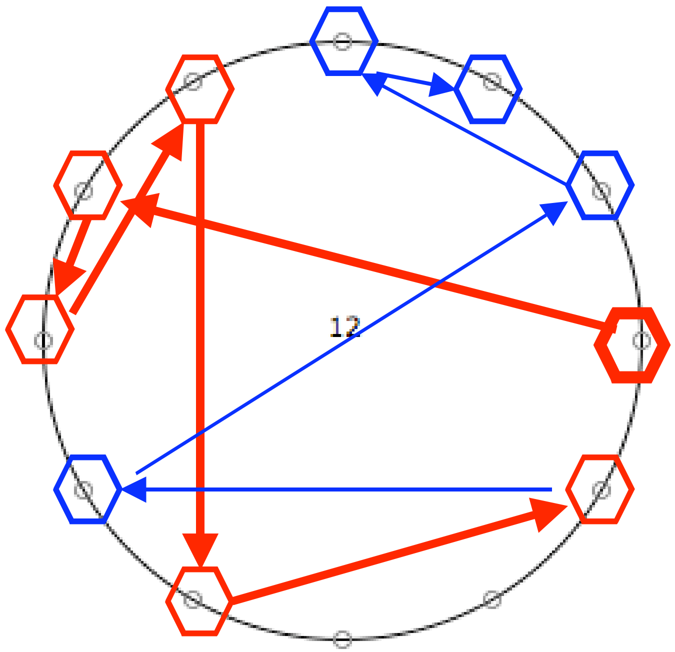
**S = (0 10 11 3 5 2 8 1 9 4 7 6)**
**S*= (10 1 4 2 9 6 5 8 7 3 11)**

# Elliott Carter : 90+ (1994) : Link–chords (mm. 49–68)



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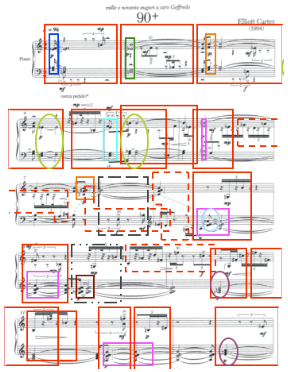


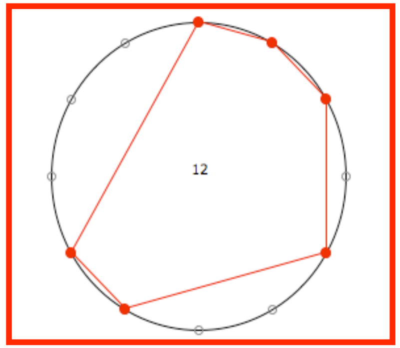


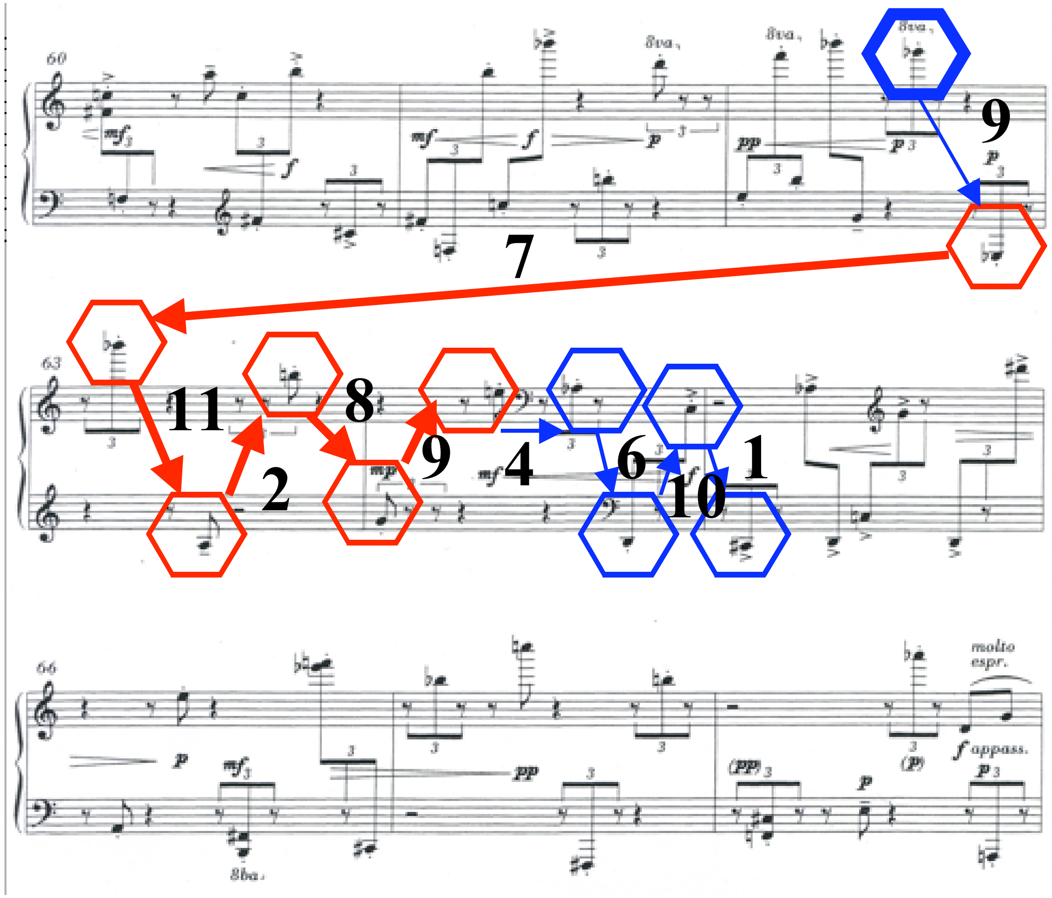


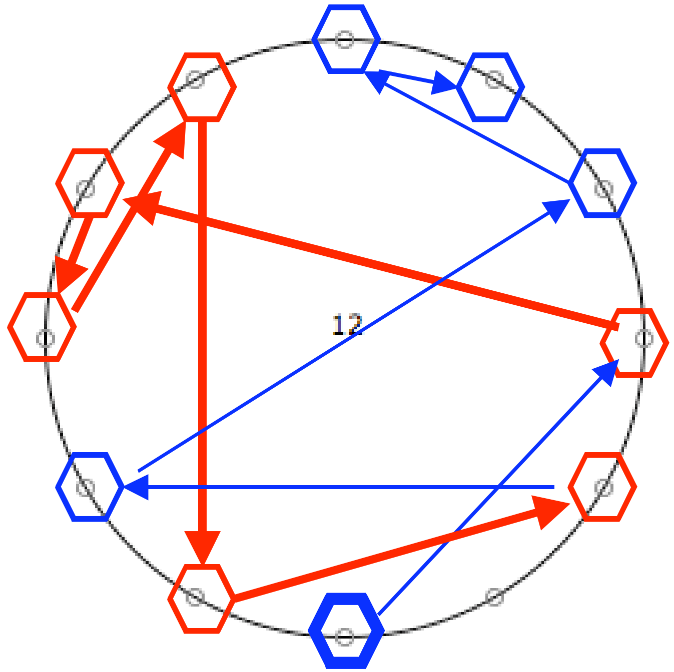

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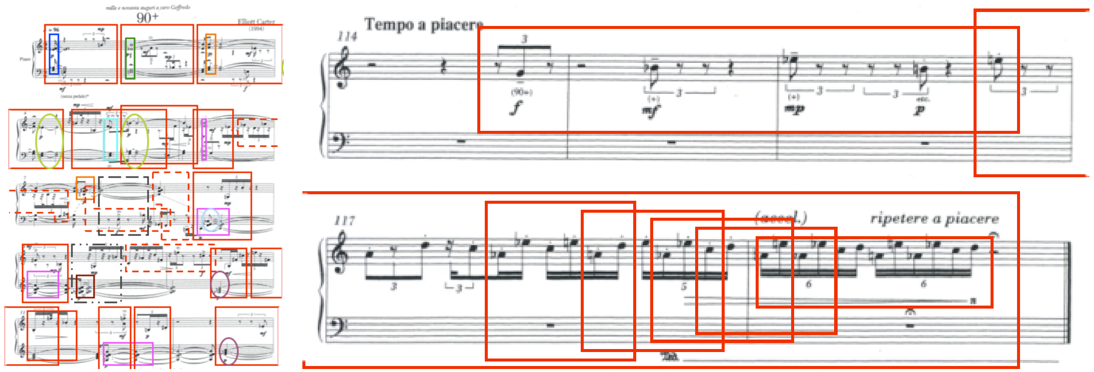
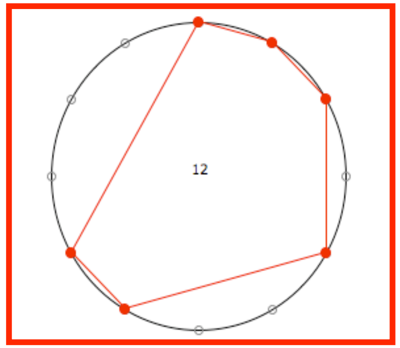
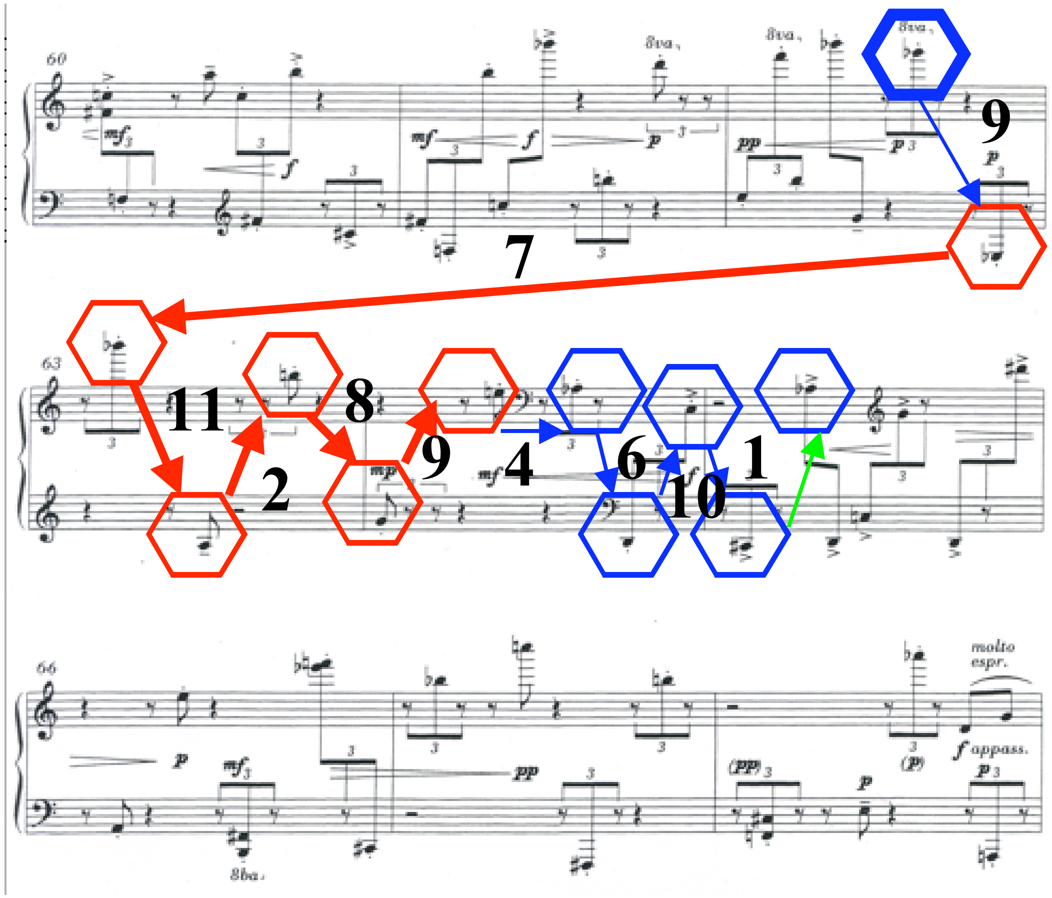
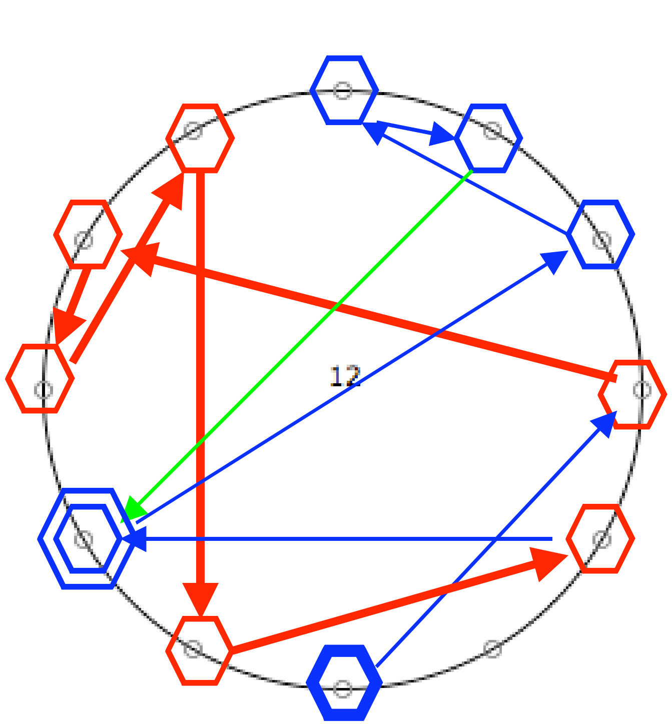
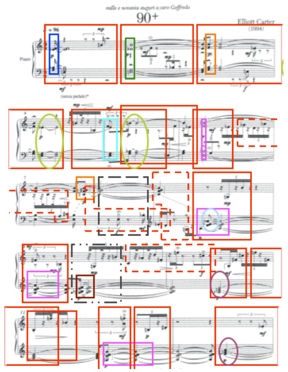


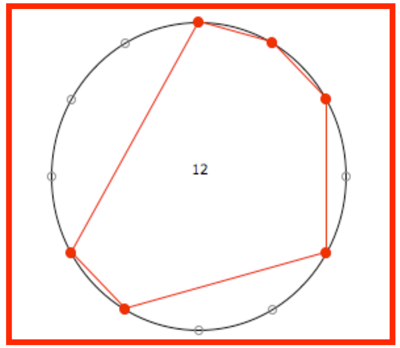



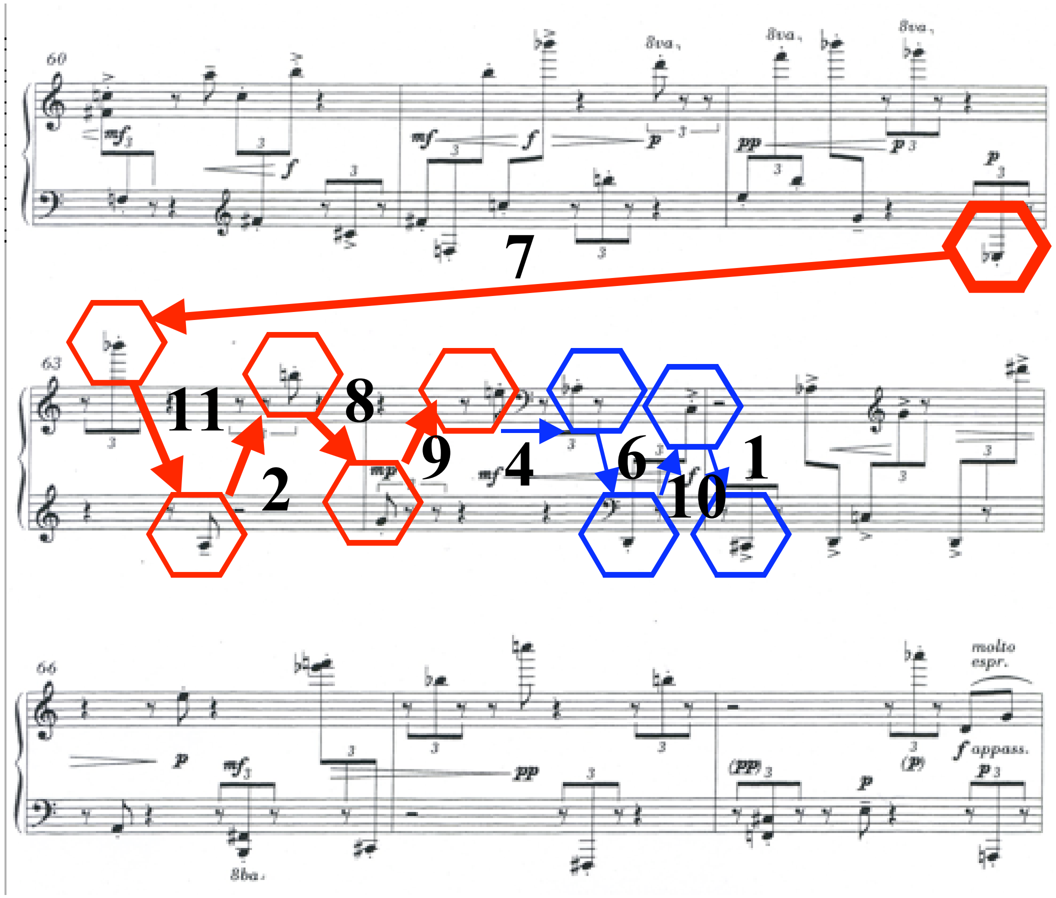



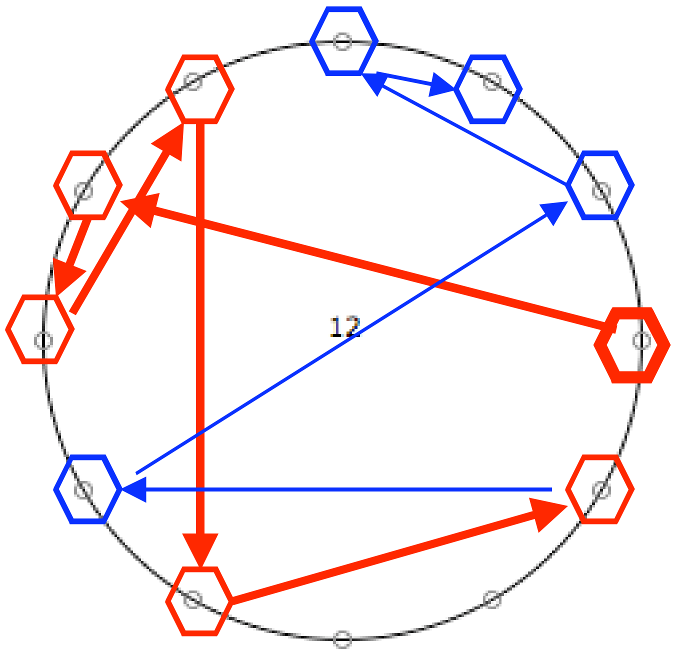



# Elliott Carter : 90+ (1994) : Link–chords (mm. 49–68)



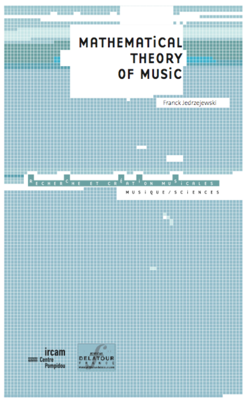



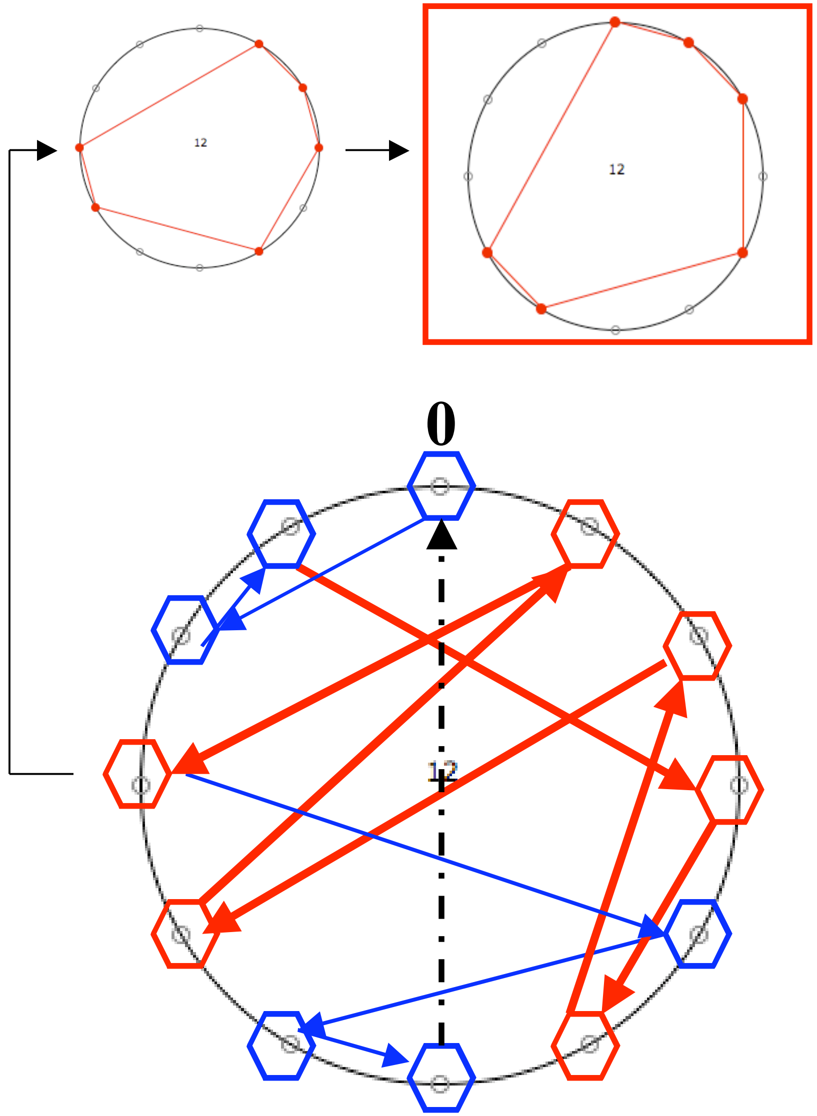



# Classificazione dei Link–chords attraverso i diagrammi di corde

**Risultato: bastano 37 diagrammi di corde per classificare tutte le 194 serie omni–intervallari contenenti l'esacordo omni–triadico (Link chords)**
**[Franck Jedrzejewski, Dicembre 2008]**

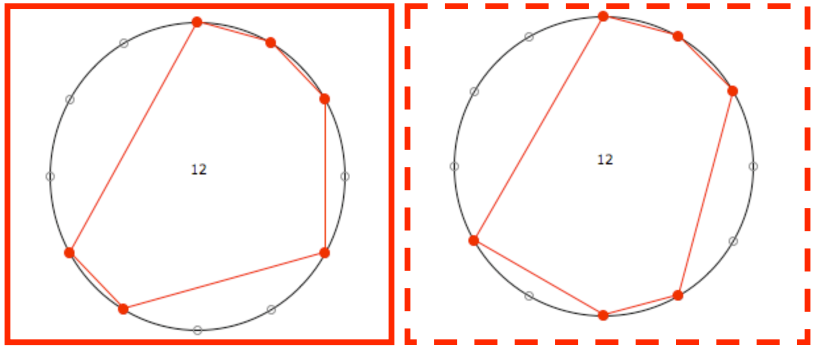



**S = (0 10 11 3 5 2 8 1 9 4 7 6)**
**S*= (10 1 4 2 9 6 5 8 7 3 11)**

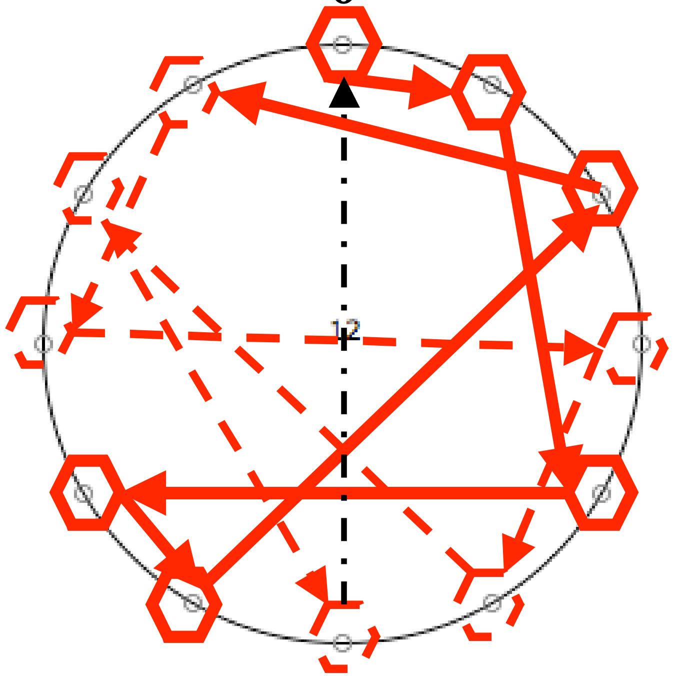
# Link–chords contenenti i due esacordi (H e H')

**Risultato: bastano 29 diagrammi di corde per classificare tutte le 44 serie omni–intervallari contenenti l'esacordo omni–triadico e il suo complementare**
**[Franck Jedrzejewski, Dicembre 2008]**

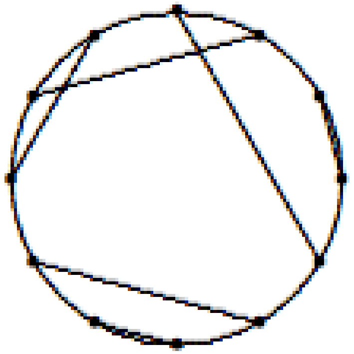



H H'

0




## Description of knot 173

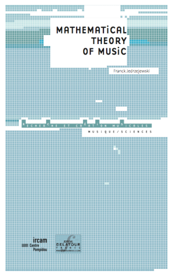


$D_{173}$ X = $acd^{-1}bca$

Gauss word 112342456653

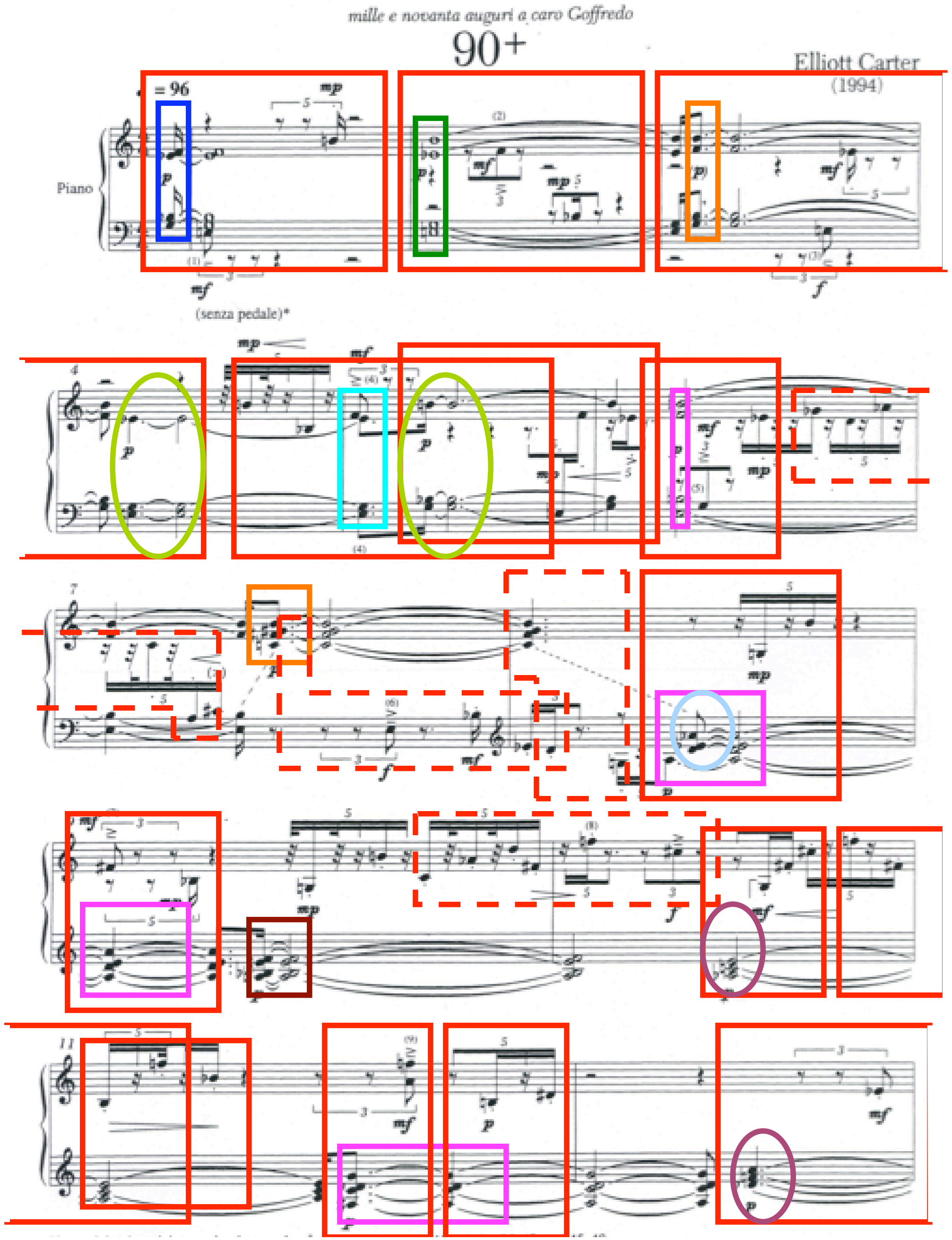
Structural vector 212100

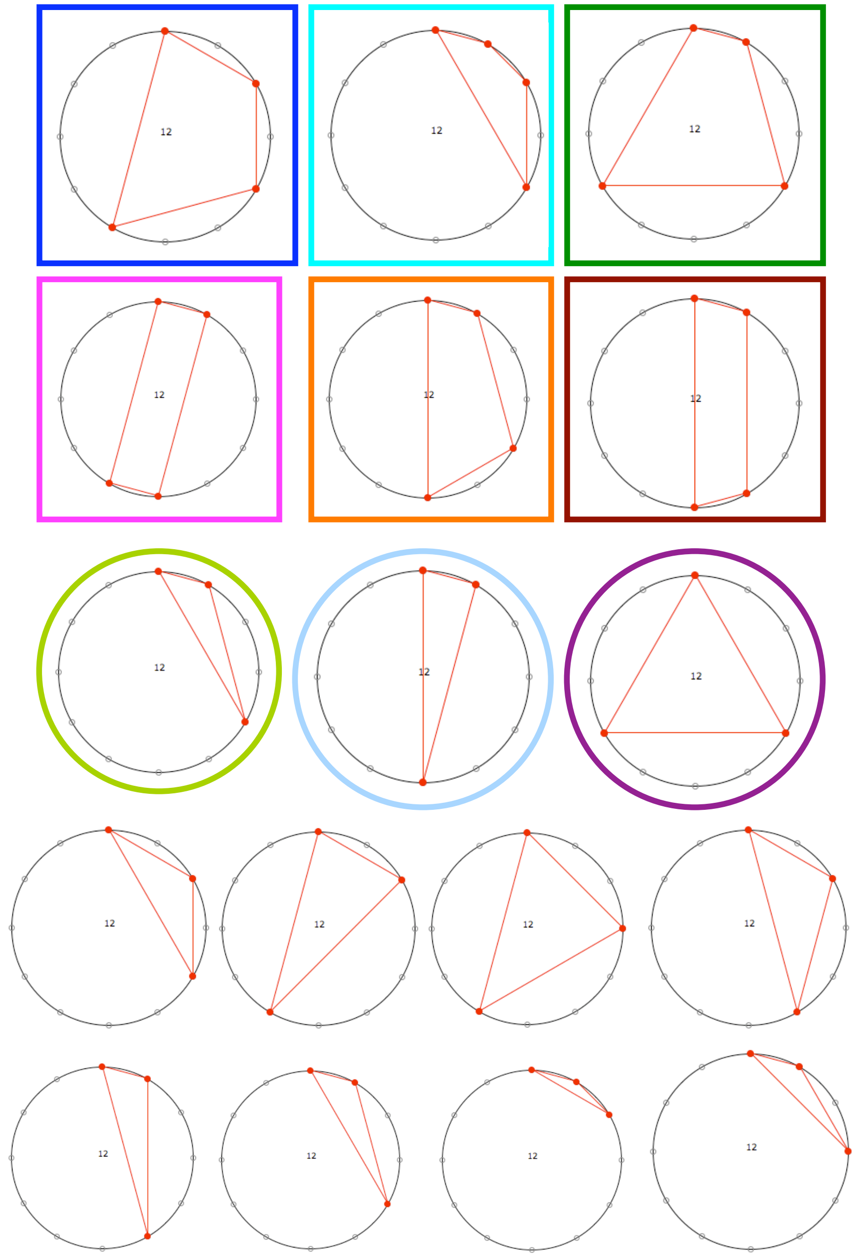
(0 1) (2 5) (3 11) (4 6) (7 10) (8 9)



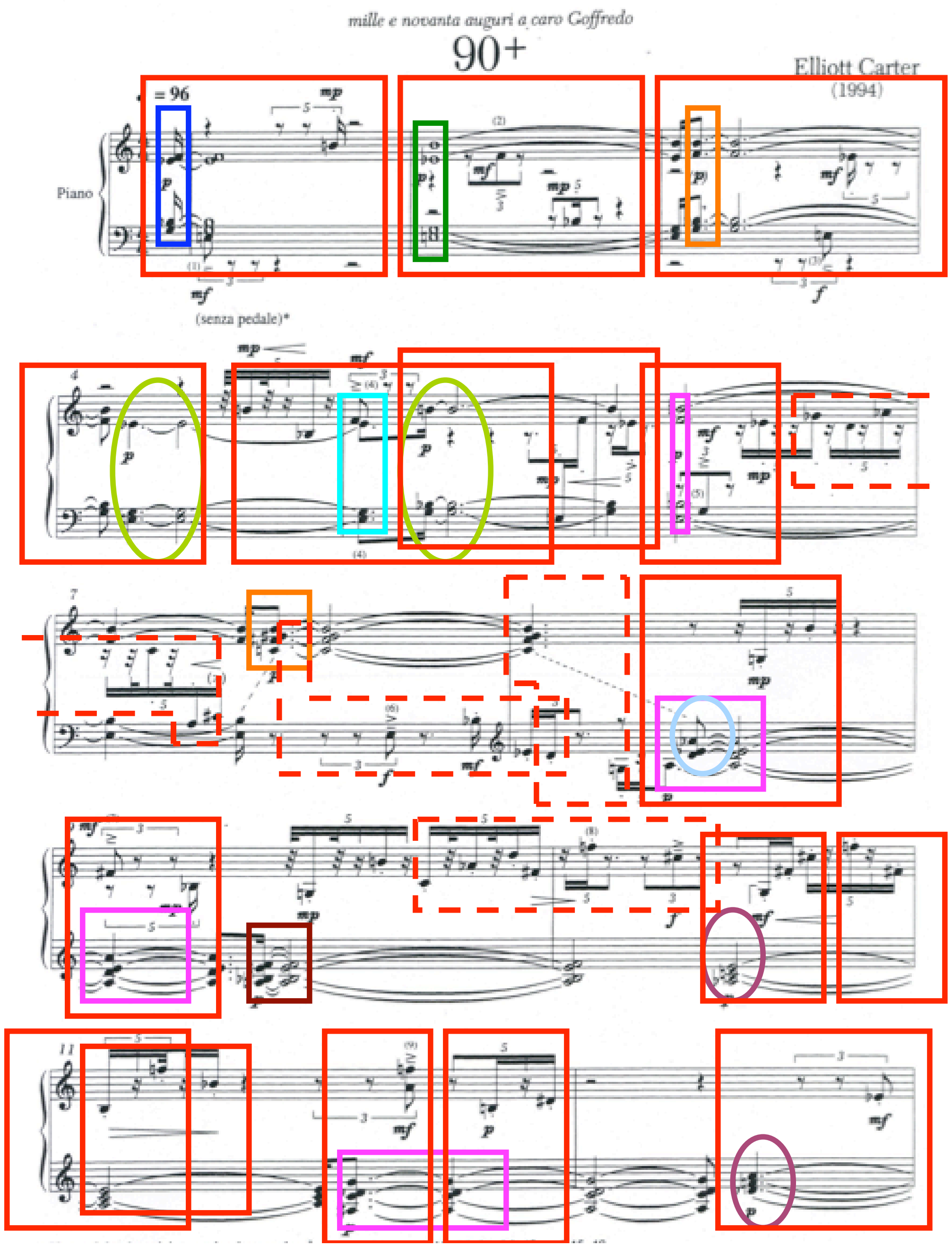

**S = (0 1 4 8 7 2 11 9 3 5 10 6)**
**S*= (1 3 4 11 7 9 10 6 2 5 8)**

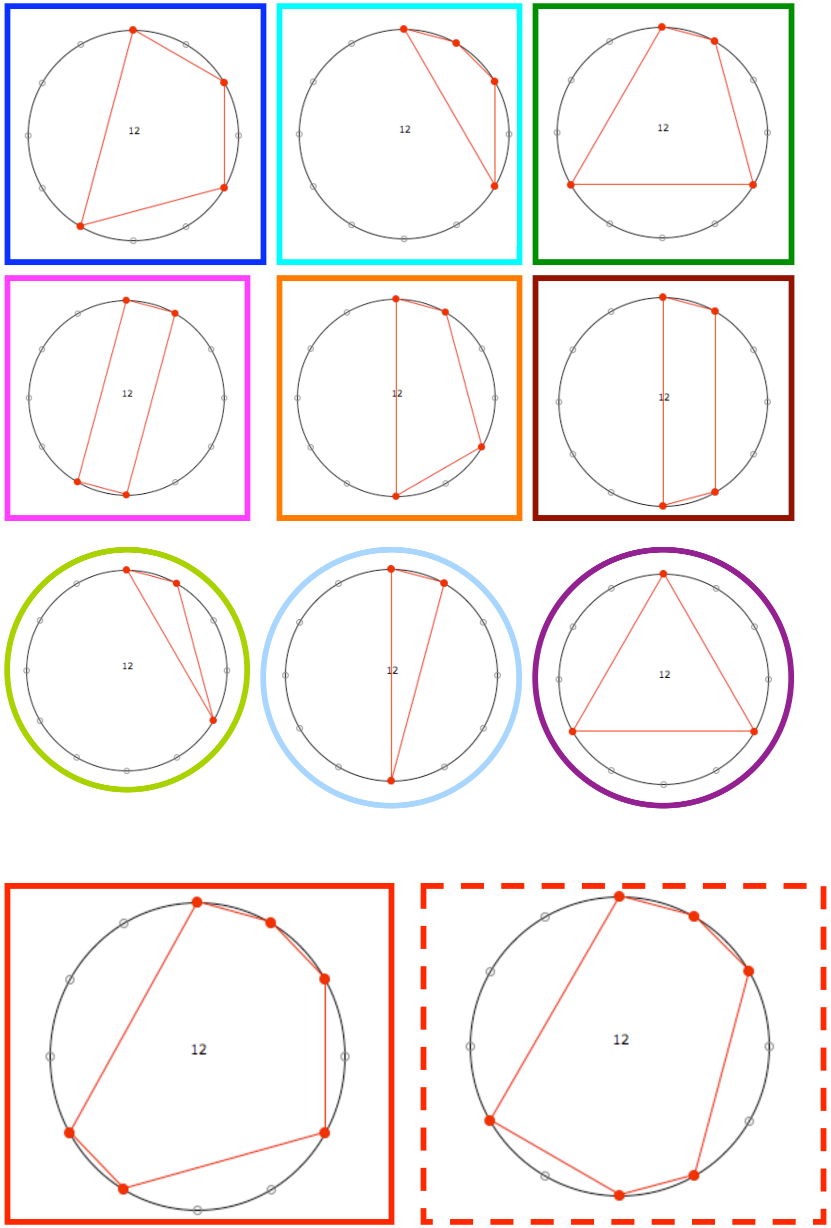
# Elliott Carter : 90+ (1994) : combinatoire tetra/tricordale



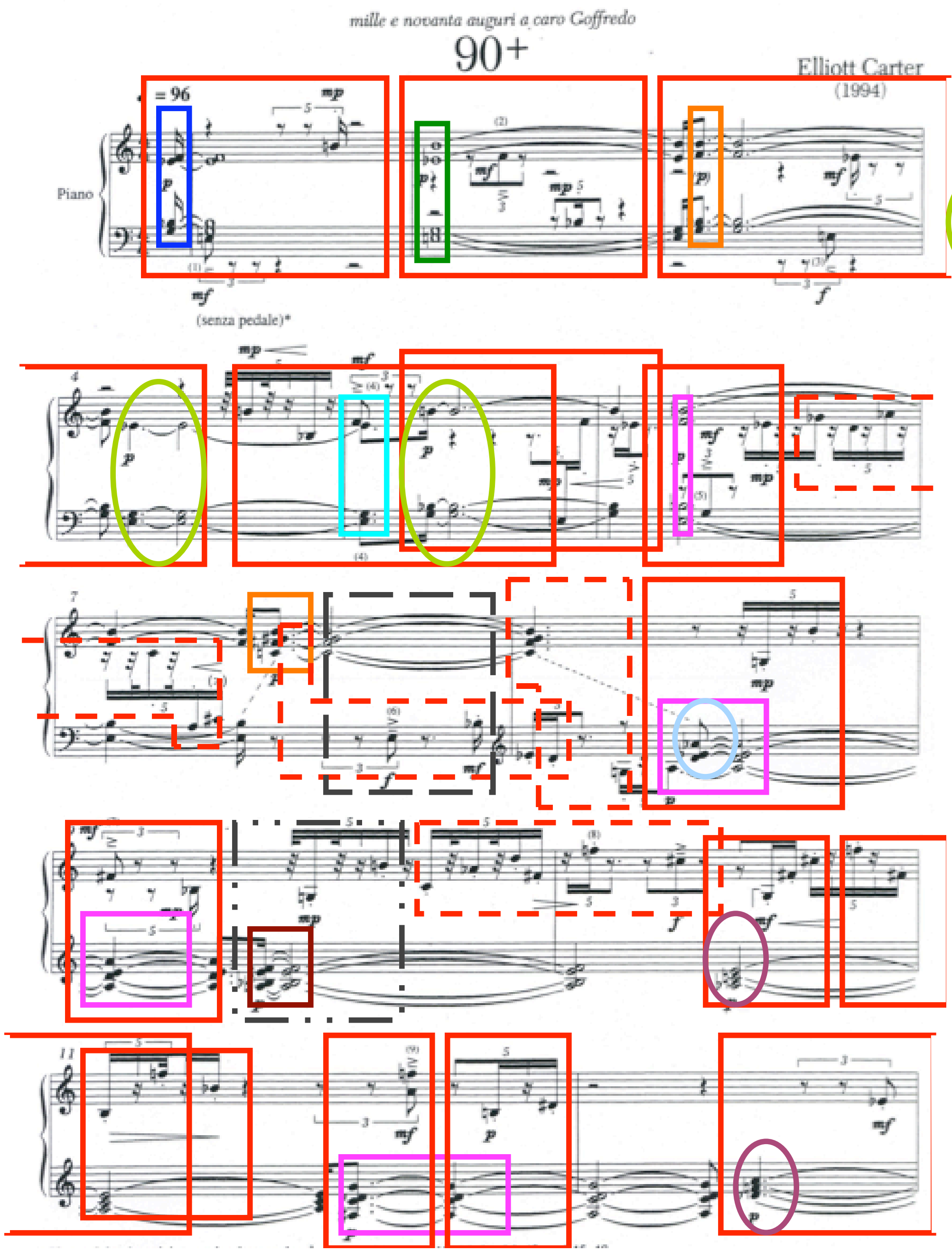



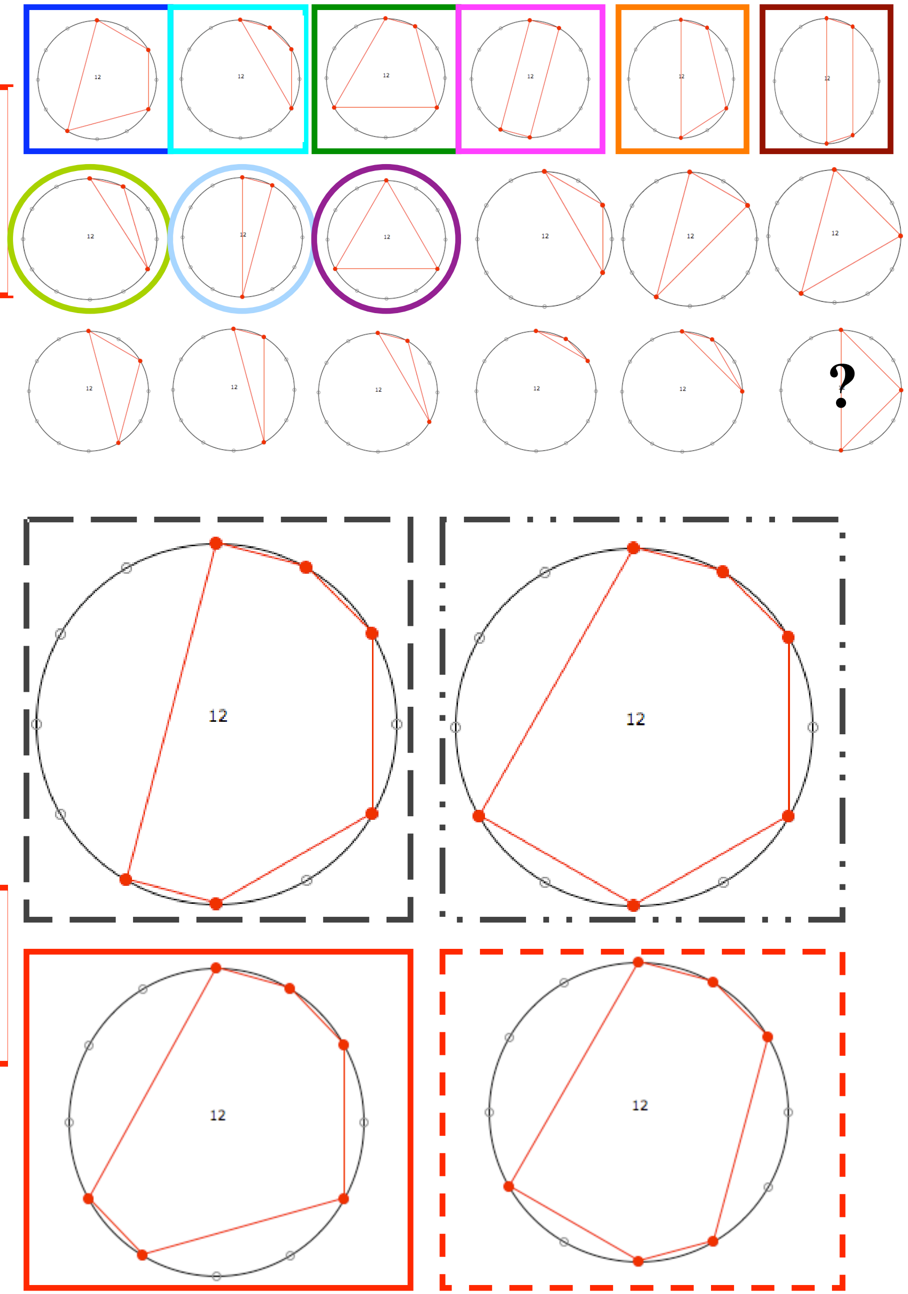
# Elliott Carter : 90+ (1994) : combinatoire tetra/tricordale



# Elliott Carter : 90+ (1994) : combinatoire tetra/tricordale

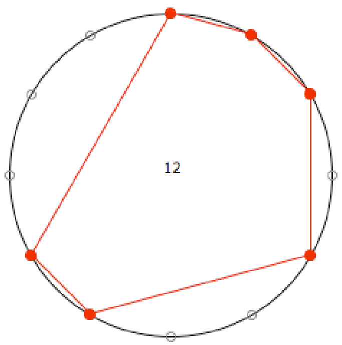


# Contenu des tricordes/tetracordes

| | |
|---|---|
| (0 1 2 4) | (0 1 2) (0 1 4) (0 2 4) (0 1 3) |
| *(0 1 2 5)* | *(0 1 2) (0 1 5) (0 2 5) (0 1 4)* |
| *(0 1 4 5)* | *(0 1 5) (0 1 4)* |
| (0 1 2 7) | (0 1 2) (0 2 7) (0 1 6) |
| (0 1 2 6) | (0 1 2) (0 1 6) (0 2 6) (0 1 5) |
| (0 1 5 6) | (0 1 6) (0 1 5) |
| *(0 2 3 6)* | *(0 1 3) (0 2 6) (0 3 6) (0 1 4)* |
| *(0 2 3 7)* | *(0 1 3) (0 2 7) (0 3 7) (0 1 5)* |
| (0 2 4 7) | (0 2 4) (0 2 7) (0 3 7) (0 2 5) |
| (0 2 4 8) | (0 2 4) (0 4 8) (0 2 6) |
| (0 1 5 7) | (0 1 5) (0 1 6) (0 2 6) |
| *(0 1 5 8)* | *(0 1 5) (0 3 7)* |
| *(0 2 5 8)* | *(0 2 5) (0 2 6) (0 3 7) (0 3 6)* |
| ***(0 2 6 8)*** | ***(0 2 6)*** |
| (0 1 4 8) | (0 1 4) (0 1 5) (0 4 8) (0 3 7) |
| (0 1 3 6) | (0 1 3) (0 1 6) (0 3 6) (0 2 5) |
| (0 1 3 7) | (0 1 3) (0 1 6) (0 3 7) (0 2 6) |
| (0 1 6 7) | (0 1 6) |
| (0 1 4 7) | (0 1 4) (0 1 6) (0 3 7) (0 3 6) |
| (0 1 4 6) | (0 1 4) (0 1 6) (0 2 6) (0 2 5) |

| | |
|---|---|
| (0 1 2) | (0 1 2 4)(0 1 2 7)(0 1 2 6) |
| (0 2 4) | (0 1 2 4)(0 2 4 7)(0 2 4 8) |
| (0 2 7) | (0 1 2 7)(0 2 4 7)(0 2 3 7) |
| (0 4 8) | (0 2 4 8)(0 1 4 8) |
| (0 1 5) | (0 1 2 6)(0 1 5 6)(0 1 5 7)(0 1 4 8)(0 1 2 5)(0 1 4 5)(0 2 3 7)(0 1 5 8) |
| (0 1 3) | (0 1 2 4)(0 1 3 6)(0 1 3 7)(0 2 3 6)(0 2 3 7) |
| (0 3 6) | (0 1 3 6)(0 1 4 7)(0 2 3 6)(0 2 5 8) |
| (0 3 7) | (0 2 4 7)(0 1 4 8)(0 1 3 7)(0 1 4 7)(0 2 3 7)(0 1 5 8)(0 2 5 8)(0 1 3 7) |
| (0 2 5) | (0 2 4 7)(0 1 3 6)(0 1 4 6)(0 1 2 5)(0 2 5 8) |
| (0 2 6) | (0 1 2 6)(0 2 4 8)(0 1 5 7)(0 1 3 7)(0 1 4 6) (0 2 3 6)(0 2 5 8)(0 2 6 8) |
| (0 1 6) | (0 1 2 7)(0 1 2 6)(0 1 5 6)(0 1 5 7)(0 1 3 6)(0 1 3 7)(0 1 6 7)(0 1 4 7)(0 1 4 6) |
| (0 1 4) | (0 1 2 4)(0 1 4 8)(0 1 4 7)(0 1 4 6) (0 1 2 5)(0 1 4 5)(0 2 3 6) |

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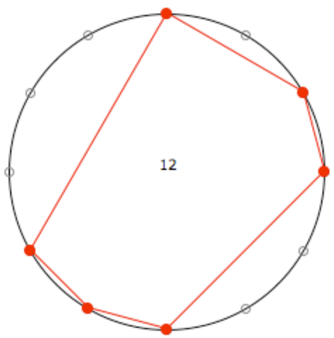


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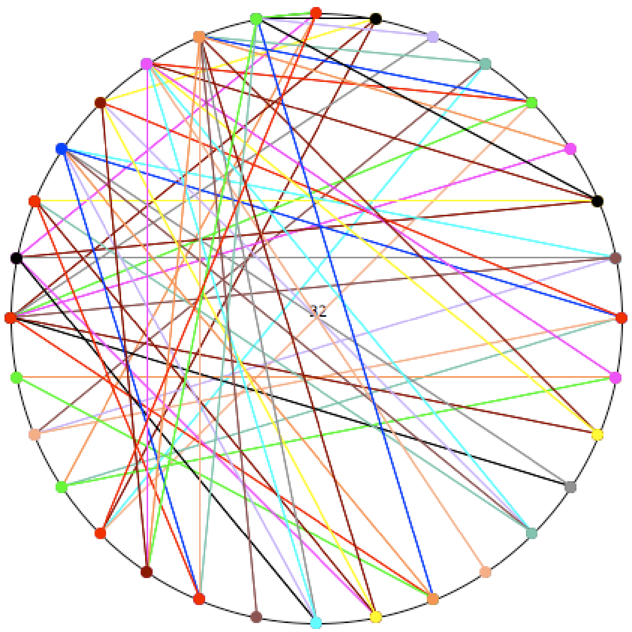
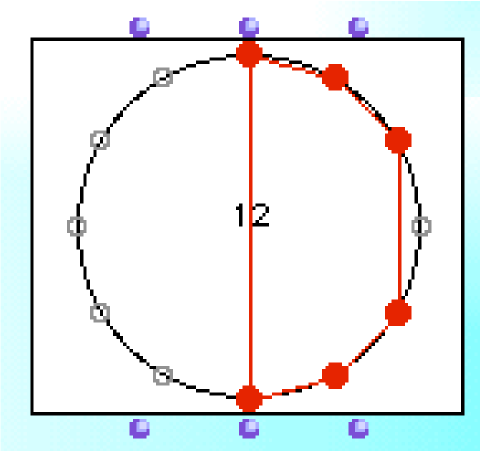
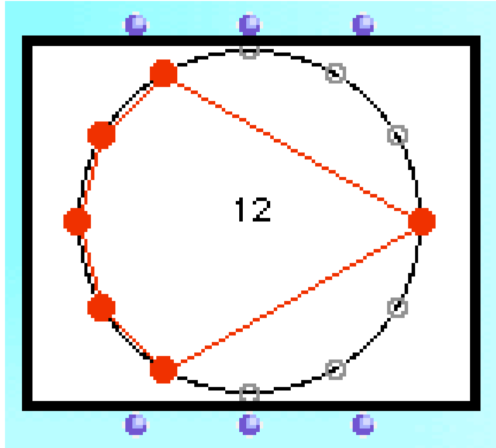


# Teorema dell'esacordo (o teorema di Babbitt)




*Un esacordo A e il suo complementare A' hanno lo stesso vettore (contenuto) intervallare*




IC(*A*) = [4, 3, 2, 3, 2, 1 ] = [4, 3, 2, 3, 2, 1 ] = IC(*A'*)

$$IC_A(k) = \mathrm{Card}\{(x,y) \in A \times A \mid x + k = y\}$$

# David Lewin e la transformata di Fourier discreta

**E. Amiot, T. Noll, M. Andreatta, C. Agon : « Oracles for Computer-Aided Improvisation », *ICMC*, New Orleans, novembre 2006**

- Il contenuto intervallare di due accordi *A* e B è il prodotto di convoluzione delle loro funzioni caratteristiche

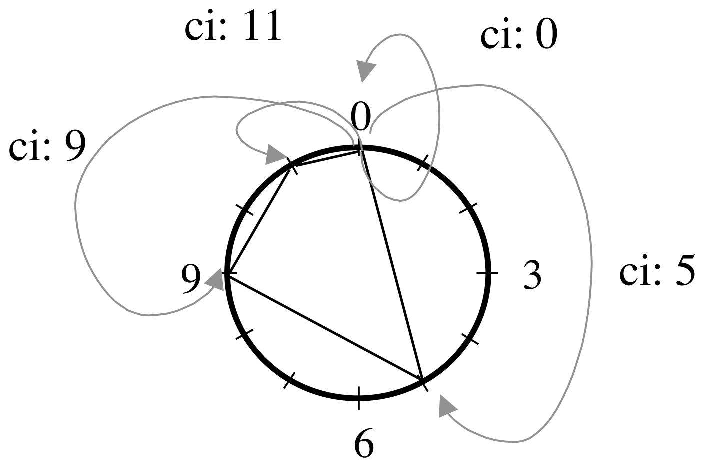
$$IC_A(k) = \text{Card}\{(x,y) \in A \times A \mid x + k = y\}$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

$$1_A * \tilde{1}_B(k) = \sum_i 1_A(i) \times 1_B(i-k) = \sum_{\substack{i \in A \\ i-k \in B}} 1$$

$$\mathcal{F}(1_A * \tilde{1}_B) = \mathcal{F}(1_A) \times \mathcal{F}(\tilde{1}_B)$$

$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

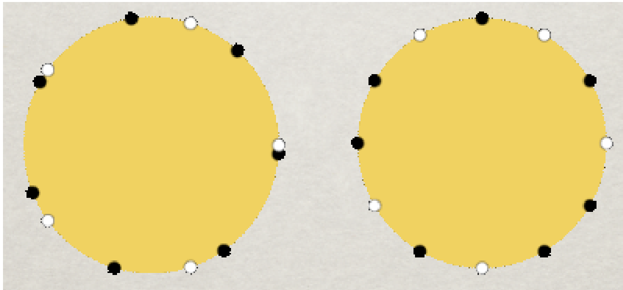



A = {0, 5, 9, 11}
$IC_A(k) = 1 \;\forall\; k = 1\ldots11$

$$\forall \mathrm{k}\; \mathcal{F}(\mathrm{IC}_{\mathbb{Z}_c \setminus A})(\mathrm{k}) = \mathcal{F}(\mathrm{IC}_A)(\mathrm{k})$$

(Teorema dell’esacordo )

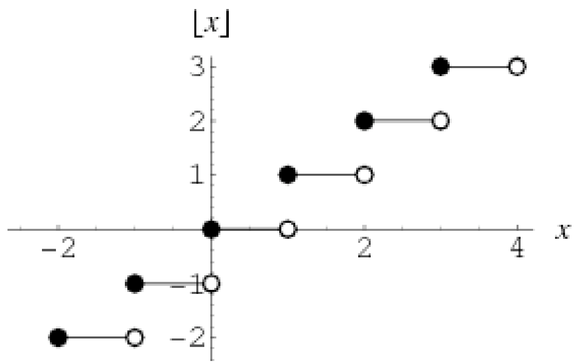
# La scala diatonica come insieme di ripartizione massimale
# *(Maximally Even Sets)*



**Definition** (Clough-Myerson-Douthett ) A set $A$ with cardinality $d$ in a given equal tempered space $\mathbf{Z}_c$ is maximally even if $A = \{a_k\}$

$$a_k = J_{c,d}^{\alpha}(k) = \left\lfloor \frac{kc+\alpha}{d} \right\rfloor$$

where $\alpha \in \mathbf{R}$
$\lfloor x \rfloor$ is the integer part of $x$



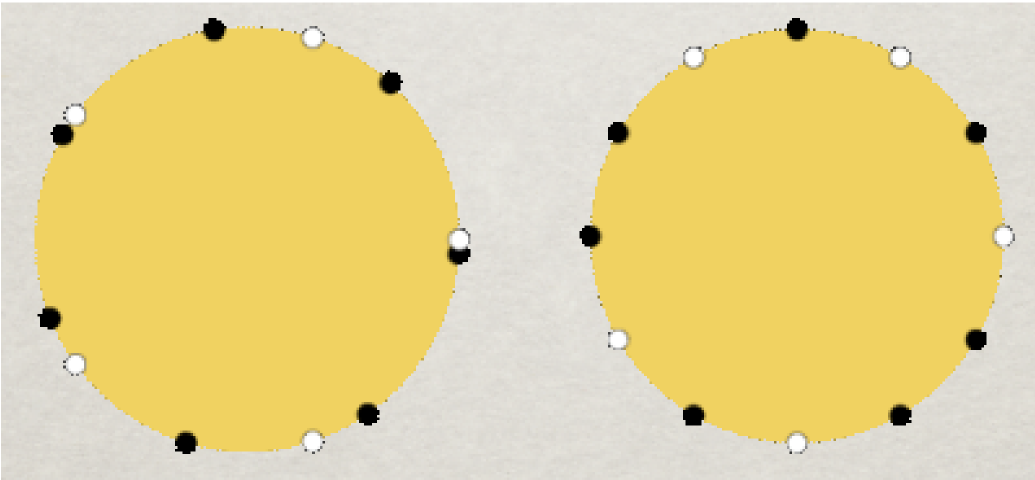

$$J_{12,7}^{5} = \left\{ \left\lfloor \frac{12k+5}{7} \right\rfloor \right\}_{k=0}^{6} = \{0,2,4,5,7,9,11\}$$

**Definition** (Amiot, 2005 ) A set $A$ with cardinality $d$ given equal tempered space $\mathbf{Z}_c$ is maximally even if $|F_A(d)| \geq |F_B(d)|$ for all subsets $B$ of cardinality $d$ in $\mathbf{Z}_c$.

where $F_{set}(t) := \sum_{k \in set} e^{2i\pi kt/12}$

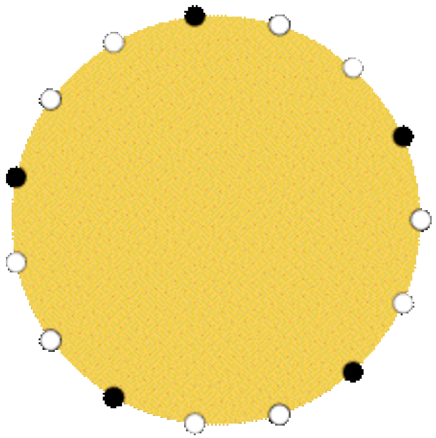
# La Transformata di Fourier e i *ME Sets*

$$\text{fourier}(\text{set},\, t) := \sum_{k \in set} e^{2i\pi kt/12}$$



Scala diatonica :
{0, 2, 4, 5, 7, 9, 10}

Scala pentatonica :
{0, 2, 5, 7, 10}



$F_A(5) = 1+1+1+1+1 = 5$

In generale, $|\, F_A(t)\, | \leq \#A$

# Tassellazioni musicali: la costruzione dei canoni a mosaico

- Teoria del ritmo periodico (Vuza 1985, 1988, 1991)
  - Un ritmo periodico come un sottoinsieme periodico localmente finito di **Q**
  - La famiglia dei ritmi periodici come anello di insiemi (chiusa per intersezione e differenza simmetrica)

- Canoni ritmici a mosaico come fattorizzazione di gruppi ciclici
  - Primi esempi di canoni a mosaico: 4 famiglie
  - Gruppi di Hajos e gruppi non-Hajos
  - Teorema di Hajos e teorema di Redei

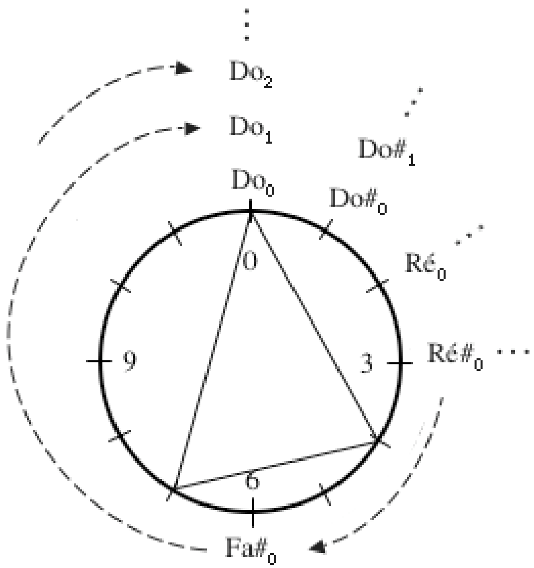
- Fattorizzazioni polinomiali (polinomi ciclotomici)
  - Condizioni di Coven-Meyerowitz
  - Dimostrazione del teorema di Hajos con la teoria dei polinomi ciclotomici

- Congetture geometrico-algebriche
  - Congettura di Minkowski
  - Congettura di Keller
  - Congettura di Fuglede (congettura spettrale)

# Plan :

## -Modèle géométrico/algébrique du rythme : la représentation circulaire

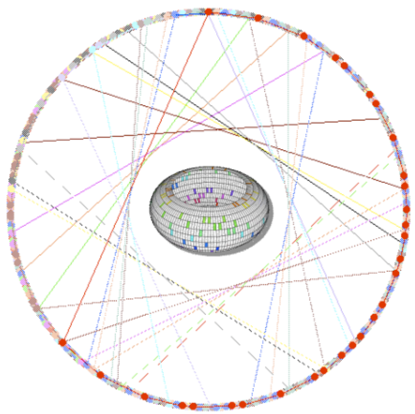
- Quelques exemples dans la musique africaine et cubaine
  - Cinquillo (Cuba)
  - Tresillo (Cuba)
  - Clave son
- Premières opérations :
  - Inversion d'un rythme
  - Complémentation d'un rythme
  - *Shadow* (G. Toussaint) d'un rythme
  - Multiplications des rythmes (à la Boulez)
- Premières propriétés :
  - Rythmes auto–inverses
  - Rythmes auto–complémentairs
  - (Rythmes en relation Z)
  - Imparité rythmique et généralisations



(4,3,5)

## -Pavage de l'axe du temps :

- Canons mosaïques par translation
  - Canons par entrés regulières
  - Canons « redondants »
  - Canons de Vuza
- Canons mosaïques par inversion et augmentation
  - Canons par inversion (theorème de Wild)
  - Canons de Noll
  - Canons parfaits (Tom Johnson)



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- **Iannis Xenakis** : la théorie des cribles et ses extensions au niveau rythmique
  - I. Xenakis : *Formalized Music* (1971. Rev. edition Pendragon Press, 1990)
  - E. Amiot, G. Assayag, C. Malherbe, A. Riotte, « Duration structure generation and recognition in music writing », Proceedings of the ICMC, La Haye, 1986
  - C. Ariza, « The Xenakis Sieve as Object: A New Model and a Complete Implementation », *Computer Music Journal*, v.29 n.2, p.40-60, June 2005
  - A. Riotte & M. Mesnage : *Formalismes et modèles musicaux*, Delatour/Ircam, 2006

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  - D. T. Vuza : « Propriétés des suites périodiques utilisées dans la pratique modale », Muzica, 2, pp. 44-48, 1984.
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- **Godfried Toussaint** : approche géométrique / informatique (http://cgm.cs.mcgill.ca/~godfried/)
  - P. Taslakian and G. T. Toussaint, "Geometric properties of musical rhythm," Proceedings of the 16th Fall Workshop on Computational and Combinatorial Geometry, Smith College, Northampton, Massachussetts, November 10-11, 2006.
  - G. Toussaint, "The  geometry of musical rhythm," Proceedings of the Japan Conference on Discrete and Computational Geometry, (JCDCG 2004), LNCS 3742, Springer-Verlag, Berlin-Heidelberg, 2005, pp. 198-212.

# BIBLIO Canoni ritmici a mosaico e tassellazioni


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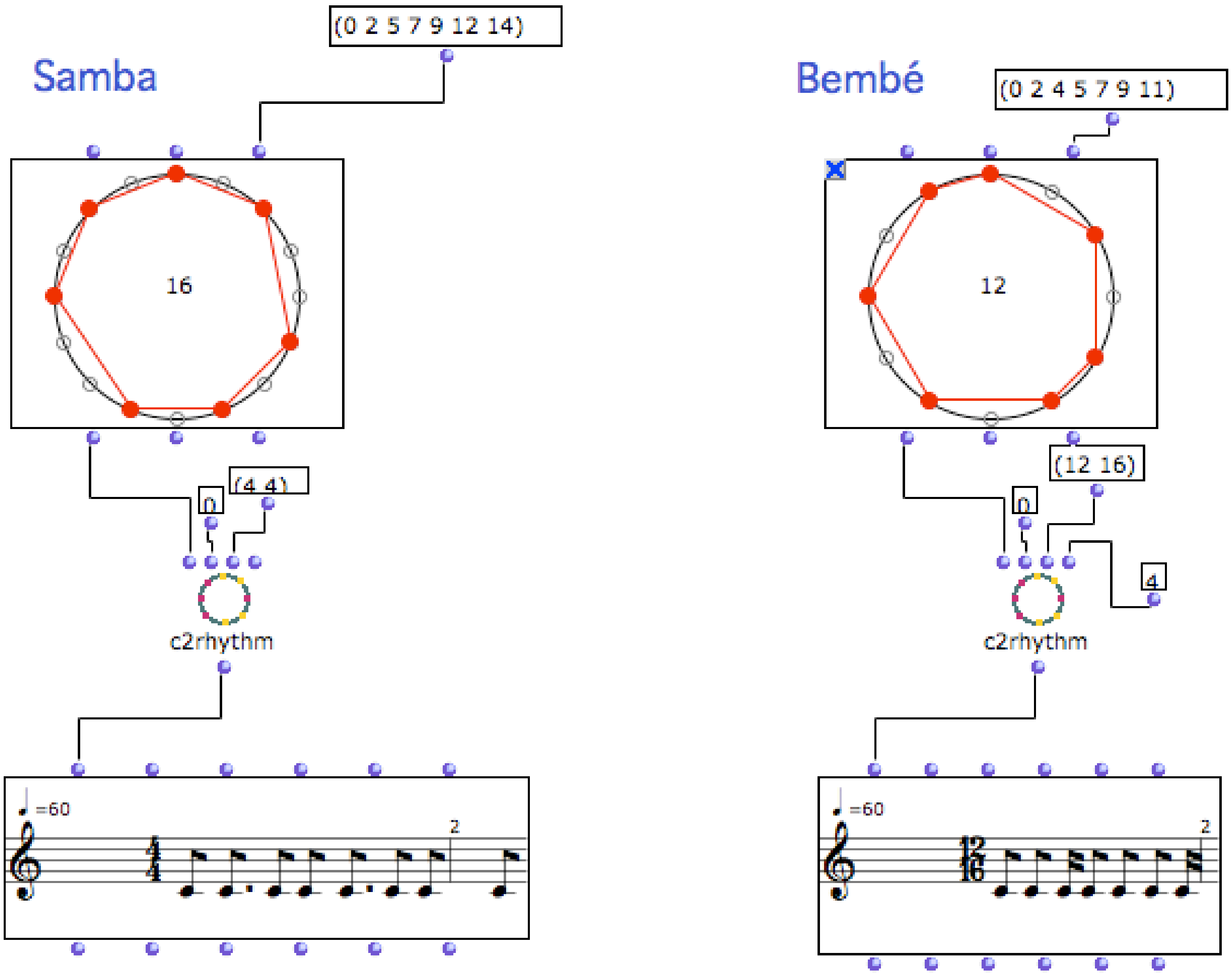
  - *Traité de rythme, de couleurs et d'ornithologie* (Tome 2)

- **Modèle mathématique des canons mosaiques par translation**
  - D. T. Vuza : « Supplementary Sets and Regular Complementary Unending Canons », en quatre parties, dans *Perspectives of New Music*, Part 1 29(2), p. 22-49 ; Part 2 30(1), p. 184-207 ; Part 3 30(2), p. 102-125 ; Part 4 31(1), p 270-305, 1991-1993.

- **Modèle informatique en *OpenMusic* et extensions du modèle théorique :**
  - M. Andreatta : *Méthodes algébriques en musique et musicologie du XX siècle. Aspects théoriques, analytiques et compositionnels »,* thèse, EHESS, 2003.
  - M. Andreatta, C. Agon, T. Noll et E. Amiot, « Towards Pedagogability of Mathematical Music Theory : algebraic Models and Tiling Problems in computer-aided composition », *Proceedings Bridges. Mathematical Connections in Art, Music and Science*, London, 2006, p. 277-284.
  - M. Andreatta et M. Chemillier, « Modèles mathématiques pour l'informatique musicale (MMIM): Outils théoriques et strategies pédagogiques », *Actes des Journées d'Informatique Musicale*, Lyon, avril 2007, p. 113-123.

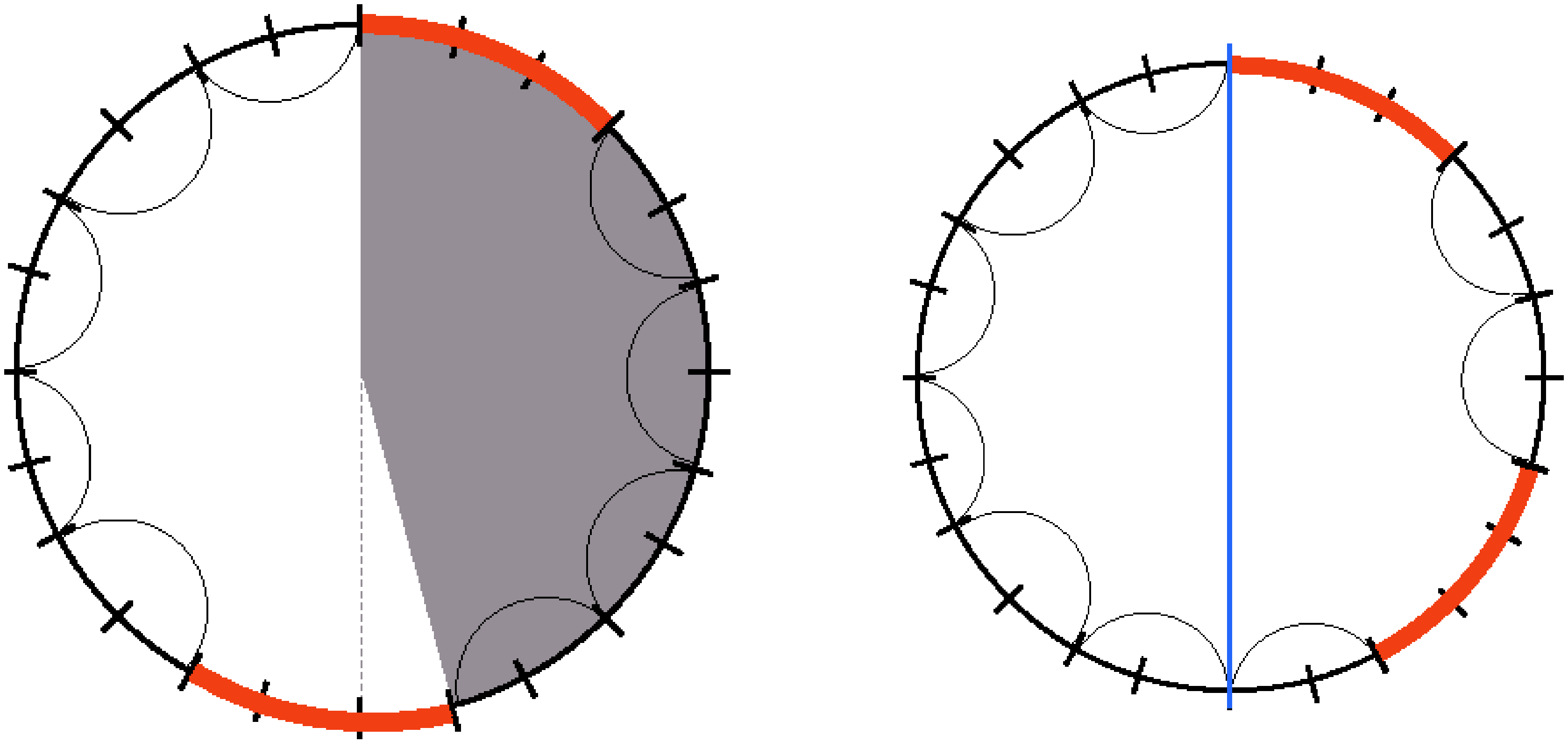
- [Fid08] G. Fidanza: *Canoni ritmici a mosaico,* tesi di laurea in matematica, 2006/2007
- [Gilb07] E. Gilbert: *Polynômes cyclotomiques, canons mosaïques et rythmes k-asymétriques*, mémoire de Master ATIAM, maggio 2007

# Teoria del ritmo periodico



# Circular Representation and Aka Pygmies rhythms

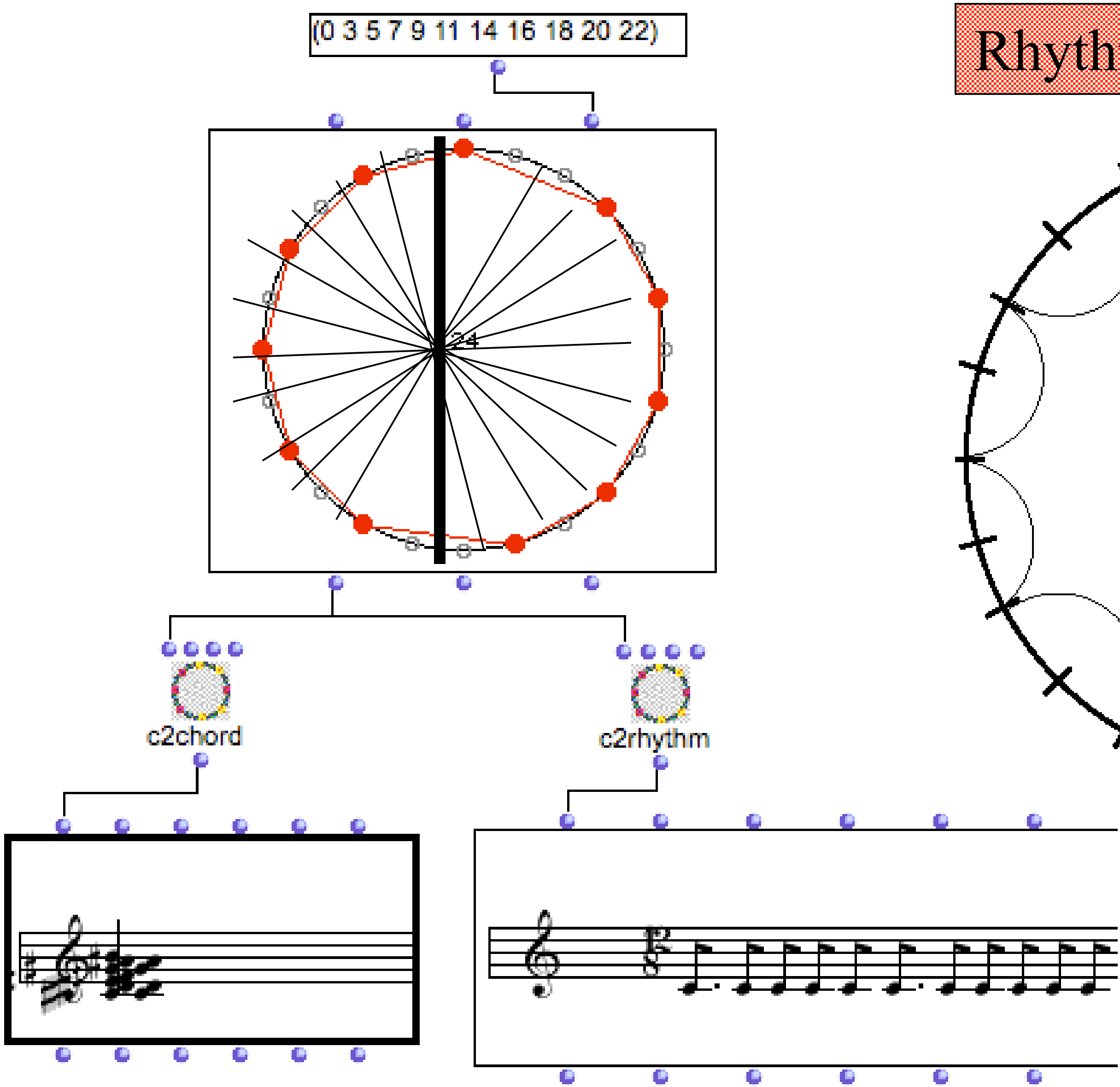
Marc Chemillier, Periodic musical sequences and Lyndon words, *Soft Computing*, Sept. 2004



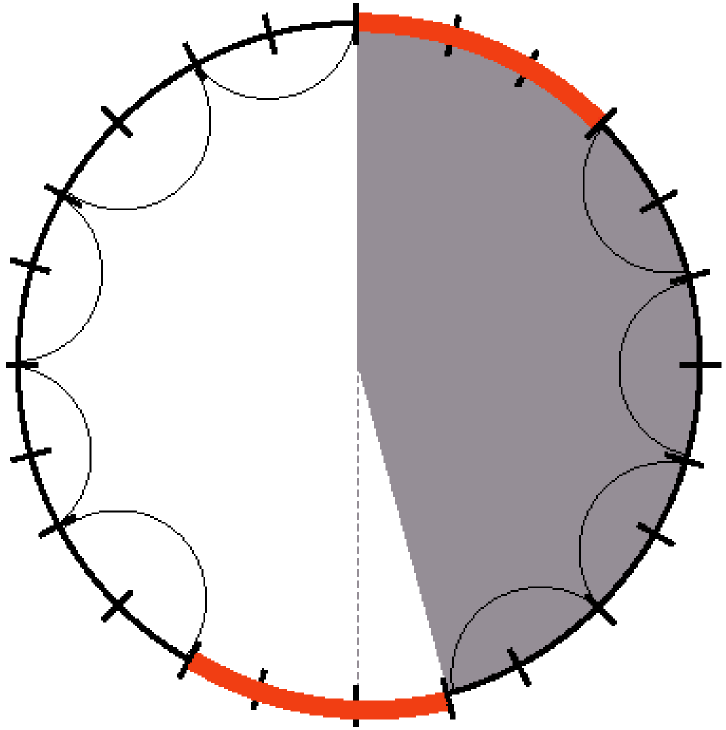
Rhythmic-Oddity Property

# Circular Representation and Aka Pygmies rhythms

Marc Chemillier, Periodic musical sequences and Lyndon words, *Soft Computing*, Sept. 2004

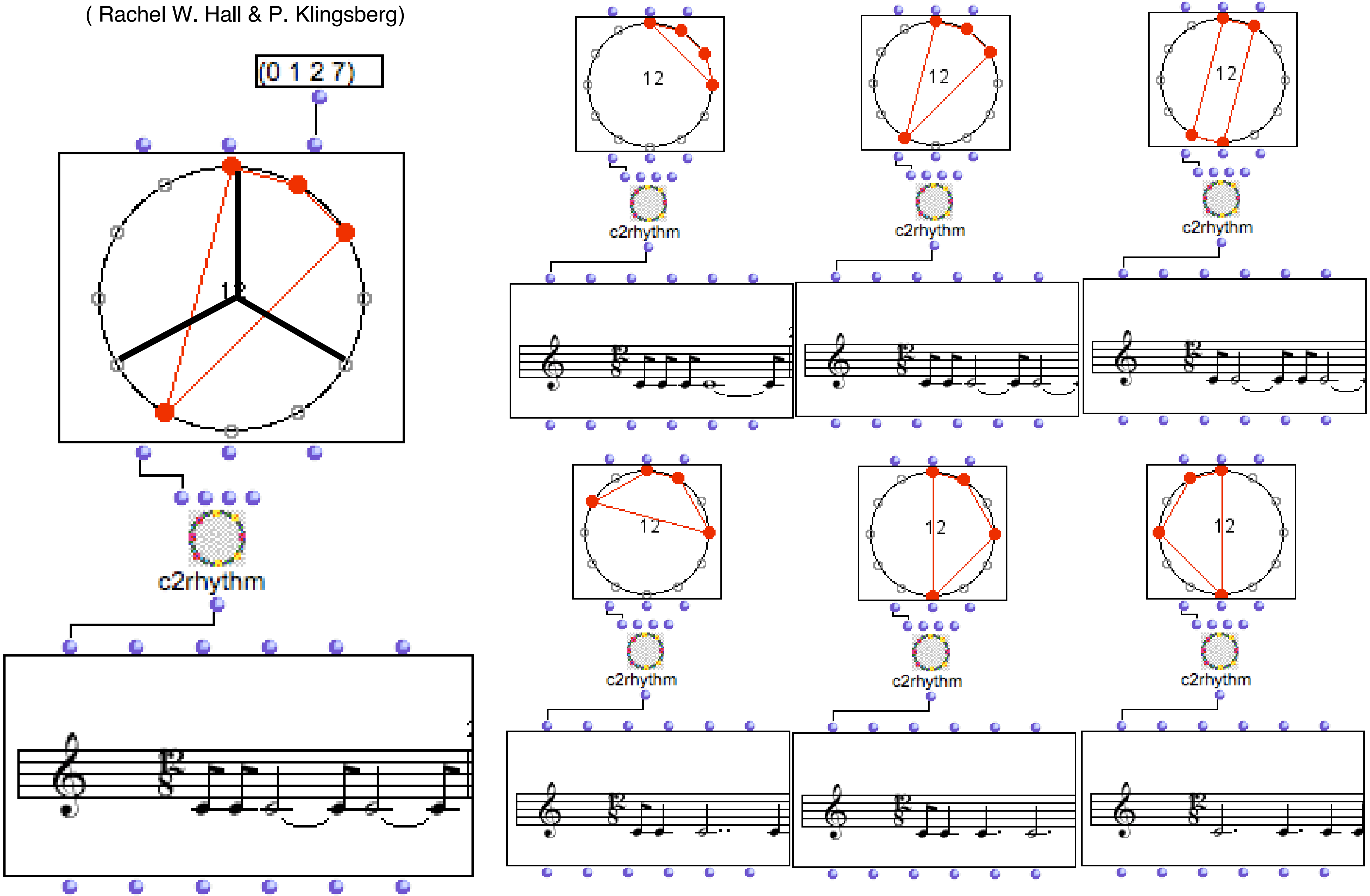



**Rhythmic-Oddity Property**



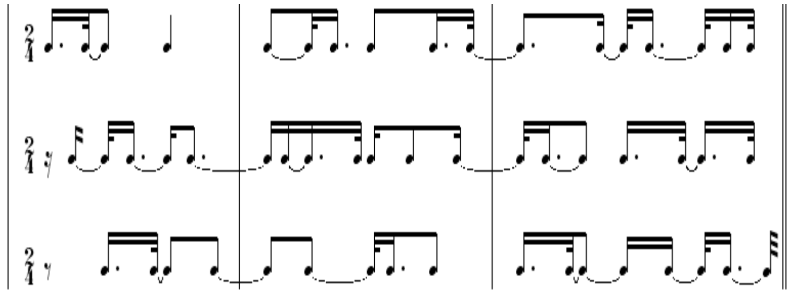
# The 3-Oddity Property

( Rachel W. Hall & P. Klingsberg)

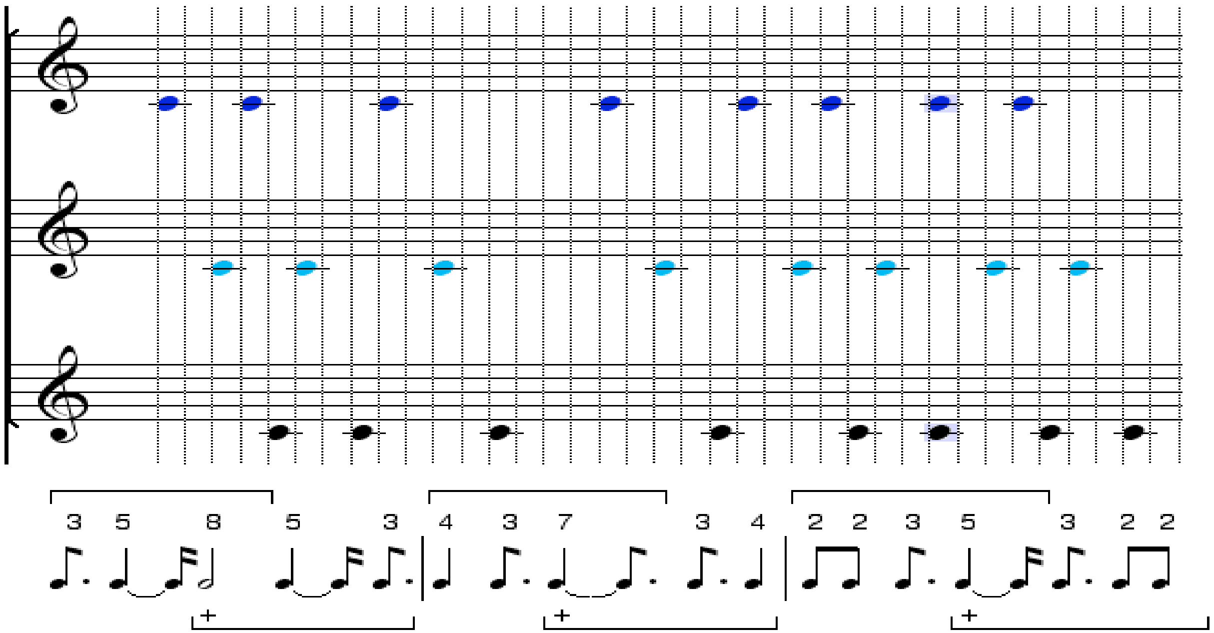


# Olivier Messiaen e i canoni ritmici


*Harawi* (1945)



*Visions de l'Amen* (1943)

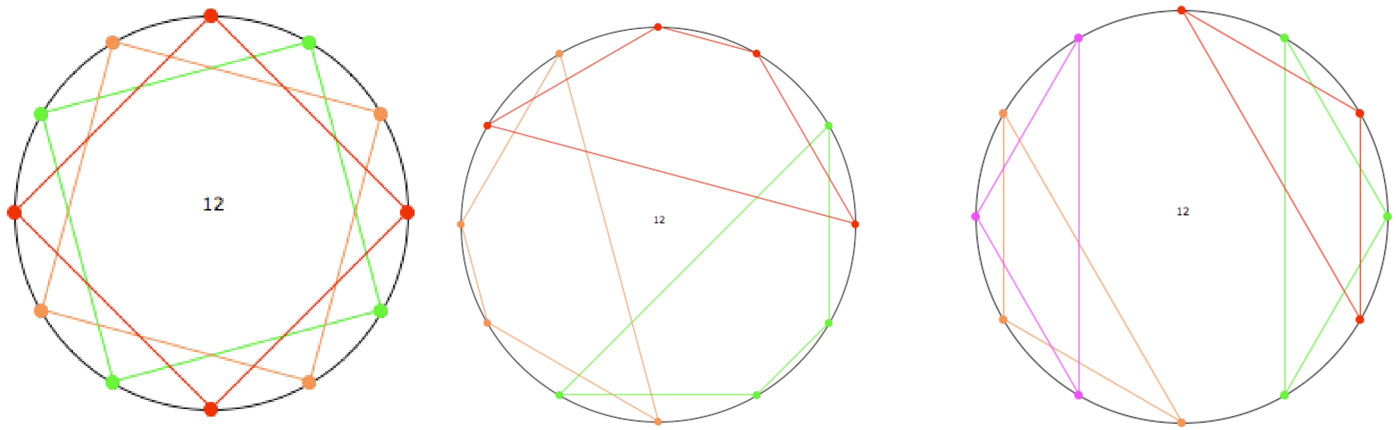


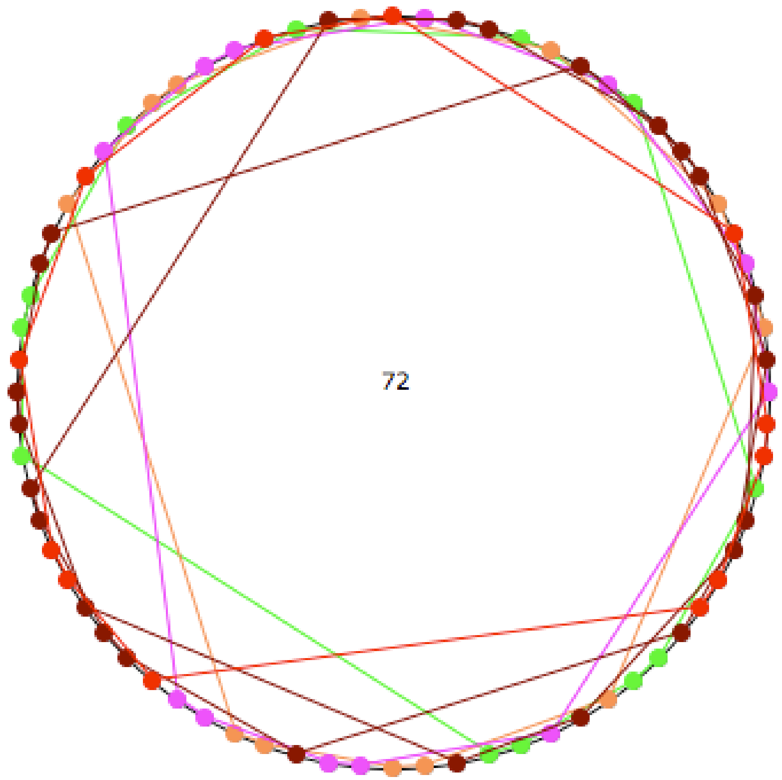

Modèle rythmique

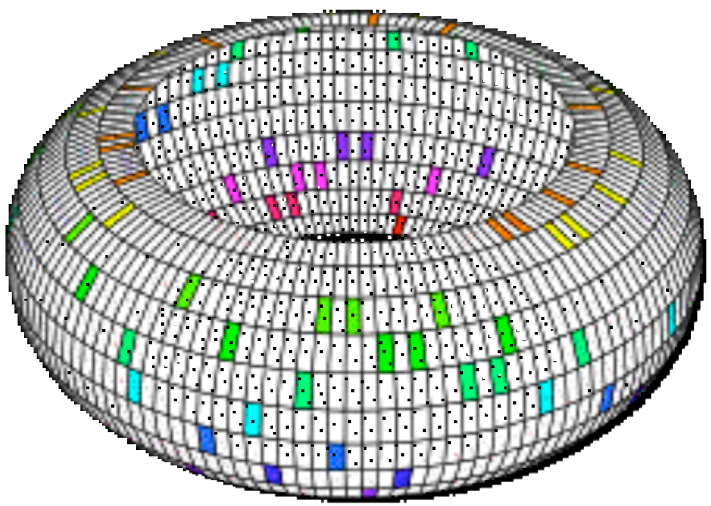
« …il résulte de tout cela que les différentes sonorités se mélangent ou s'opposent de manières très diverses, **jamais au même moment ni au même endroit […]. C'est du désordre organisé** »

O. Messiaen : *Traité de Rythme, de Couleur et d'Ornithologie,* tome 2, Alphonse Leduc, Paris, 1992

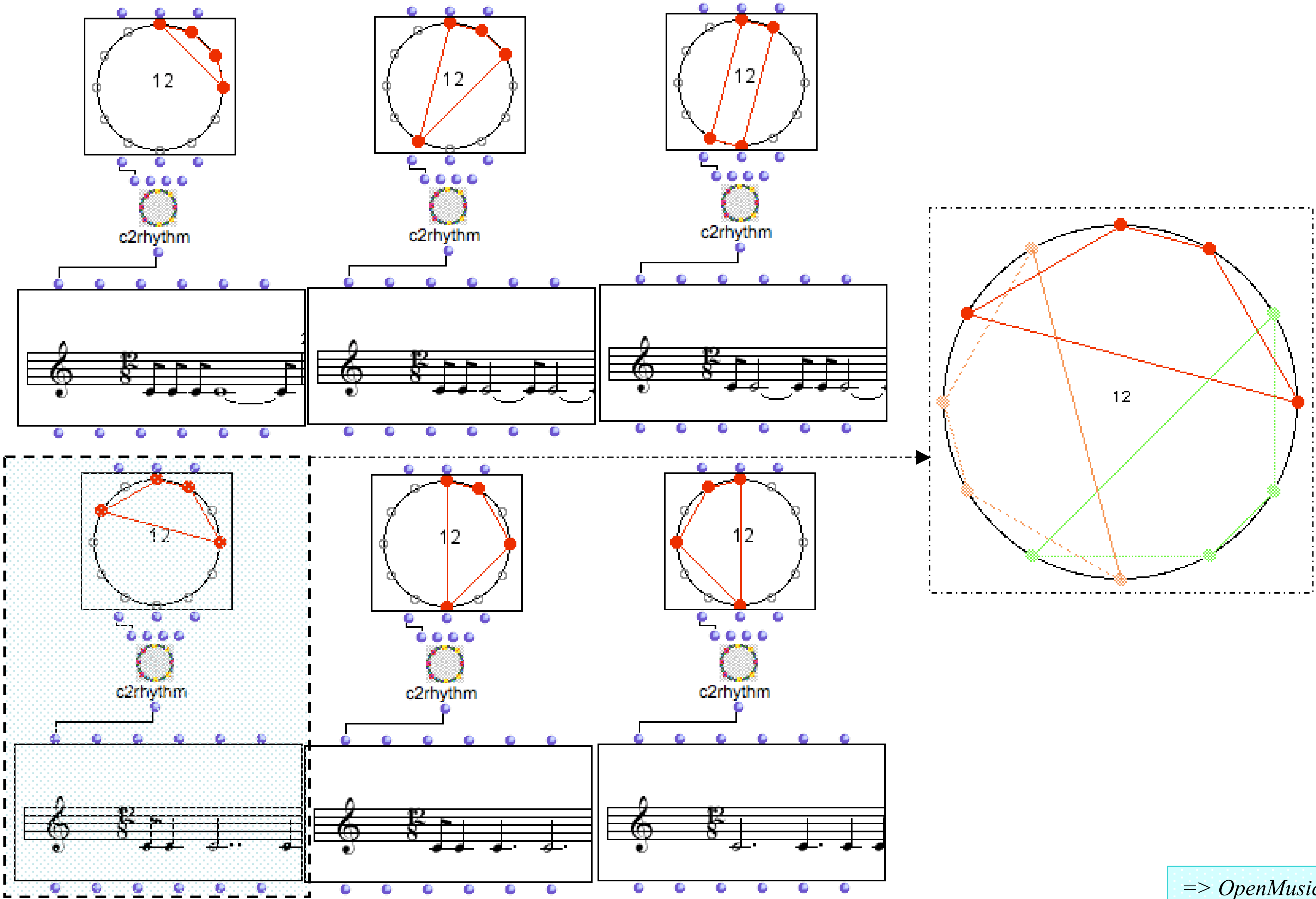
# Cos'è un canone ritmico a mosaico: 4 famiglie



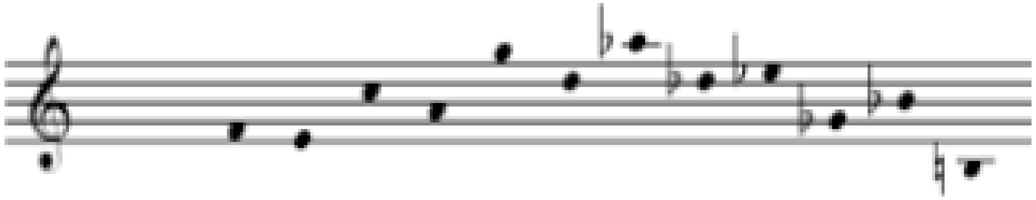



# 3-asymmetric rhythmic pattern and tiling process

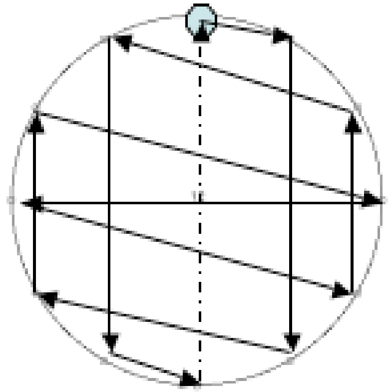



=> *OpenMusic*

# Canoni ritmici a mosaico e serie omni-intervallari

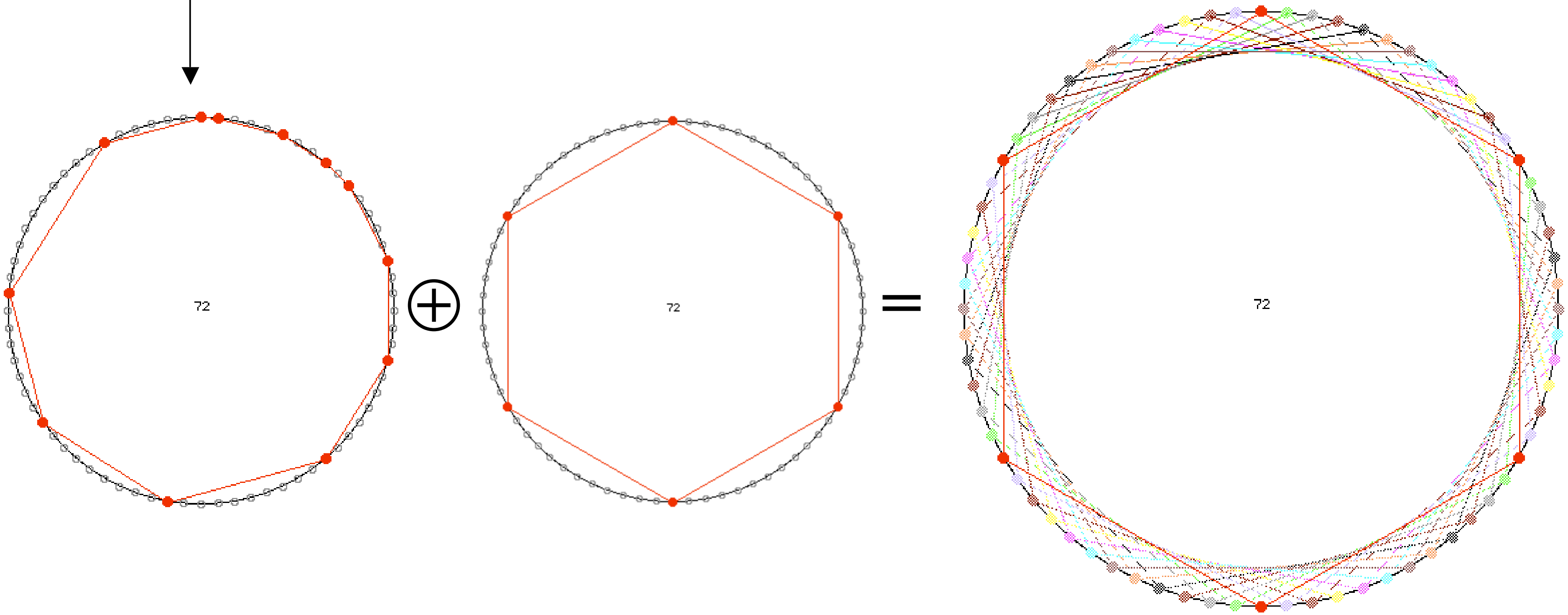


$S = \{0,11,7,4,2,9,3,8,10,1,5,6)$



$S^* = (1,4,3,2,5,6,7,10,9,8,11)$

mod 72



# Canoni ritmici a mosaico e serie dodecafoniche

. . .
. . . 25
**24**
13
**12**

$S$={**0**, 1, 3, 6, 7, 9, 8, 10, 11, 2, 4, 5)

↑

$S^*$=(1,2,3,1,2,11,2,1,3,2,1,45) ←

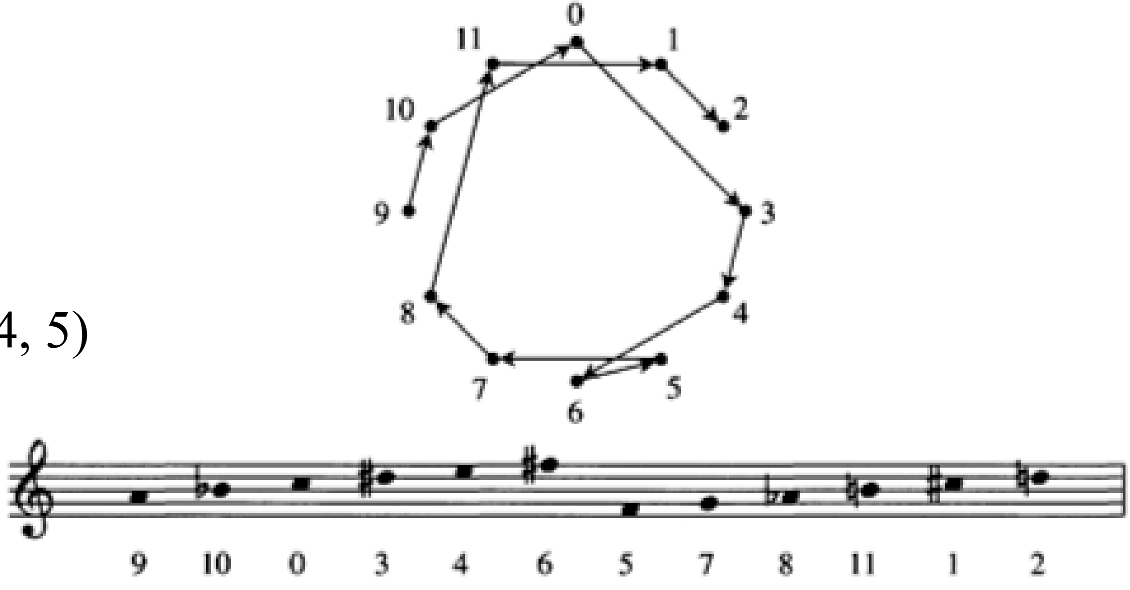
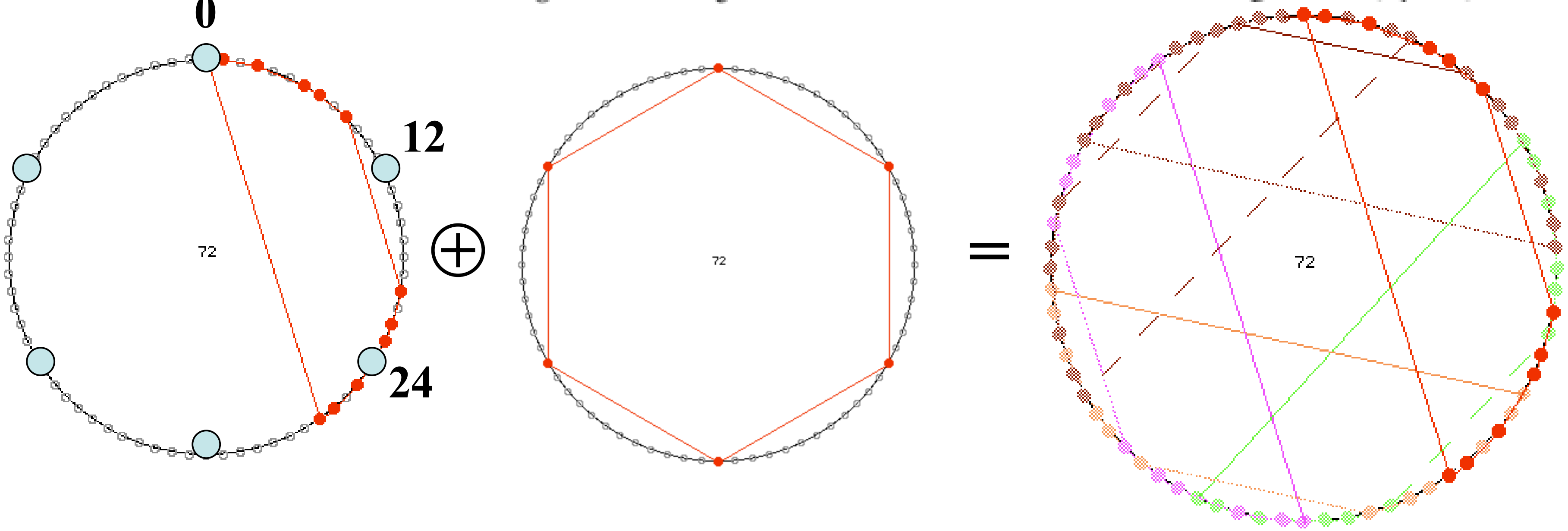
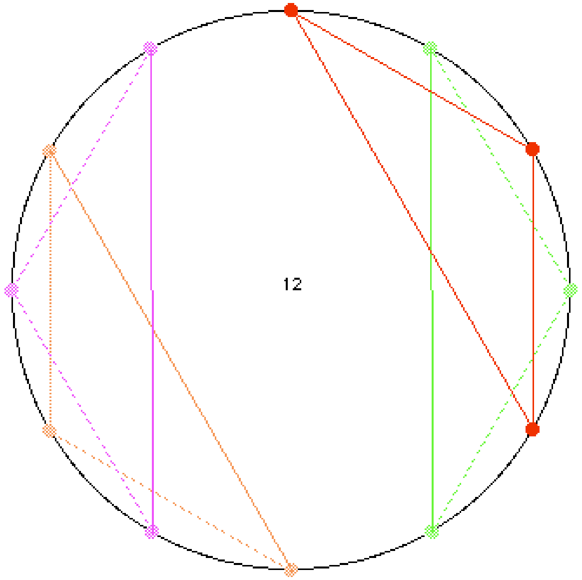
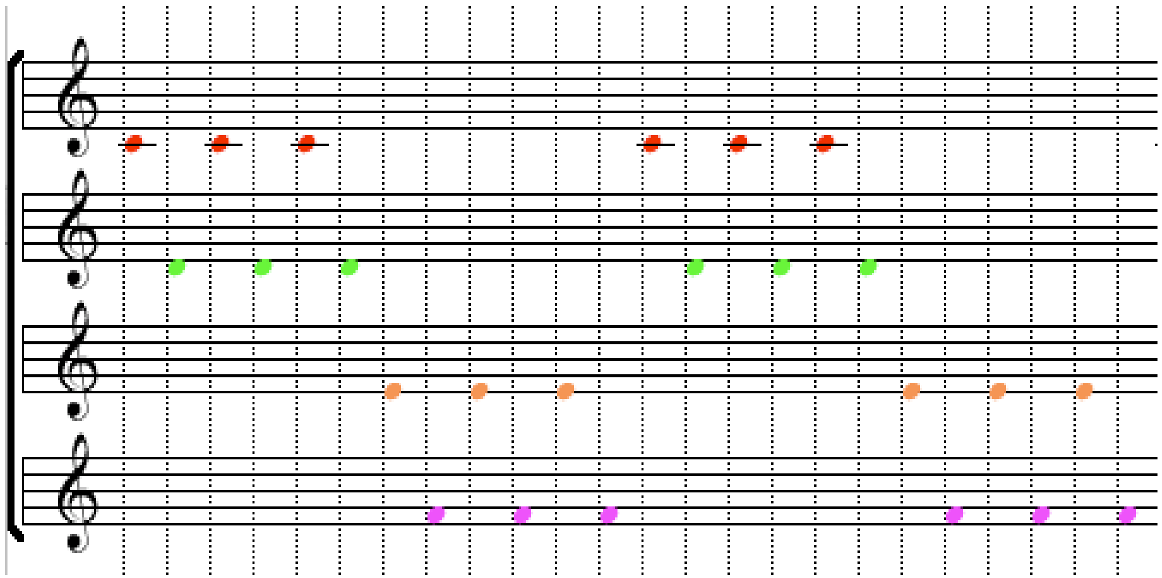


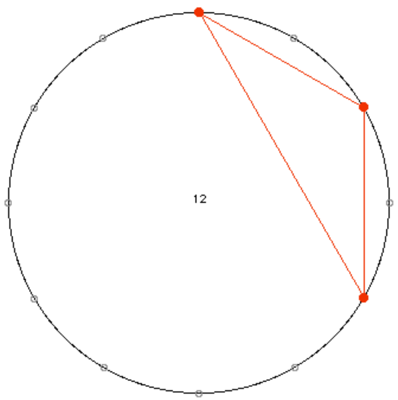

**Figure 1.** Clock diagram and musical notation for the row from Schoenberg's Serenade, opus 24, movement 5.



# Canoni a mosaico a simmetria trasposizionale



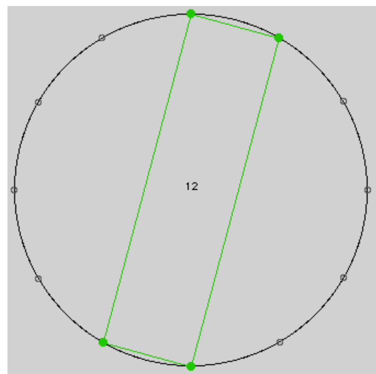





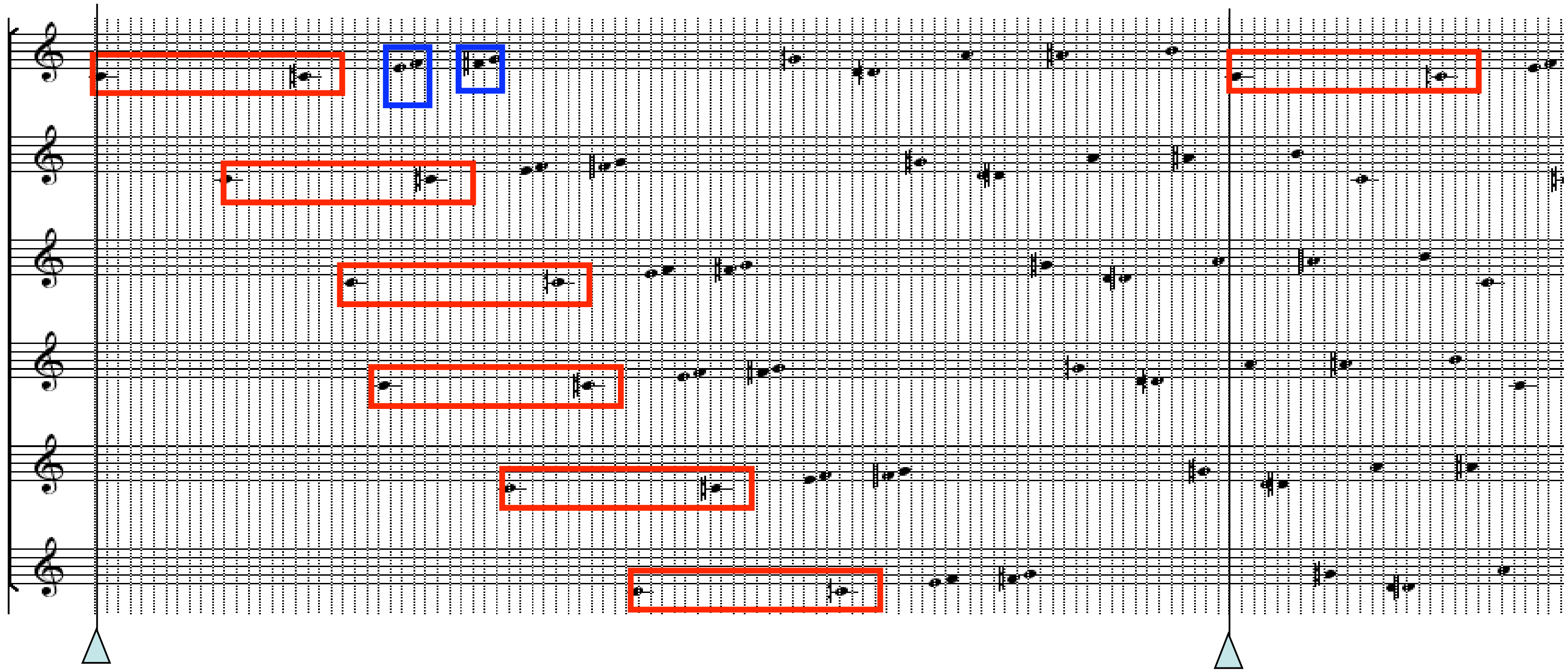

$\mathbf{Z}_{12} = A \oplus B$

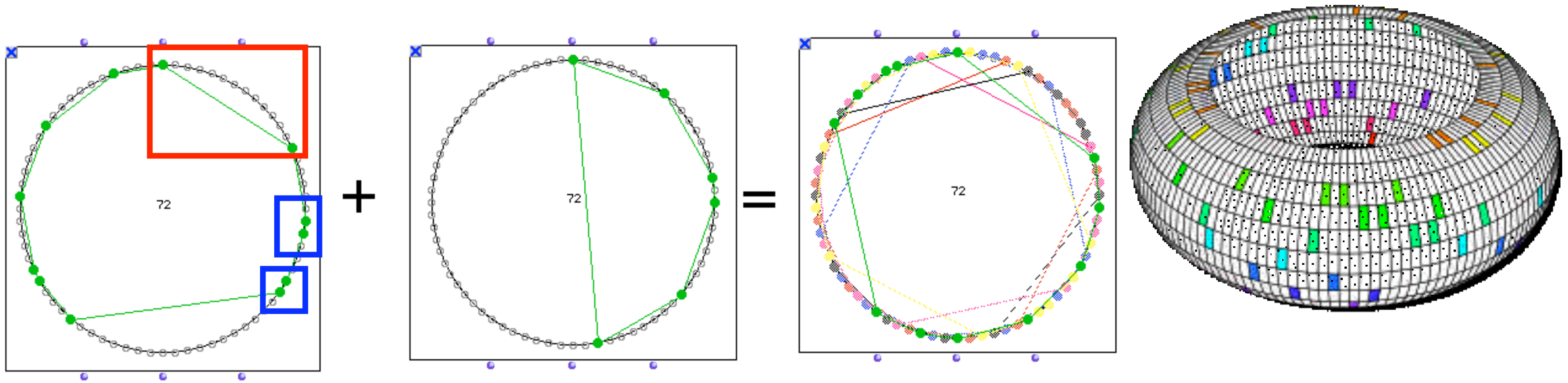
$A = \{0,2,4\}$

$B = \{0,1,6,7\}$



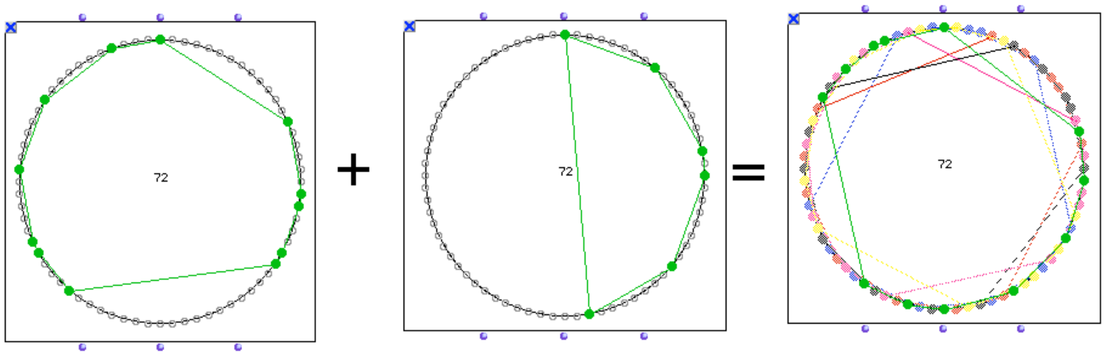
# *Vuza Canons* : canoni a mosaico senza periodicità interne

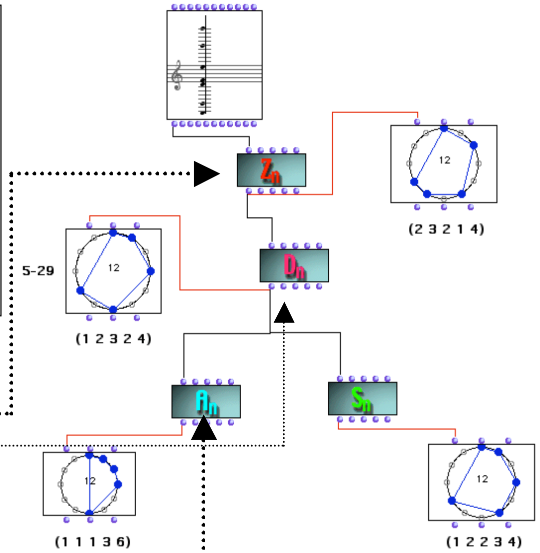




# Paradigmatic classification of Vuza Canons

=> *OpenMusic*




**R**

(1 3 3 6 11 4 9 6 5 1 3 20)
(20 3 1 5 6 9 4 11 6 3 3 1)
(1 4 1 19 4 1 6 6 7 4 13 6)
(6 13 4 7 6 6 1 4 19 1 4 1)
(1 5 15 4 5 6 6 3 4 17 3 3)
(3 3 17 4 3 6 6 5 4 15 5 1)

(8 8 2 8 8 38)
(16 2 14 2 16 22)
(14 8 10 8 14 18)

**S**

**R**

(1 3 3 6 11 4 9 6 5 1 3 20)
(1 4 1 19 4 1 6 6 7 4 13 6)
(1 5 15 4 5 6 6 3 4 17 3 3)

(8 8 2 8 8 38)
(16 2 14 2 16 22)
(14 8 10 8 14 18)

**S**

**R**

(1 3 3 6 11 4 9 6 5 1 3 20)
(1 4 1 19 4 1 6 6 7 4 13 6)

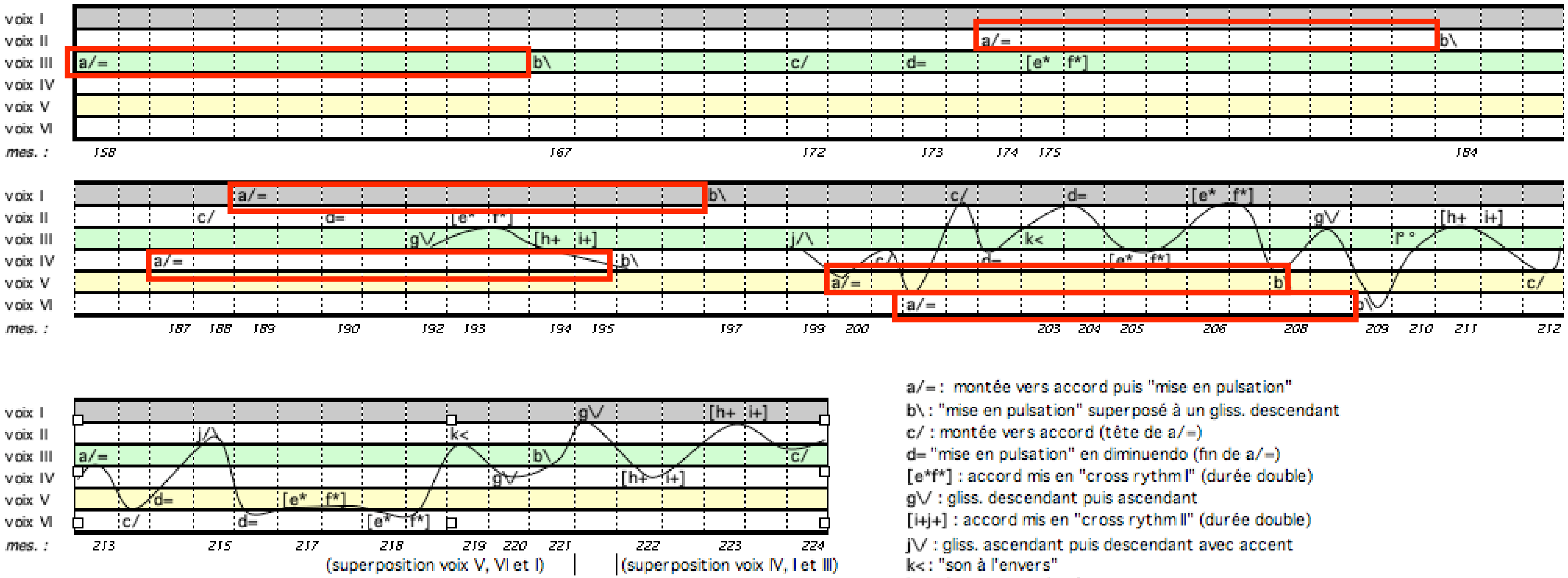
(14 8 10 8 14 18)

**S**

# Fabien Lévy

## Canoni di Vuza su gesti strumentali complessi

• *Coïncidences* (pour 33 musiciens, 1999-2007)



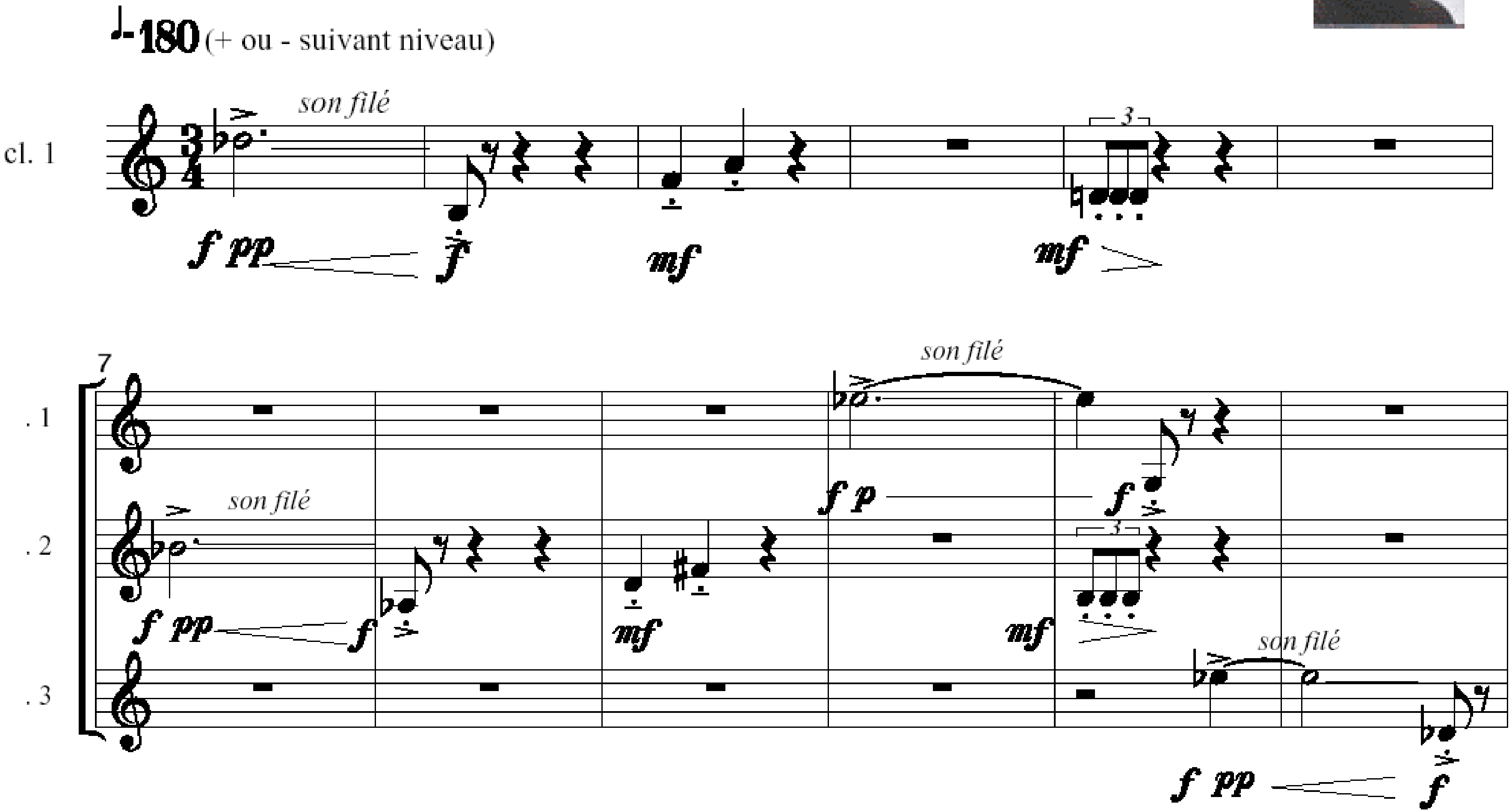

a/= : montée vers accord puis "mise en pulsation"
b\ : "mise en pulsation" superposé à un gliss. descendant
c/ : montée vers accord (tête de a/=)
d= "mise en pulsation" en diminuendo (fin de a/=)
[e*f*] : accord mis en "cross rythm I" (durée double)
g\/ : gliss. descendant puis ascendant
[i+j+] : accord mis en "cross rythm II" (durée double)
j/\ : gliss. ascendant puis descendant avec accent
k< : "son à l'envers"
l° ° : deux impacts brefs et piano

Coïncidences - Fabien Levy : déroulement du canon (mes. 158 à 226)
(chaque impact fait 3 temps)

**Interprètes : Tokyo Symphony Orchestra, Dir.: Kazuyoshi Akiyama, 05/09/2007, Suntory Hall, Tokyo, Japon**

# Fabien Levy

## Canones de Vuza come strumento pedagogico



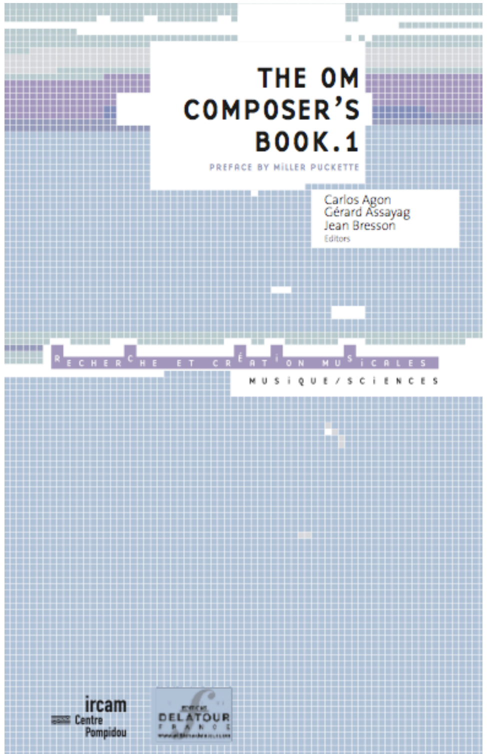

• *Où niche l’Hibou* (1999-2006)

# Georges Bloch

## Strategie compositive a partire da un modello formale

- Organisation métrique d'un canon mosaïque
- Réduction d'un canon par auto-similarité
- Modulation métrique entre canons
- Transformation d'un canon dans une texture
- Canons mosaïques et IAO (*OMax*)


- ***Projet Beyeler*** (2001)
- *Projet Hitchcock*
- *Visite des tours de la cathédrale de Reims*
- *Noël des Chasseurs*
- *Canons à marcher*
- *Canon à eau*
- *Harawun* (2004)
- *L'Homme du champ* (2005)
- *A piece based on Monk* (2007)
- *Peking Duck Soup* (2008)




- *A piece based on Monk* (2007) (« Well You Need'nt »)

# Georges Bloch (2000-2007)

## Compositional strategies starting from the formal model

- Metrical organization of a tiling canon
- Reduction of a tiling canon into self-similar canons
- Metric modulation between canons
- Transformation of a tiling canon into a texture


- *Projet Beyeler* (2001)
- *Projet Hitchcock*
- *Visite des tours de la cathédrale de Reims*
- *Noël des Chasseurs*
- *Canons à marcher*
- *Canon à eau*
- *Harawun* (2004)
- *L'Homme du champ* (2005)

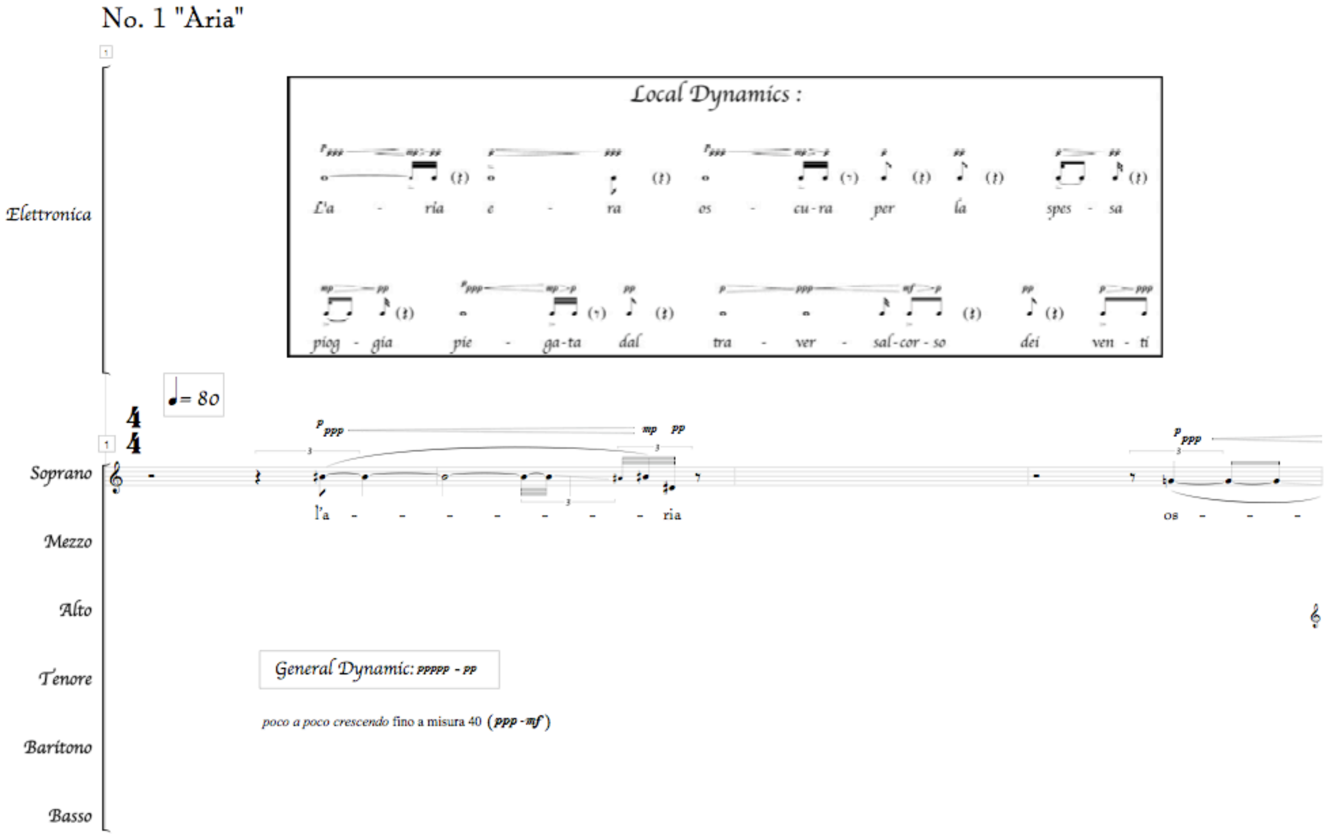
- *A piece based on Monk* (2005-2007)

# Mauro Lanza

## Canoni di Vuza e periodicità locali

• *La descrizione del diluvio* (Ricordi, 2007-2008)

Canon à 14 voix sur le pattern rythmique :
**(1 3 25 27 1 3 11 14 27 1 3 25 27 4 25 27 1 3 25 14 13 1 3 25 27 1 3 52)**




*6 voix sont en live et 8 dans l'électronique. L'unité est la double-croche de triolet. Le choix des notes et des durées est fait en cherchant à souligner certaines* ***quasi-périodicités*** *du canon de Vuza, et cela donne à chaque voix un caractère beaucoup plus "redondant".*

# Mauro Lanza

## Canons de Vuza et périodicités locales

- *La descrizione del diluvio* (Ricordi, 2007-2008)

*[…] Le choix des notes et des durées est fait en cherchant à souligner certaines **quasi-périodicités** du canon de Vuza, et cela donne à chaque voix un caractère beaucoup plus "redondant".*

**(1 3 25 27 1 3 11 14 27 1 3 25 27 4 25 27 1 3 25 14 13 1 3 25 27 1 3 52)**

**(*1 3 25 27 1 3* 11 14 27 1 3 25 27 4 25 27 1 3 25 14 13 *1 3 25 27 1 3* 52)**

**(*1 3 25 27* 1 3 11 14 27 *1 3 25 27* 4 25 27 1 3 25 14 13 *1 3 25 27* 1 3 52)**

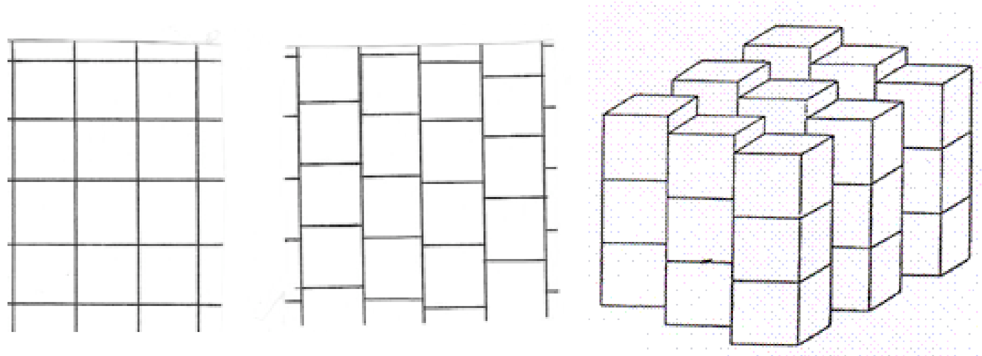
**(*1 3 25* 27 1 3 11 14 27 *1 3 25* 27 4 25 27 *1 3 25* 14 13 *1 3 25* 27 1 3 52)**

**(*1 3* 25 27 *1 3* 11 14 27 *1 3* 25 27 4 25 27 *1 3* 25 14 13 *1 3* 25 27 *1 3* 52)**

# Canoni ritmici a mosaico: un problema « matemusicale »

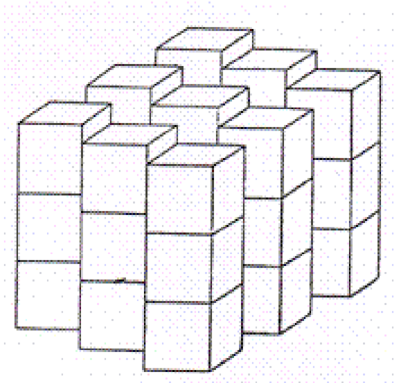
- Olivier Messiaen's 'formalization' of rhythmic canons
- Dan Tudor Vuza's model of Regular Complementary Canons of Maximal Category (*Perspectives of New Music*, 1991-1993)
- The computer-aided model of Vuza Canons and first catalogues of solutions (Agon&Andreatta, 1999)
- Compositional applications of the model (by Fabien Levy, Georges Bloch)
- Enumeration and classifications of Vuza canons (Fripertinger, Amiot, Noll, Andreatta, Tangian, Jedrzejewski)
- Thomas Noll's generalized model of augmented tiling canons
- Emmanuel Amiot's model of cyclotomic tiling canons
- The *MathTools* environment in *OpenMusic* (Agon&Andreatta)

- Minkowski's Conjecture (1896/1907)
- Hajos algebraic solution (1942)
- The classification of Hajos groups (Hajos, de Bruijn, Sands, …)
- The Tiling of the line problem and Fuglede's Conjecture (Tijdeman,Lagarias, Laba, Coven-Meyerowitz, Kolountzakis…)
- Fuglede's Conjecture and Vuza's Canons (Amiot)



*In a simple lattice tiling of the n-dimensional space by unit cubes, at least one couple of cubes share a complete n-1 dimensional face*
(Cf. S. Stein, S. Szabó : *Algebra and Tiling,* 1994)

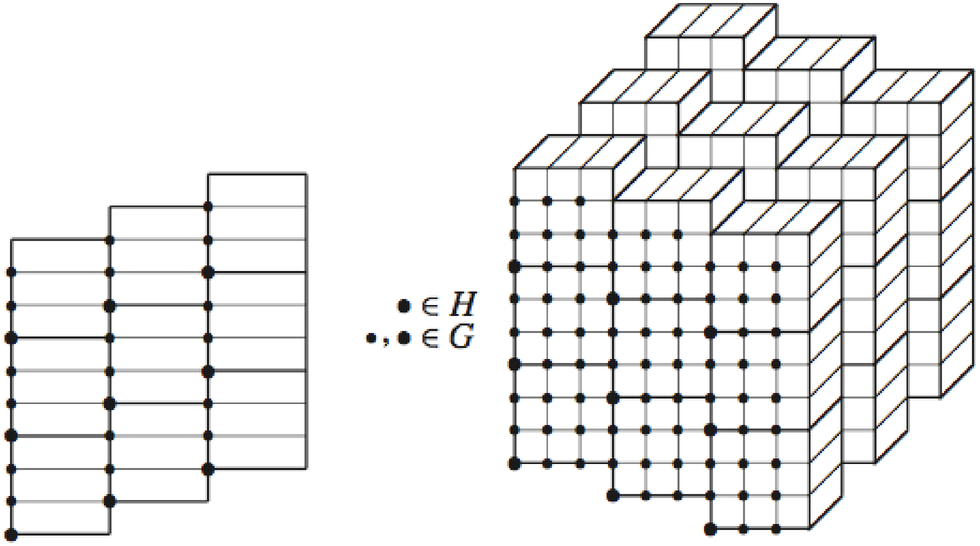
# Congettura di Minkowski e formulazione algebrica



***Congettura di Minkowski (1896/1907)***

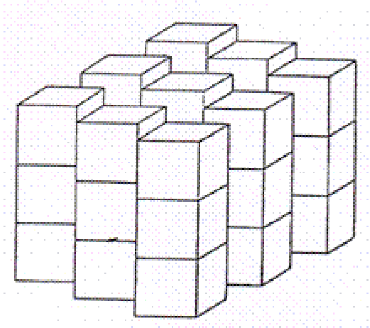
Dato un ricoprimento **reticolare** (= l'insieme delle straslazioni è un reticolo) a cubi dello spazio euclideo n-dimensionale, esiste una coppia di cubi (*twins*) che hanno in comune una faccia di dimensione n-1.

G. Fidanza, *Canoni ritmici a mosaico*, tesi di laurea, 2006/2007




- $H$ = reticolo formato dai vertici di minor coordinate (a valori in **Q** senza perdita di generalità)
- $G$ = vertici di minor coordinate dei parallelepipedi che dividono ogni cubo in un numero finito
- $H<G$
- $\exists\{a_1, \dots, a_n\}$ base di $G$ tale che $m_i a_i = e_i$ $\forall i=1,\dots,n$ ove $m_i$ è il numero di fette in cui viene diviso ogni cubo lungo l'i-esima coordinata
- *Considero* $G/H$ e per ogni i costruisco $A_i =\{0, a_i, 2a_i , \dots, m_i -1)a_i\}$
- $G/H= A_1 \oplus A_2 \oplus \dots \oplus A_n$

# Conjecture de Minkowski et théorème de Hajós



***Conjecture de Minkowski (1896/1907)***
*Dans un pavage simple [simple lattice tiling] d'un espace à n dimensions par des cubes unités, il y a au moins un couple de cubes qui ont en commun une face entière de dimension n-1.*

***Théorème de Hajós (1942)***

Soit $G$ un groupe abélien fini et soient $a_1, a_2, \ldots, a_n$ n éléments de $G$. Si l'on suppose que le groupe admet comme factorisation la somme directe des sous-ensembles $A_1 \ldots A_n$

$$A_1 = \{1, a_1, \ldots, a_1^{m_1-1}\}, A_2 = \{1, a_2, \ldots, a_2^{m_2-1}\}, \ldots, A_n = \{1, a_n, \ldots, a_n^{m_n-1}\}$$

avec $m_i > 0$ pour tout i=1, 2, …, $n$, alors un des $A_i$ est un groupe

***Théorème de Redei (1965)***

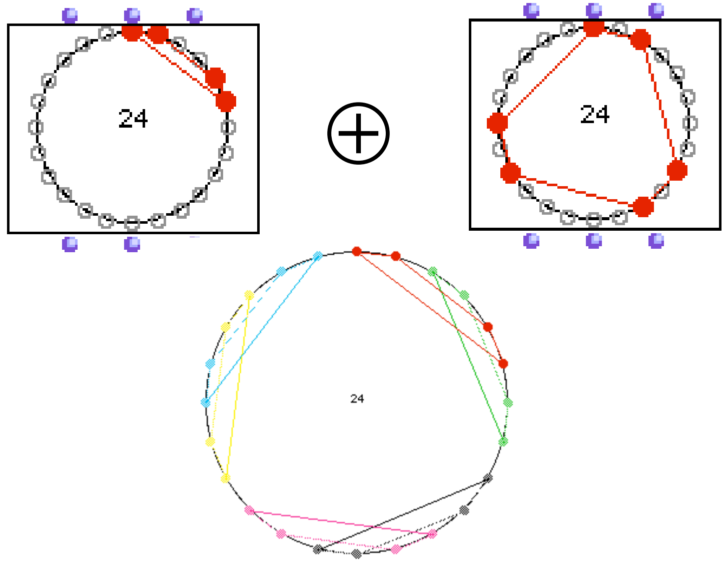
Soit $G$ un groupe abélien fini et soient $A_1, A_2, \ldots, A_n$ $n$ sous-ensembles de $G$, chacun contenant l'élément neutre du groupe et chacun ayant un nombre premier d'éléments et supposons que le groupe admette comme factorisation la somme directe des sous-ensembles $A_i$, i=1, …, $n$.
Alors, un des sous-ensembles $A_i$ est **périodique**

Cf. S. Stein, S. Szabó : *Algebra and Tiling. Homomorphisms in the service of Geometry,* Carus Math. Monographs, 1994.

# Gruppi di Hajos e periodicità dei fattori

A group $G$ is an "Hajós group" if for all factorisation of $G$ into a direct sum of subsets $A_1, A_2, \ldots, A_k$, at least one of the factors is periodic.

| | |
|---|---|
| Rédei 1947 | $(p, p)$ |
| Hajós 1950 | **Z** |
| | **Z/nZ** avec n= $p^{\alpha}$ |
| De Brujin 1953 | $(p^{\alpha}, q)$ |
| | $(p, q, r)$ |
| Sands 1957 | $(p^2, q^2)$ |
| | $(p^2, q, r)$ |
| | $(p, q, r, s)$ |
| Sands 1959 | $(2^2, 2^2)$ |
| | $(3^2, 3)$ |
| | $(2^n, 2)$ |
| Sands 1962 | $(p, 3, 3)$ |
| | $(p, 2^2, 2)$ |
| | $(p, 2, 2, 2, 2)$ |
| | $(p^2, 2, 2, 2)$ |
| | $(p^3, 2, 2)$ |
| | $(p, q, 2, 2)$ |
| Sands 1964 | **Q** |
| | **Z+Z/pZ** |
| | **Q+Z/pZ** |




**Groupes non-Hajós (bad groups)**

**72**
108 120 144 168 180
200 216 240 252 264 270 280 288
300 312 324 336 360 378 392 396
400 408 432 440 450 456 468 480
500 504 520 528 540 552 560 576 588 594
600 612 616 624 648 672 675 680 684 696
700 702 720 728 744 750 756 760 784 792
800 810 816 828 864 880 882 888…

# Radici dell'unità e polinomi ciclotomici

Racines *n*-ièmes de l'unité : $z^n = 1$

$n$=3 ⟶ $\left\{1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\right\}$

$n$=4 ⟶ $\{1, +i, -1, -i\}$

Le racines *n*-ièmes de l'unité peuvent s'écrire sous la forme :

$$e^{\frac{2k\pi i}{n}} = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \qquad (k, n \in \mathbb{N} \text{ et } 0 \leq k < n)$$

Elles sont exactement les racines du polynôme : $P(X) = X^n - 1$

Le racines *n*-ièmes primitives de l'unité : $e^{\frac{2ki\pi}{n}}$ $(n,k)=1$

Elles sont exactement les racines du polynôme cyclotomique :

$$\Phi_n(X) = \prod_{k=1}^{\varphi(n)} (X - z_k) \longleftrightarrow X^n - 1 = \prod_{d|n} \Phi_d(X).$$

# Polinomi ciclotomici e mosaici

$$\Phi_n(X) = \prod_{k=1}^{\varphi(n)} (X - z_k) \longleftrightarrow X^n - 1 = \prod_{d|n} \Phi_d(X).$$

$\Phi_1(X) = X - 1 \longleftrightarrow (-1, 1)$

$\Phi_2(X) = 1 + X \longleftrightarrow (1, 1)$

$\Phi_3(X) = 1 + X + X^2 \longleftrightarrow (1, 1, 1)$

$\Phi_4(X) = 1 + X^2 \longleftrightarrow (1, 0, 1)$

$\Phi_5(X) = 1 + X + X^2 + X^3 + X^4 \longleftrightarrow (1, 1, 1, 1, 1)$

$\Phi_6(X) = 1 - X + X^2 \longleftrightarrow (1, -1, 1)$

$$\Delta_n = 1 + X + X^2 + \ldots + X^{n-1} = \prod_{\substack{d|n \\ d \neq 1}} \Phi_d(X)$$

$$\Delta_4 = 1 + X + X^2 + X^3 = \Phi_2(X) \times \Phi_4(X)$$

$$A(x) \times B(x) = (A \oplus B)(x) \equiv 1 + x + \ldots x^{n-1} \pmod{X^n - 1}.$$

# Buone e cattive fattorizzazioni

$$\Delta_n = 1 + X + X^2 + \ldots + X^{n-1} = \prod_{\substack{d \mid n \\ d \neq 1}} \Phi_d(X)$$

$\Phi_2(X) = 1 + X$ ⟷ $(1, 1)$

$\Phi_3(X) = 1 + X + X^2$ ⟷ $(1, 1, 1)$

$\Phi_4(X) = 1 + X^2$ ⟷ $(1, 0, 1)$

$\Phi_6(X) = 1 - X + X^2$ ⟷ $(1, -1, 1)$

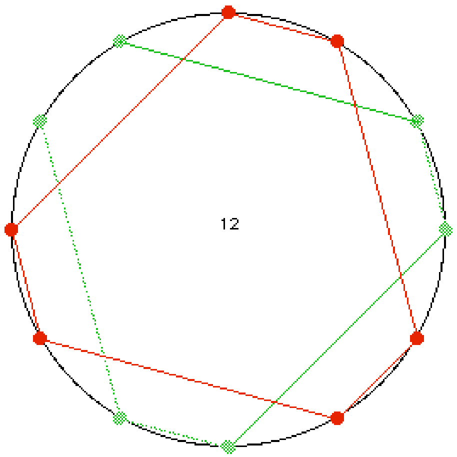
$$\Delta_{12} = 1 + X + \ldots + X^{11} = \Phi_2 \times \Phi_3 \times \Phi_4 \times \Phi_6 \times \Phi_{12}$$

$A(X) = \Phi_2 \times \Phi_3 \times \Phi_6 \times \Phi_{12} = 1 + X + X^4 + X^5 + X^8 + X^9$

$B(X) = \Phi_4 = 1 + X^2$

→ $S = \{0, 2\}$

→ $R = \{0, 1, 4, 5, 8, 9\}$




$A^*(X) = \Phi_2 \times \Phi_3 \times \Phi_{12}$

$B^*(X) = \Phi_4 \times \Phi_6$

Questa decomposizione non funziona!

# Condizioni di Coven-Meyerowitz

- E. Coven & A. Meyerowitz : "Tiling the integers with translates of one finite set", *J. Algebra*, 212, pp.161-174, 1999

There is no loss of generality in restricting attention to translates of a finite set $A$ of *nonnegative* integers. Then $A(x) = \sum_{a \in A} x^a$ is a polynomial such that $\#A = A(1)$. Let $S_A$ be the set of prime powers $s$ such that the $s$-th cyclotomic polynomial $\Phi_s(x)$ divides $A(x)$. Consider the following conditions on $A(x)$.

(T1) $A(1) = \prod_{s \in S_A} \Phi_s(1)$.

(T2) If $s_1, \ldots, s_m \in S_A$ are powers of distinct primes, then $\Phi_{s_1 \cdots s_m}(x)$ divides $A(x)$.

**Theorem A.** *If $A(x)$ satisfies (T1) and (T2), then $A$ tiles the integers.*

**Theorem B1.** *If $A$ tiles the integers, then $A(x)$ satisfies (T1).*

**Theorem B2.** *If $A$ tiles the integers and $\#A$ has at most two prime factors, then $A(x)$ satisfies (T2).*

**Corollary.** *If $\#A$ has at most two prime factors, then $A$ tiles the integers if and only if $A(x)$ satisfies (T1) and (T2).*

# Condizioni di Coven-Meyerowitz

(T1) $A(1) = \prod_{s \in S_A} \Phi_s(1)$.
(T2) If $s_1, \ldots, s_m \in S_A$ are powers of distinct primes, then $\Phi_{s_1 \cdots s_m}(x)$ divides $A(x)$.

**Theorem A.** *If $A(x)$ satisfies (T1) and (T2), then A tiles the integers.*

$$A(X) = \Phi_2 \times \Phi_3 \times \Phi_6 \times \Phi_{12} = 1 + X + X^4 + X^5 + X^8 + X^9$$

$$\Phi_2(X) = 1 + X$$

$$\Phi_3(X) = 1 + X + X^2$$

(T1) $A(1) = 6 = \Phi_2(1) \times \Phi_3(1) = 2 \times 3$

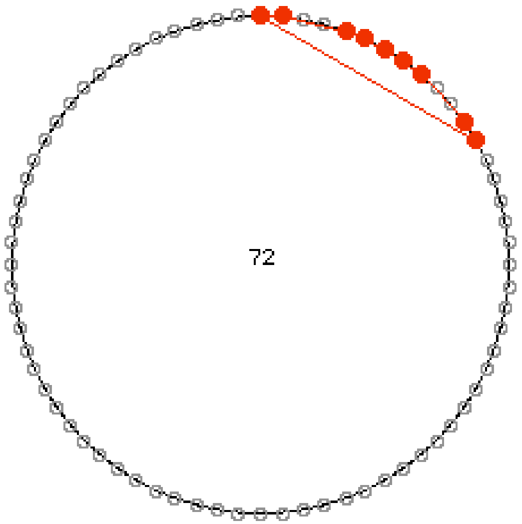
(T2) $\Phi_2 \mid A(X)$ et $\Phi_3 \mid A(X) \Rightarrow \Phi_{2\times 3} \mid A(X)$

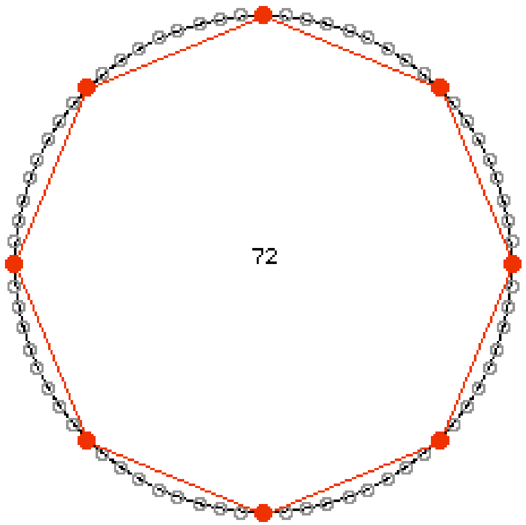
**Theorem B1.** *If A tiles the integers, then $A(x)$ satisfies (T1).*

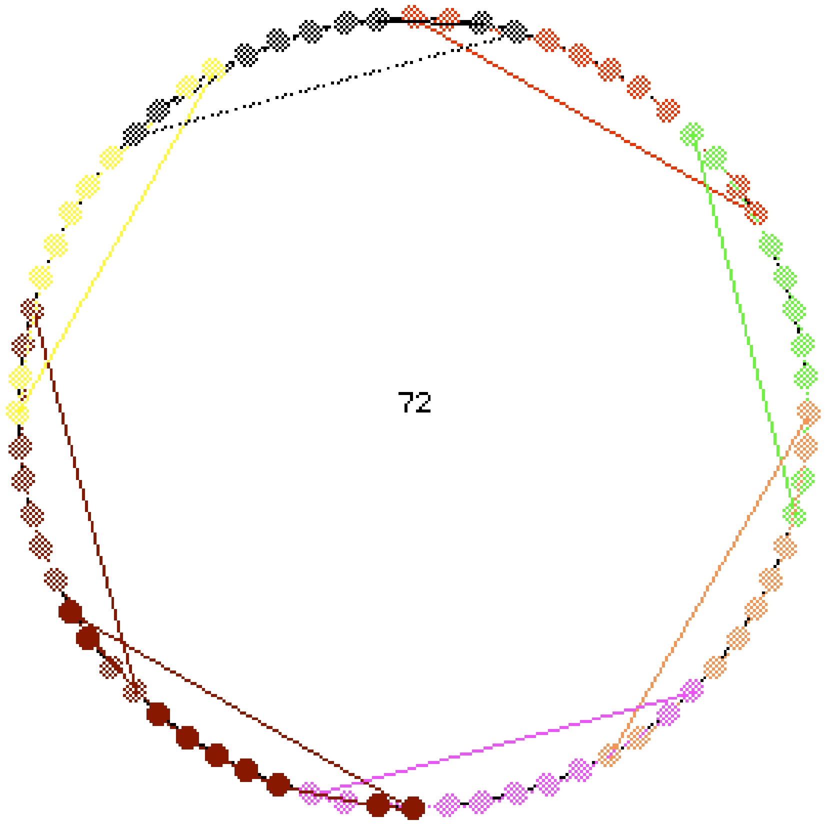
$$A^*(X) = \Phi_2 \times \Phi_3 \times \Phi_{12} = 1 + 2X + 2X^2 - X^3 - X^4 + X^5 + 2X^6 + X^7$$

$$A^*(1) = 7 \neq \Phi_2 \times (1) \times \Phi_3(1) = 6$$

# La classe dei canoni ciclotomici




- E. Amiot, M. Andreatta, C. Agon: « Tiling the (musical) line with polynomial: some theoretical and implementational aspects », *ICMC*, Barcelona, 2005, pp.227-230.

# Le congetture di Minkowski/Fuglede e i canoni ritmici a mosaico

- Minkowski's conjecture (1896/1907)
- Hajos algebraic solution (1942)
- Hajos quasi-periodic conjecture
- The classification of Hajos groups (Hajos, de Bruijn, Sands, …)
- Classification of factorizations for non-Hajos groups (Vuza, Andreatta, Agon, Amiot, Fripertinger, …)
- …

- The Tiling of the line problem and Fuglede's Conjecture (**Tijdeman**, **Coven-Meyerowitz**, Lagarias, **Laba**, Kolountzakis…)
- Given a finite set that tiles **Z**, what will be the period (Kolountzakis, Steinberger, …)
- Fuglede's Conjecture and Vuza's Canons (Amiot, 2004)
- …

• R. Tijdeman: "Decomposition of the Integers as a direct sum of two subsets", *Number Theory*, Cambridge University Press, 1995. The fundamental Lemma:
A tiles $\mathbf{Z}_n$ => $p$A tiles $\mathbf{Z}_n$ when $<p,n>=1$

• I. Laba : "The spectral set conjecture and multiplicative properties of roots of polynomials", *J. Lond Math Soc*, 2002
T1 + T2 => spectral
T2 => spectral
spectral => T1

• E. Coven & A. Meyerowitz: "Tiling the integers with translates of one finite set", *J. Algebra*, 212, pp.161-174, 1999
T1 + T2 => tile
Tile => T1

• E. Amiot : "A propos des canons rythmiques", *Gazette des Mathématiciens,* n°106, Octobre 2005.
if A tiles with period $n$ and $\mathbf{Z}_n$ is Hajos => A has T2 (=> A is spectral)

Se A tassella ma non è spettrale => A è il ritmo di un canone di Vuza

# Congettura di Fuglede e canoni di Vuza

WOLFRAM RESEARCH mathworld.wolfram.com

## Fuglede's Conjecture

CONTRIBUTE TO THIS ENTRY

*Portions of this entry contributed by Emmanuel Amiot*

Fuglede (1974) conjectured that a domain $\Omega$ admits an operator spectrum iff it is possible to tile $\mathbb{R}^d$ by a family of translates of $\Omega$. Fuglede proved the conjecture in the special case that the tiling set or the spectrum are lattice subsets of $\mathbb{R}^d$ and Iosevich *et al.* (1999) proved that no smooth symmetric convex body $\Omega$ with at least one point of nonvanishing Gaussian curvature can admit an orthogonal basis of exponentials.

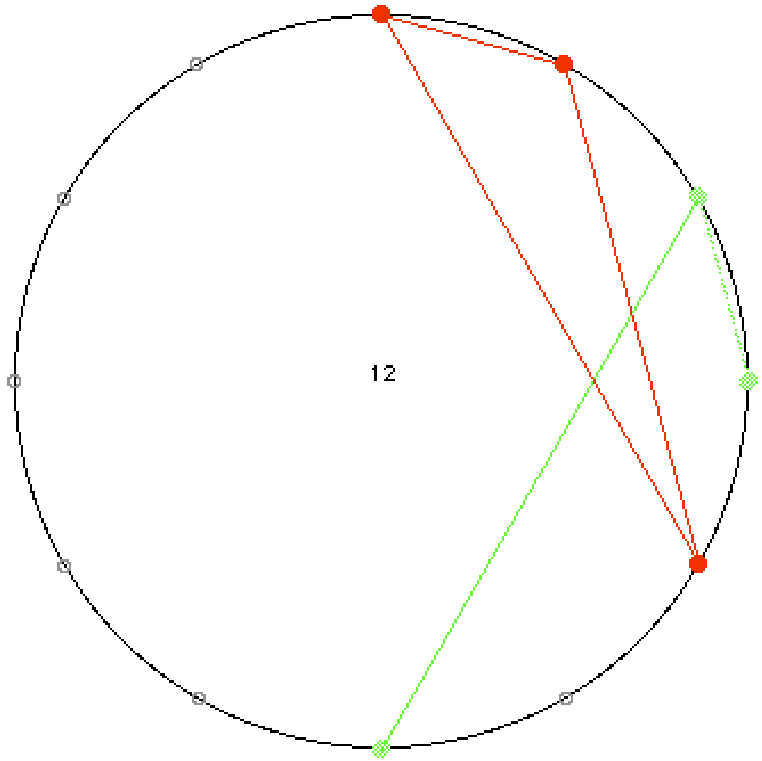
Using complex Hadamard matrices of orders 6 and 12, Tao (2003) constructed counterexamples to the conjecture in some small Abelian groups, and lifted these to counterexamples in $\mathbb{R}^5$ or $\mathbb{R}^{11}$.

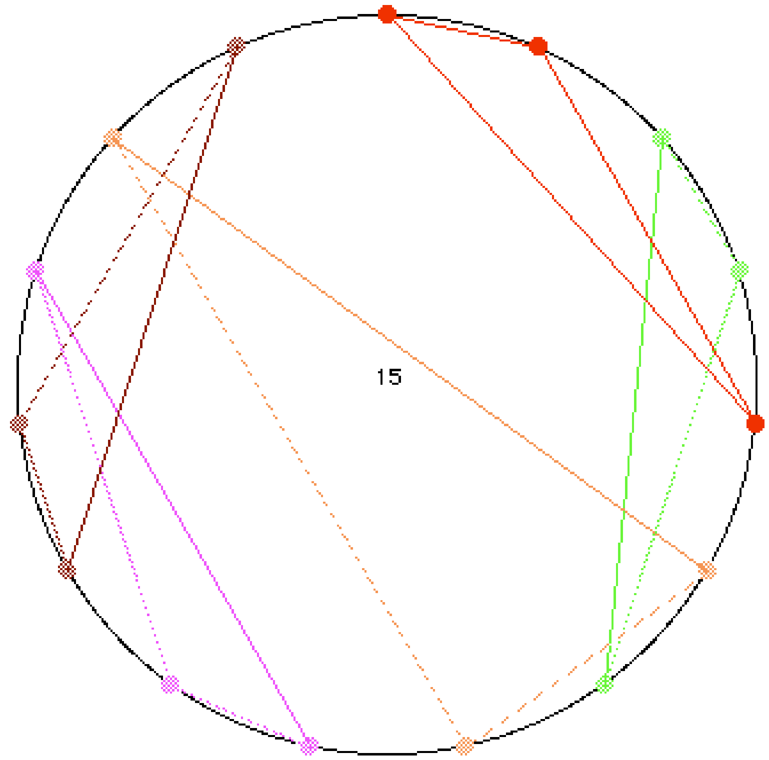
However, the conjecture has been proved in a great number of special cases (e.g., all convex bodies) and remains an open problem in small dimensions. For example, it has been shown in dimension 1 that a nice algebraic characterization of finite sets tiling $\mathbb{Z}$ indeed implies one side of Fuglede's conjecture (Coven-Meyerowitz 1998). Furthermore, it is sufficient to prove these conditions when the tiling gives a factorization of a non-Hajós cyclic group (Amiot).

# Tiling the line and/or circle with augmentations

- Tom Johnson (2001): tiling the line with a given rhythmic pattern
  - ex. (0 1 4). Does it tile? With augmentations? With which period?

• **Theorem (Amiot, 2002) :** *Any tiling of the line with the pattern* (0 1 4) *and its augmentations is periodic and the period is equal to a multiple of 15*



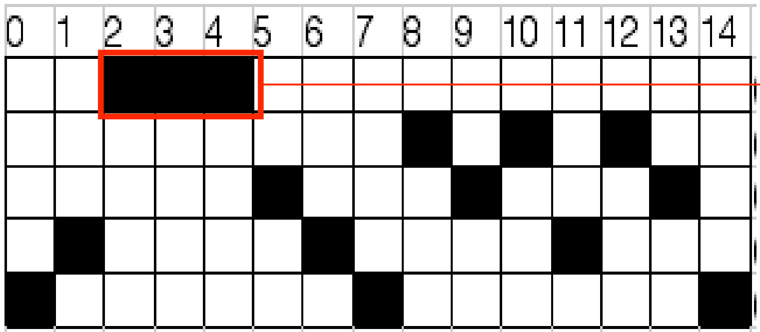



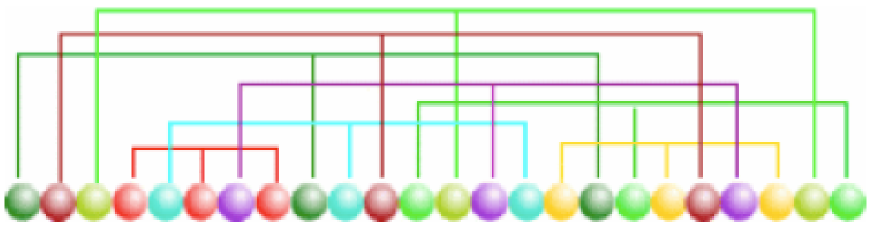

$n = 12$

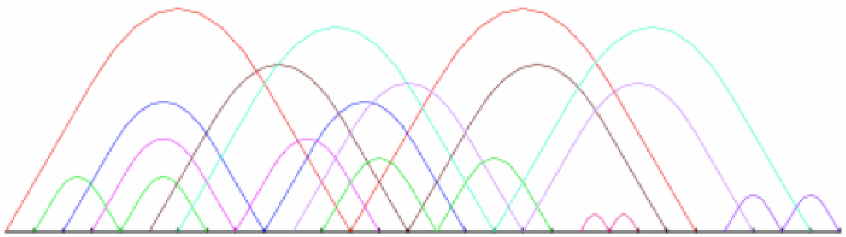
$n = 15$

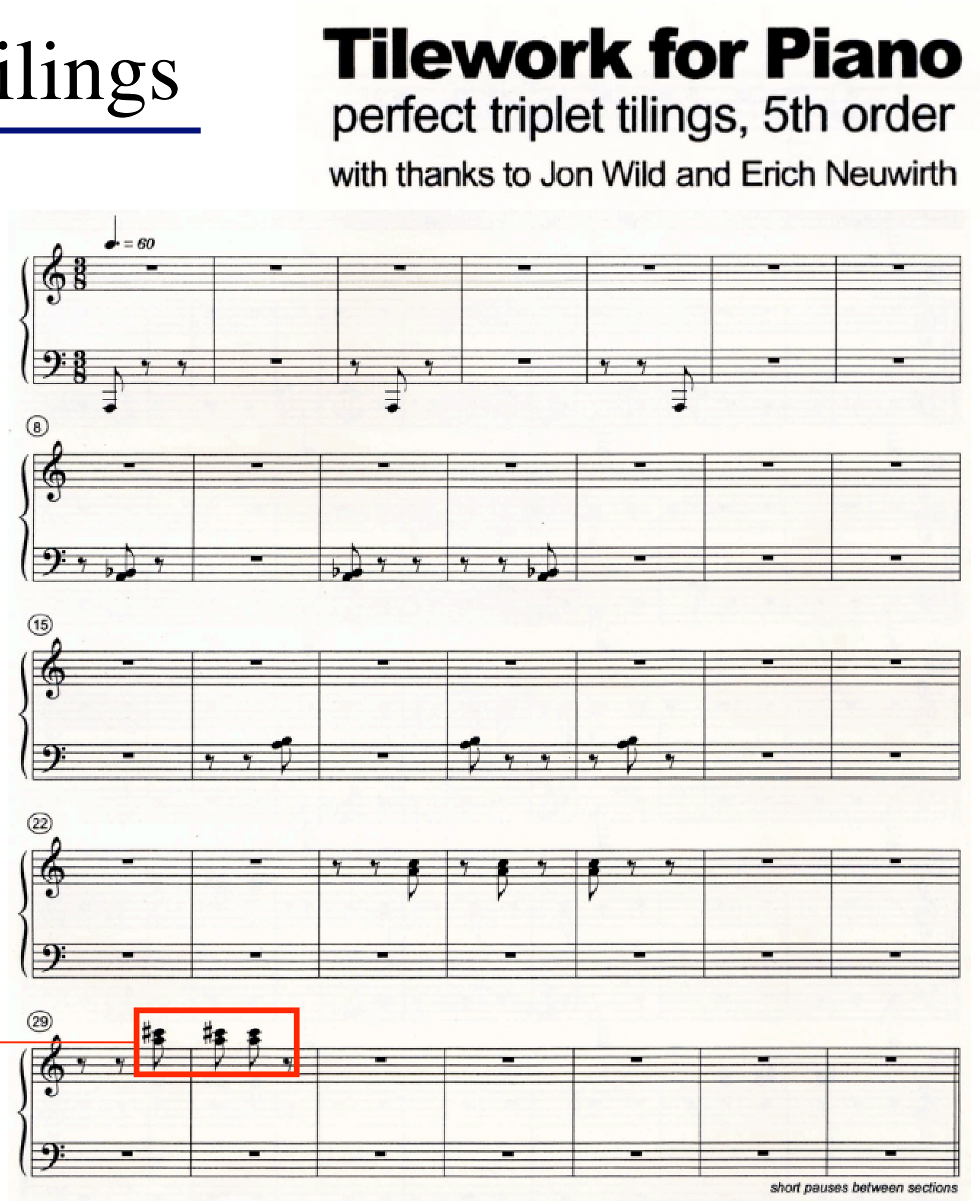
# Tom Johnson's « Self-Similar Melodies »

# Tom Johnson's Perfect Tilings



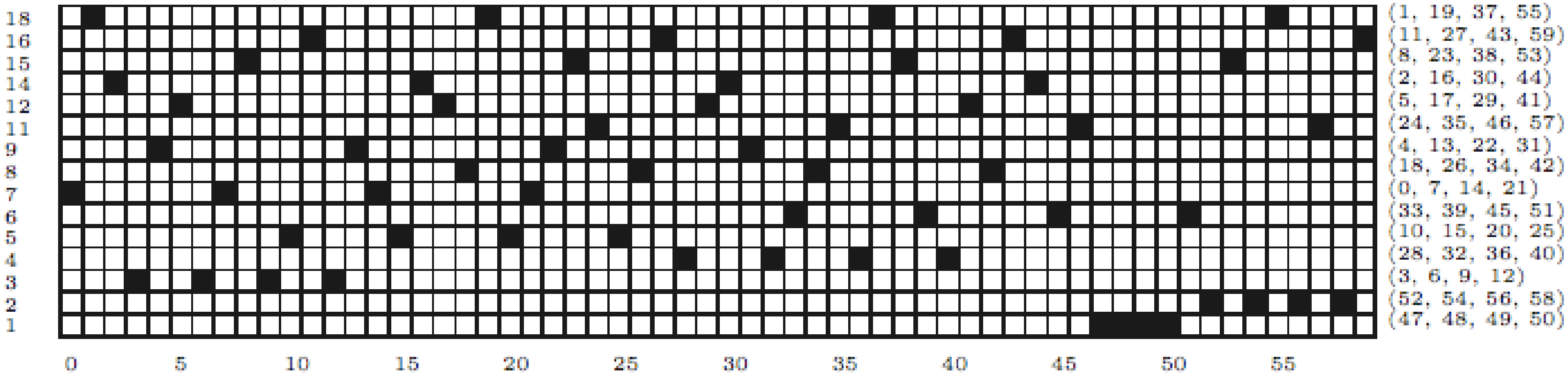



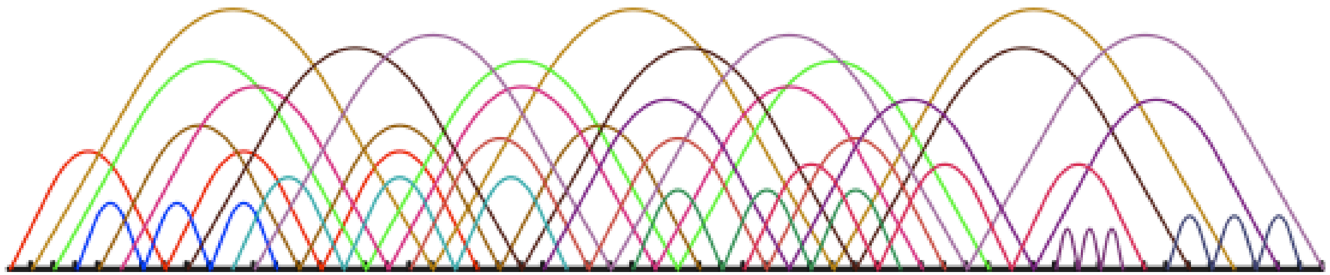




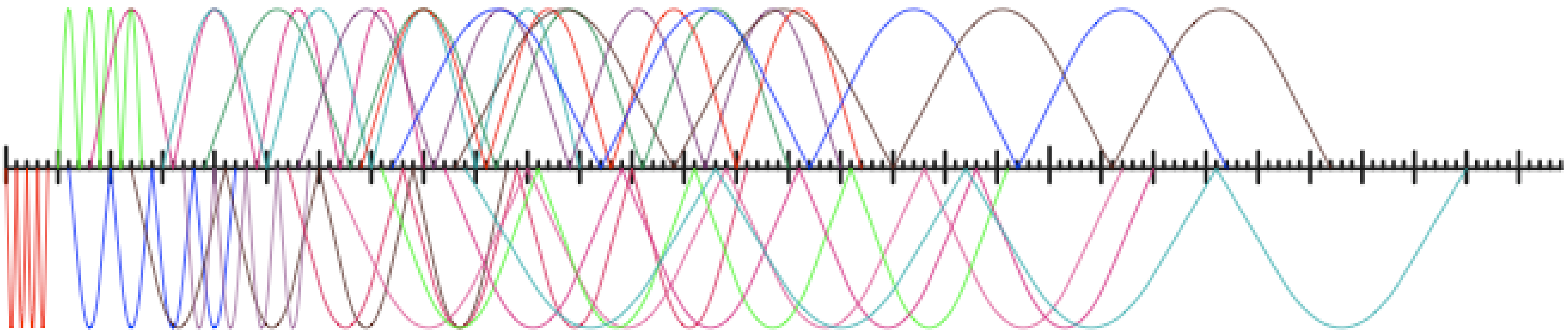

Jean-Paul Davalan, « Perfect Rhythmic Tilings » (to appear in *Tiling Problems in Music,* M. Andreatta & C. Agon eds., Collection « Musique/Sciences », 2008)

# Perfect Rhythmic Tilings and open problems

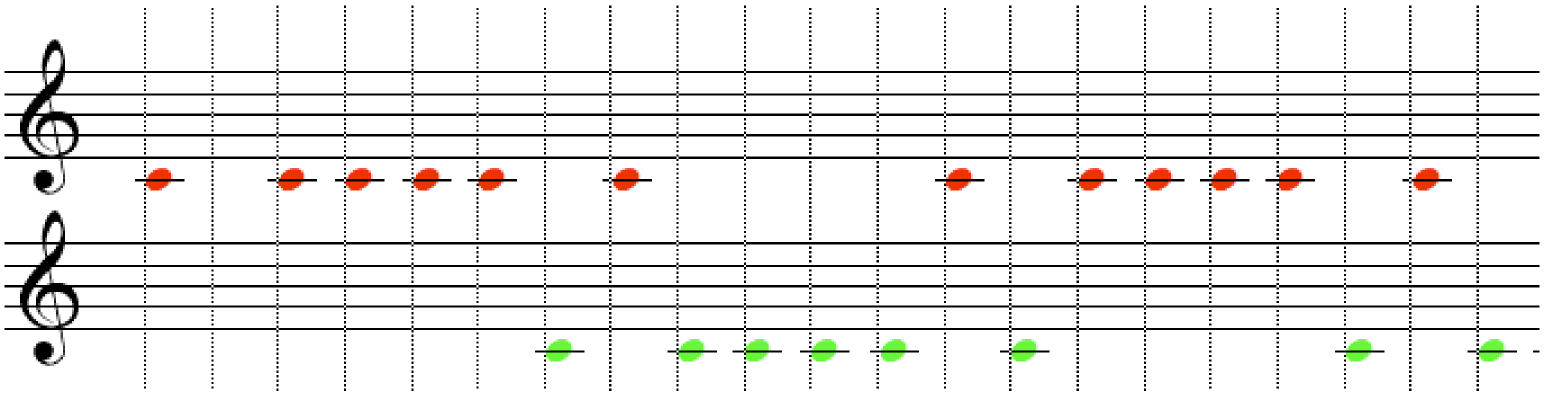


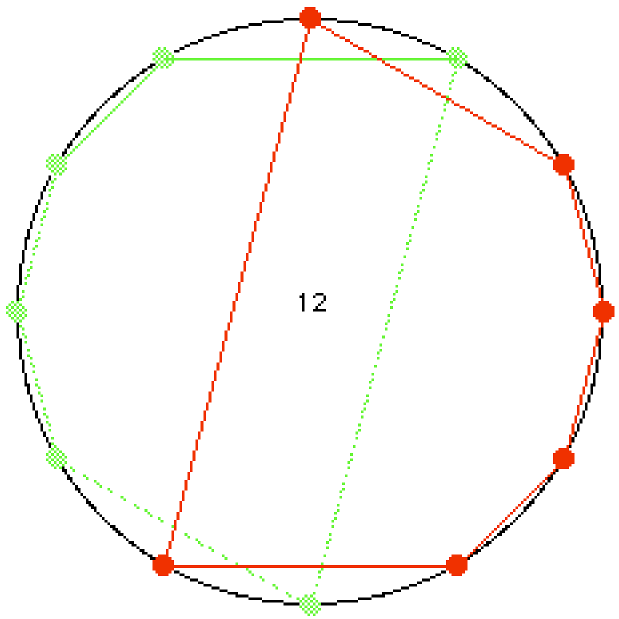



Does it exist a **quintuple** perfect tiling canon?



# Canons mosaïques par translation et augmentation



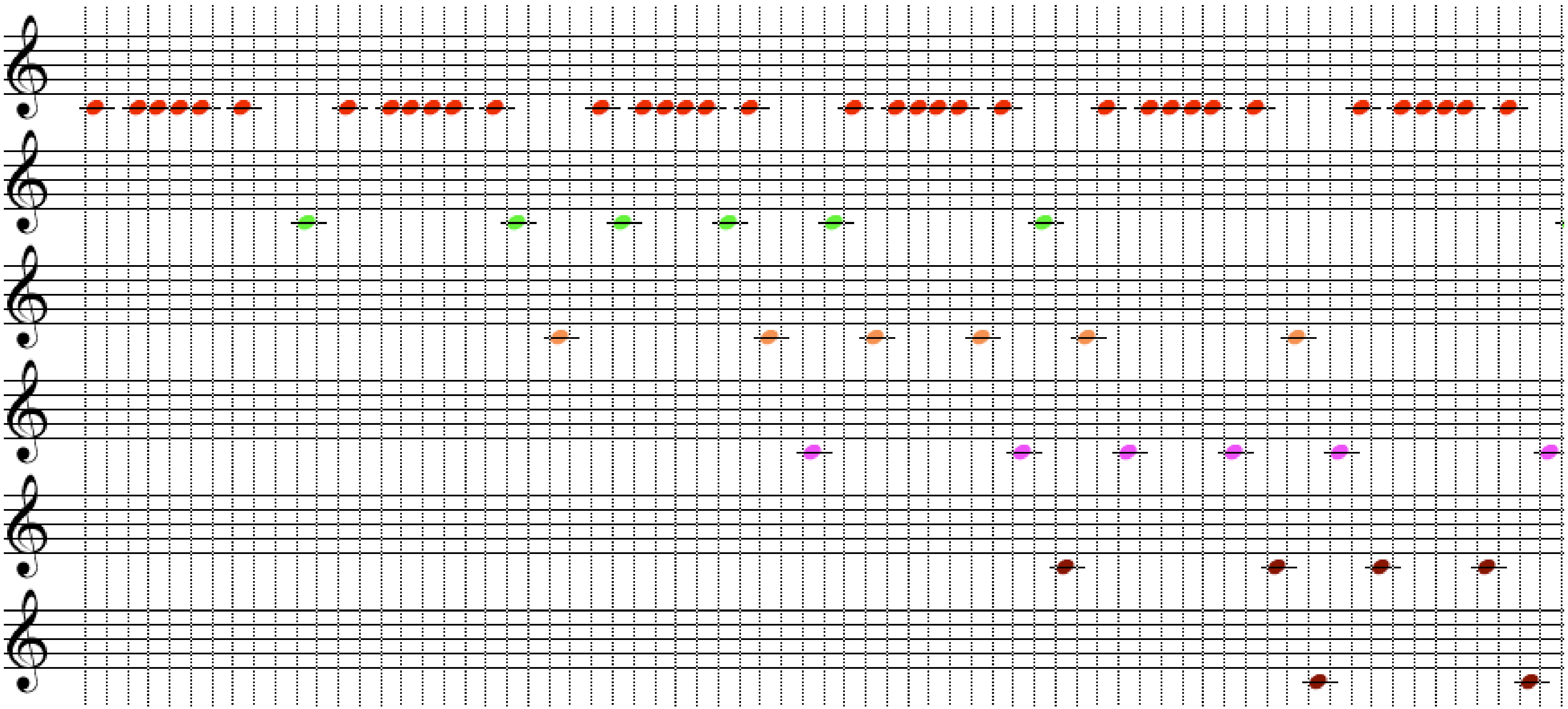



```
(((0 1 2 3 4 6) ((1 11)))
((0 1 2 3 4 5) ((1 11) (1 1)))
((0 1 2 3 5 7) ((1 11) (1 7)))
((0 1 3 4 7 8) ((1 5)))
((0 1 2 3 6 7) ((1 11)))
((0 1 3 4 6 9) ((1 11) (1 5)))
((0 1 3 6 7 9) ((1 11) (1 5)))
((0 1 2 6 7 8) ((1 11) (1 7) (1 5) (1 1)))
((0 1 4 5 8 9) ((1 11) (1 7) (1 5) (1 1)))
((0 1 2 5 6 7) ((1 7) (1 5)))
((0 2 3 4 5 7) ((1 11) (1 7) (1 5) (1 1)))
((0 1 4 5 6 8) ((1 11) (1 7)))
((0 1 2 4 5 7) ((1 5)))
((0 1 3 4 5 8) ((1 5) (1 1)))
((0 1 2 4 5 8) ((1 11)))
((0 1 2 4 6 8) ((1 11) (1 7)))
((0 2 3 4 6 8) ((1 11)))
((0 2 4 6 8 10) ((1 11) (1 7) (1 5) (1 1))))
```

=> *OpenMusic*

# *Augmented Tiling Canons* o l'azione del gruppo affine

(in collaborazione con Thomas Noll)



=> *OpenMusic*

# Computer-aided model of the compositional process

*Achorripsis*
by Mikhail Malt

*Herma*
by Gérard Assayag and Mikhail Malt

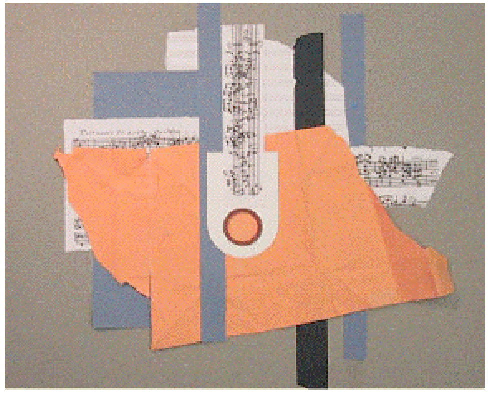
*Nomos Alpha*
by Carlos Agon and Moreno Andreatta

**Stochastic music**

***Poisson Law***
***Exponential distr.***
***Gaussian distr.***
***Pitch-Duration***

**Symbolic music**

***Boolean operations***
***Exponential distr.***



**Symbolic/Algebraic music**

***Sieve theory***
***Fibonacci process***
***Group of rotations***

OpenMusic

*ST/10-01, 48-01*
by Mikhail Malt

*Akrata* by Stephan Schaub and Mikhail Malt

*Nomos Alpha*
(real time version)
By Mikhail Malt

**Future works**

# Table (mosaic) of Coherences

*Philosophy* (in the etymological sense)

Thrust towards truth, revelation. Quest in everything, interrogation, harsh criticism, active knowledge through creativity.

*Chapters* (in the sense of the methods followed)

| Partially inferential and experimental | Entirely inferential and experimental | Other methods to come |
|---|---|---|
| ARTS (VISUAL, SONIC, MIXED . . .) | SCIENCES (OF MAN, NATURAL)<br>PHYSICS, MATHEMATICS, LOGIC | ? |

This is why the arts are freer, and can therefore guide the sciences, which are entirely inferential and experimental.

*Categories of Questions* (fragmentation of the directions leading to creative knowledge, to philosophy)

REALITY (EXISTENTIALITY); CAUSALITY; INFERENCE: CONNEXITY; COMPACTNESS; TEMPORAL AND SPATIAL UBIQUITY AS A CONSEQUENCE OF NEW MENTAL STRUCTURES; INDETERMINISM . . . ← bi-pole → . . . DETERMINISM; . .

↓

*Families of Solutions or Procedures* (of the above categories)

- ? 
- INDETERMINISM → FREE STOCHASTIC materialized by a computer program; MARKOVIAN
- bi-pole → SETS; GAMES
- DETERMINISM → GROUPS

↓

*Pieces* (examples of particular realization)

- ?
- FREE STOCHASTIC → ACHORRIPSIS, ST/10–1, 080262, ST/48–1, 240162, ATRÉES, MORSIMA-AMORSIMA
- SETS → HERMA
- MARKOVIAN → ANALOGIQUE A, ANALOGIQUE B, SYRMOS
- GAMES → DUEL, STRATÉGIE
- GROUPS → AKRATA, NOMOS ALPHA, NOMOS GAMMA

*Classes of Sonic Elements* (sounds that are heard and recognized as a whole, and classified with respect to their sources)

ORCHESTRAL, ELECTRONIC (produced by analogue devices), CONCRETE (microphone collected), DIGITAL (realized with computers and digital-to-analogue converters), . . .

*Microsounds*

Forms and structures in the pressure-time space, recognition of the classes to which microsounds belong or which microstructures produce.

Microsound types result from questions and solutions that were adopted at the CATEGORIES, FAMILIES, and PIECES levels.

# TABLE (MOSAIC) OF COHERENCES

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| | INDETERMINISM | | bi-pole | | DETERMINISM |
|---|---|---|---|---|---|
| ? | FREE STOCHASTIC materialized by a computer program | MARKOVIAN | SETS | GAMES | GROUPS |

*Pieces* (examples of particular realization)

| | | | | | |
|---|---|---|---|---|---|
| ? | ACHORRIPSIS<br>ST/10–1, 080262<br>ST/48–1, 240162<br>ATRÉES<br>MORSIMA-AMORSIMA | ANALOGIQUE A<br>ANALOGIQUE B<br>SYRMOS | HERMA | DUEL<br>STRATÉGIE | AKRATA<br>NOMOS ALPHA<br>NOMOS GAMMA |

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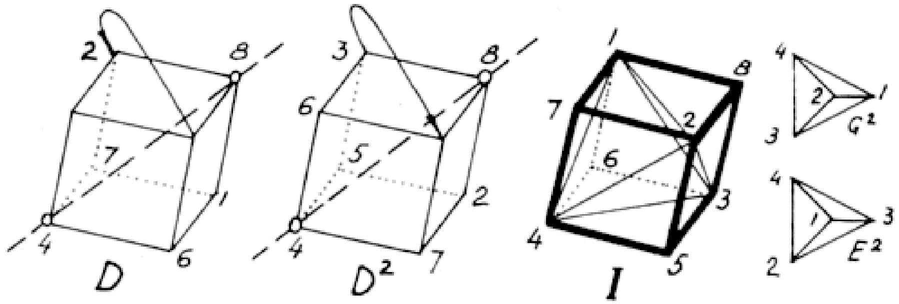
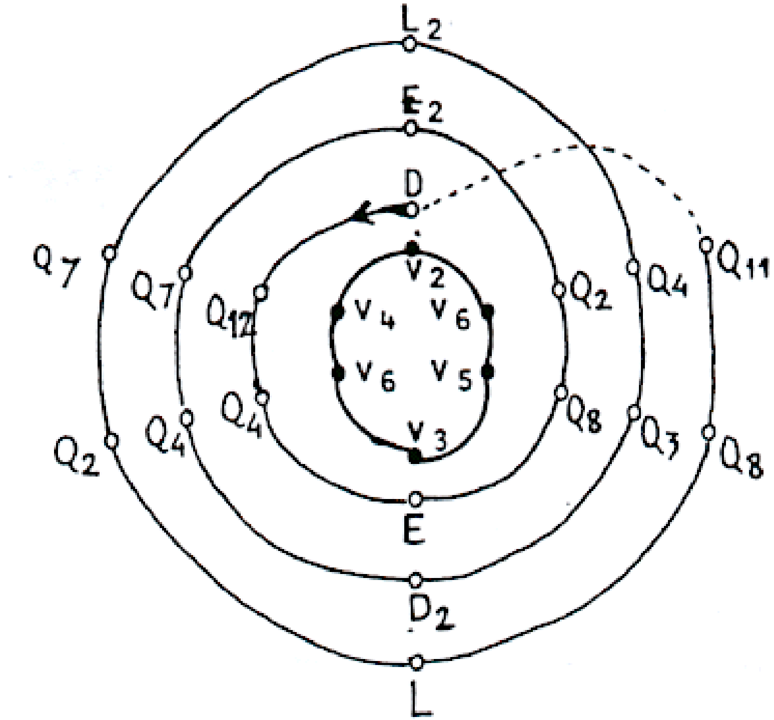
# Analisi musicale computazionale: *Nomos Alpha* di I. Xenakis

*La questione delle simmetrie (identità spaziali) e delle periodicità (identità nel tempo) ha un ruolo fondamentale nella musica, a tutti i livelli, da quello dei campioni sonori della sintesi del suono mediante computer, fino all'architettura di un intero brano musicale*




## *Nomos Alpha* (1966)

*Musique symbolique pour violoncelle seul, possède une architecture "hors-temps" fondée sur la théorie des groupes de transformations.*



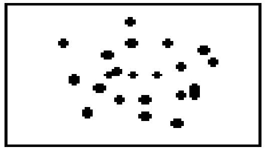
# Verso una « musique formelle »

*« La musica può […] essere definita come un’organizzazione d’**operazioni** e di **relazioni** elementari fra enti o funzioni d’enti sonori. Comprendiamo lo spazio di scelta [place de choix] che spetta alla **teorie degli insiemi**, non soltanto per la **costruzione** di nuove opere ma anche per l’**analisi** e la migliore comprensione di brani del passato. E così, è difficile comprendere a fondo anche una costruzione stocastica o una ricerca storica [investigation de l’histoire] attraverso strumenti stocastici senza l’aiuto della regina dellescienze e pure delle arti, ovvero la logica o, nella sua forma matematica, l’ **algebra** ».*

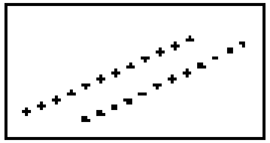
Iannis Xenakis : « La musique stochastique : éléments sur les procédés probabilistes de composition musicale », *Revue d'Esthétique*, vol. 14 n°4-5, 1961.

# Analisi musicale computazionale: *Nomos Alpha* di I. Xenakis

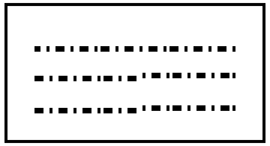
**S1**



**S2**



**S3**



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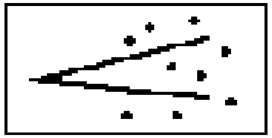
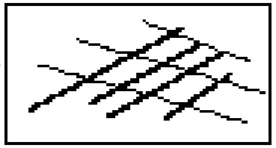


**S5**



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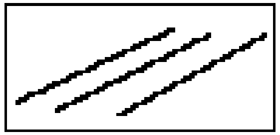

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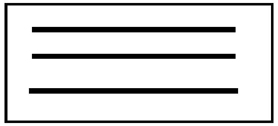

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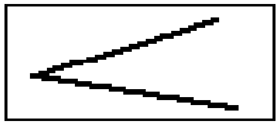

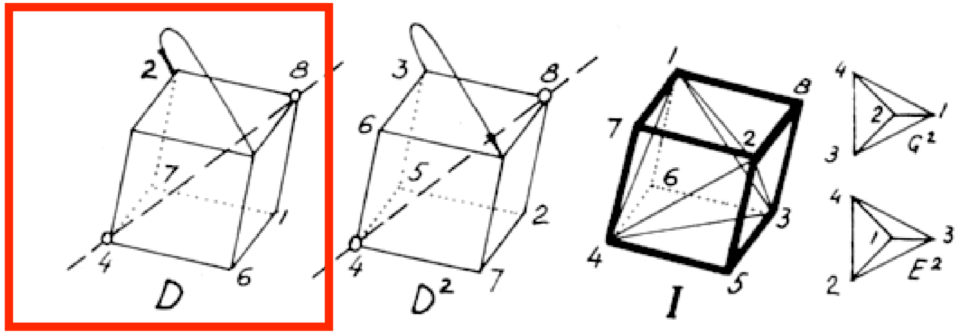


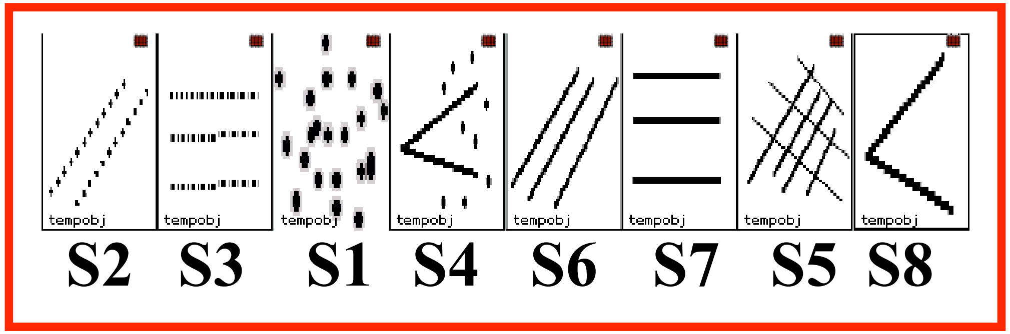


| | |
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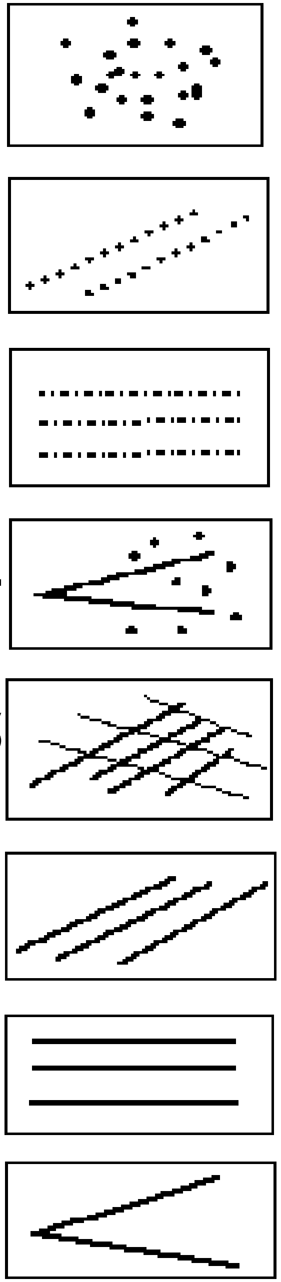
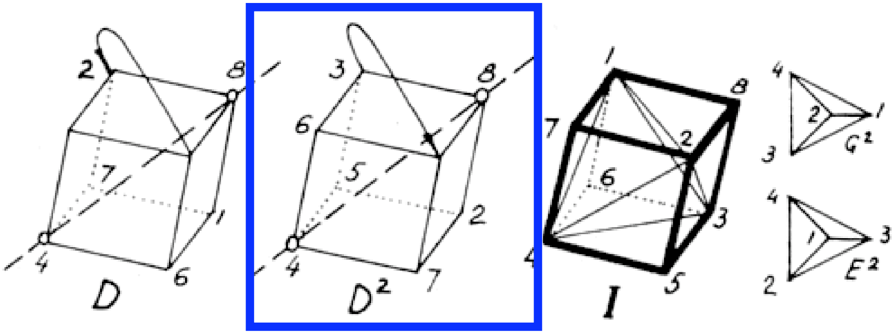
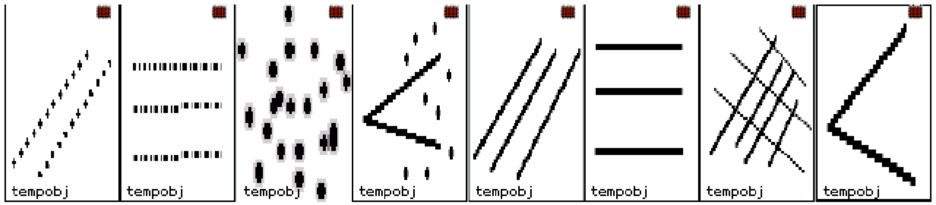
| $A$ | 21436587 |
| $B$ | 34127856 |
| $C$ | 43218765 |
| $D$ | 23146758 |
| $D^2$ | 31247568 |
| $E$ | 24316875 |
| $E^2$ | 41328576 |
| $G$ | 32417685 |
| $G^2$ | 42138657 |
| $L$ | 13425786 |
| $L^2$ | 14235867 |
| $Q_1$ | 78653421 |
| $Q_2$ | 76583214 |
| $Q_3$ | 86754231 |
| $Q_4$ | 67852341 |
| $Q_5$ | 68572413 |
| $Q_6$ | 65782134 |
| $Q_7$ | 87564312 |
| $Q_8$ | 75863142 |
| $Q_9$ | 58761432 |
| $Q_{10}$ | 57681324 |
| $Q_{11}$ | 85674123 |
| $Q_{12}$ | 56871243 |

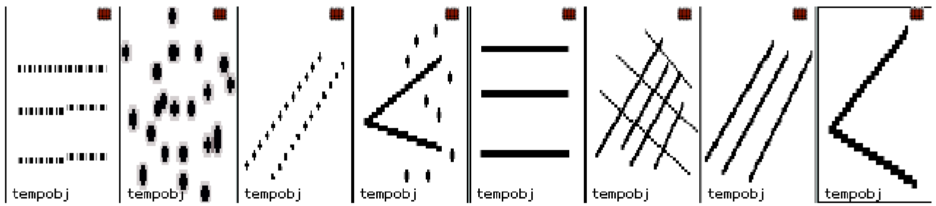
# Analisi musicale computazionale: *Nomos Alpha* di I. Xenakis

S1 S2 S3 S4 S5 S6 S7 S8






S3 S1 S2 S4 S7 S5 S6 S8

| | |
|---|---|
| $I$ | 12345678 |
| $A$ | 21436587 |
| $B$ | 34127856 |
| $C$ | 43218765 |
| $D$ | 23146758 |
| $D^2$ | 31247568 |
| $E$ | 24316875 |
| $E^2$ | 41328576 |
| $G$ | 32417685 |
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| $Q_7$ | 87564312 |
| $Q_8$ | 75863142 |
| $Q_9$ | 58761432 |
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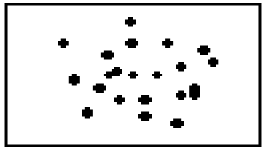
# Analisi musicale computazionale: *Nomos Alpha* di I. Xenakis

**S1**



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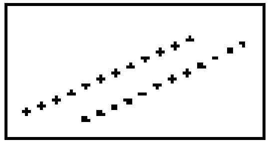

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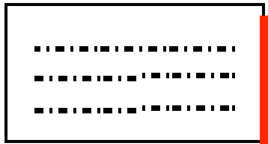
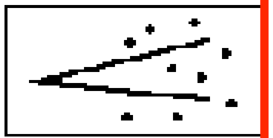
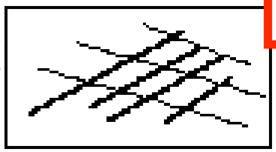


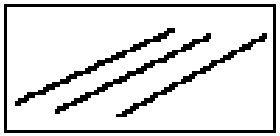
**S4**



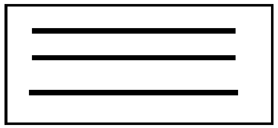
**S5**



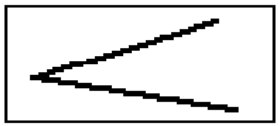
**S6**

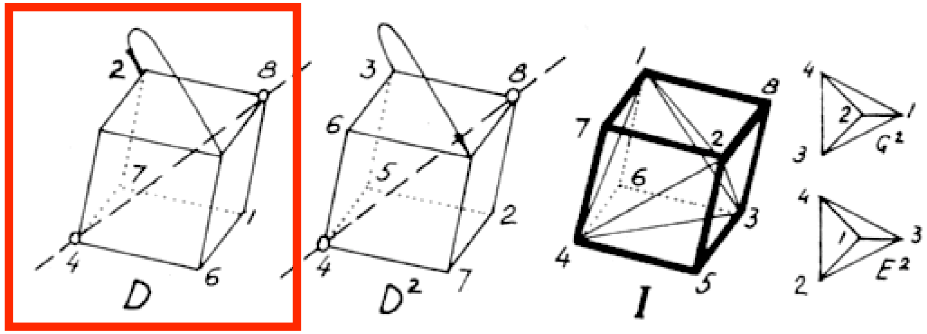


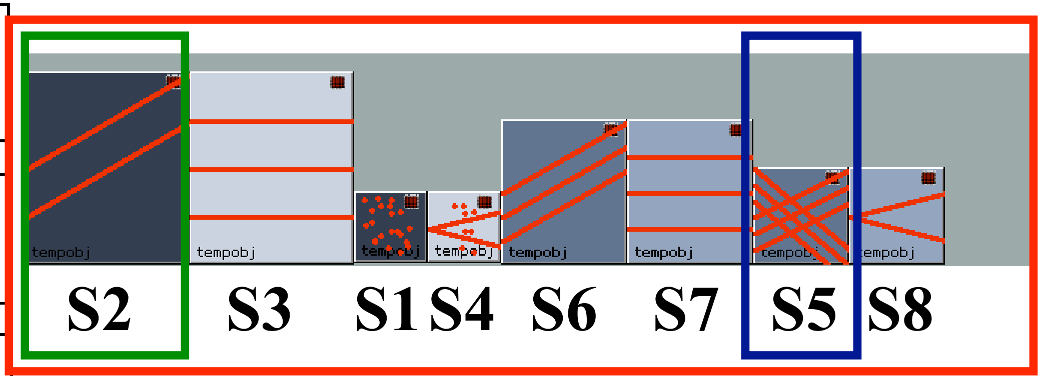
**S7**



**S8**






$\kappa^{\alpha}_{1} = 1 . mf . 2 = 2 mf$

$\kappa^{\alpha}_{2} = 1 . fff . 4.5 = 4.5 fff$

$\kappa^{\alpha}_{3} = 2.5 . fff . 4.5 = 11.25 fff$

$\kappa^{\alpha}_{4} = 2.5 . mf . 2 = 5 mf$

$\kappa^{\alpha}_{5} = 1.5 . f . 2.62 = 3.93 f$

$\kappa^{\alpha}_{6} = 1.5 . ff . 3.44 = 5.15 ff$

$\kappa^{\alpha}_{7} = 2.0 . ff . 3.44 = 6.88 ff$

$\kappa^{\alpha}_{8} = 2.0 . f . 2.62 = 5.24 f$

$\kappa^{\beta}_{1} = 0.5 . mf . 2 = 1 mf$

$\kappa^{\beta}_{2} = 0.5 . fff . 4.5 = 2.25 fff$

$\kappa^{\beta}_{3} = 5 . fff . 4.5 = 22.5 fff$

$\kappa^{\beta}_{4} = 5.0 . mf . 2 = 10.0 mf$

$\kappa^{\beta}_{5} = 1.08 . f . 2.62 = 2.83 f$

$\kappa^{\beta}_{6} = 1.08 . ff . 3.44 = 3.72 ff$

$\kappa^{\beta}_{7} = 2.32 . ff . 3.44 = 7.98 ff$

$\kappa^{\beta}_{8} = 2.32 . f . 2.62 = 6.08 f$

$\kappa^{\gamma}_{1} = 1 . mf . 2 = 2 mf$

$\kappa^{\gamma}_{2} = 1 . fff . 2 = 2 fff$

$\kappa^{\gamma}_{3} = 4.0 . fff . 4.5 = 18.0 fff$

$\kappa^{\gamma}_{4} = 4.0 . mf . 2.0 = 8.0 mf$

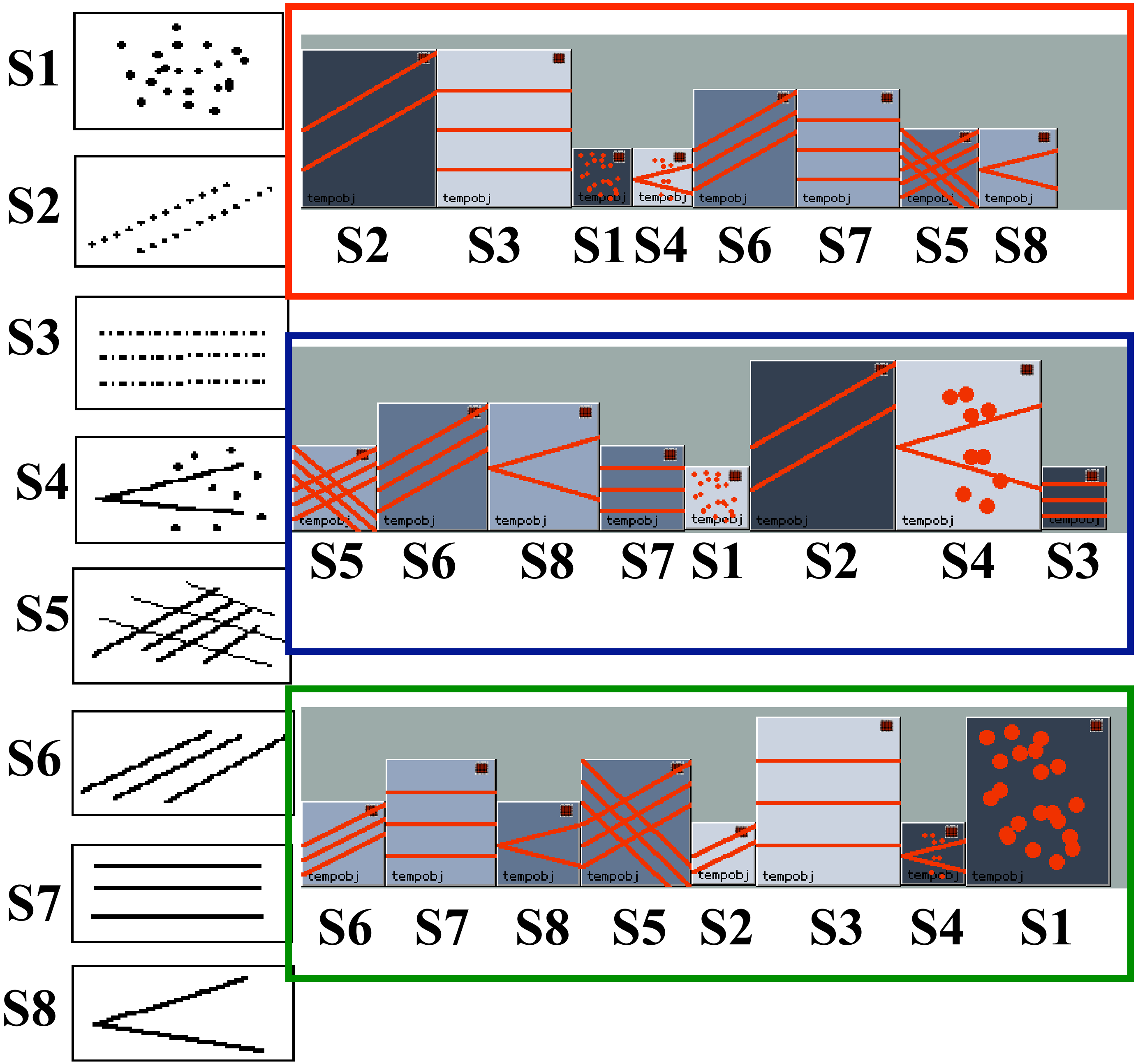
$\kappa^{\gamma}_{5} = 2.0 . f . 2.62 = 5.24 f$

$\kappa^{\gamma}_{6} = 2.0 . ff . 3.44 = 6.88 ff$

$\kappa^{\gamma}_{7} = 3.0 . ff . 3.44 = 10.32 ff$

$\kappa^{\gamma}_{8} = 3.0 . f . 2.62 = 7.86 f$

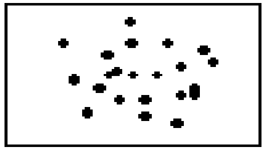
# Analisi musicale computazionale: *Nomos Alpha* di I. Xenakis



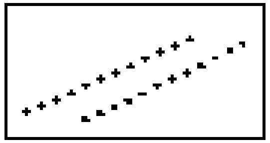

| | |
|---|---|
| $I$ | 12345678 |
| $A$ | 21436587 |
| $B$ | 34127856 |
| $C$ | 43218765 |
| $D$ | 23146758 |
| $D^2$ | 31247568 |
| $E$ | 24316875 |
| $E^2$ | 41328576 |
| $G$ | 32417685 |
| $G^2$ | 42138657 |
| $L$ | 13425786 |
| $L^2$ | 14235867 |
| $Q_1$ | 78653421 |
| $Q_2$ | 76583214 |
| $Q_3$ | 86754231 |
| $Q_4$ | 67852341 |
| $Q_5$ | 68572413 |
| $Q_6$ | 65782134 |
| $Q_7$ | 87564312 |
| $Q_8$ | 75863142 |
| $Q_9$ | 58761432 |
| $Q_{10}$ | 57681324 |
| $Q_{11}$ | 85674123 |
| $Q_{12}$ | 56871243 |

# Analisi musicale computazionale: *Nomos Alpha* di I. Xenakis

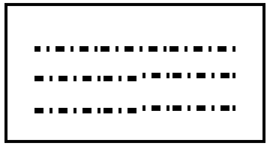
**S1**



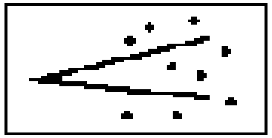
**S2**



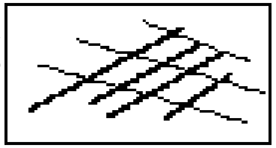
**S3**



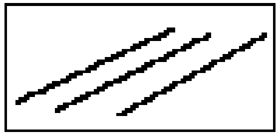
**S4**



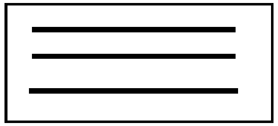
**S5**



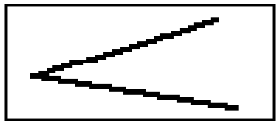
**S6**

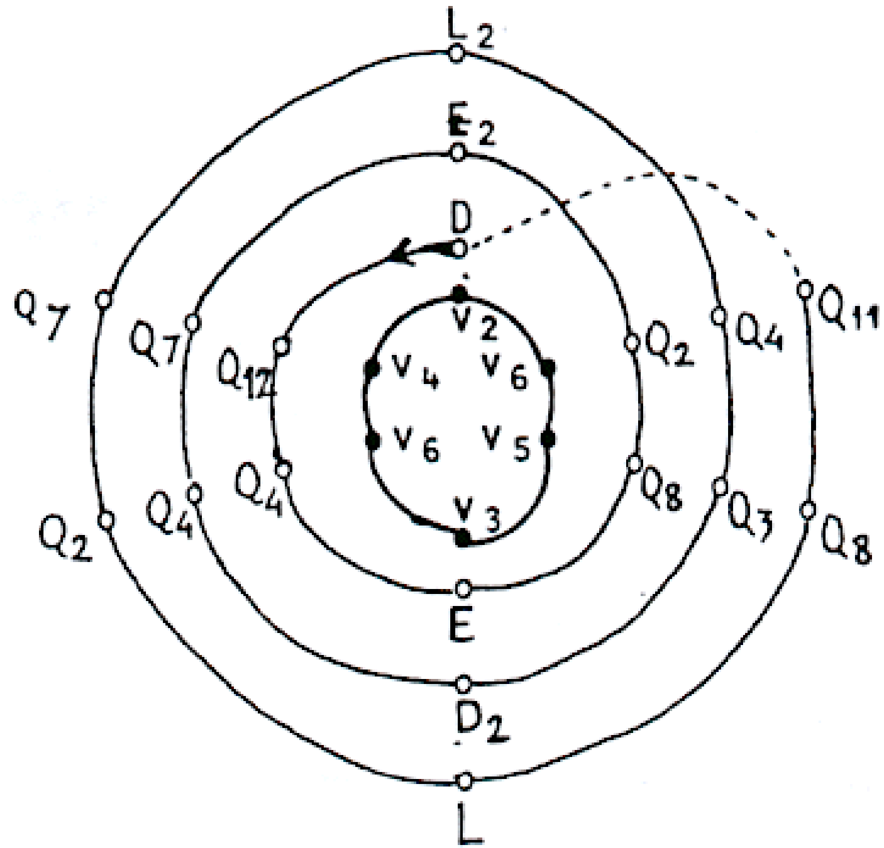


**S7**



**S8**



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| $Q_9$ | 58761432 |
| $Q_{10}$ | 57681324 |
| $Q_{11}$ | 85674123 |
| $Q_{12}$ | 56871243 |

=> *OpenMusic*