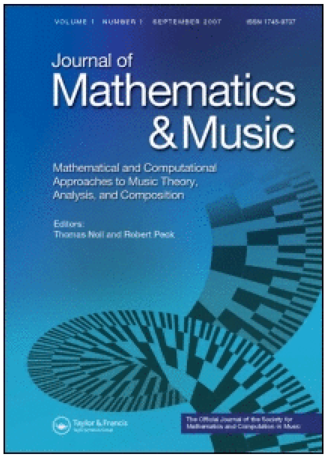

Journal of Mathematics and Music
Mathematical and Computational Approaches to Music Theory, Analysis, Composition and Performance
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t741809807

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Emmanuel Amiot [a]
[a] 1 rue du Centre, St Nazaire, France

Online Publication Date: 01 November 2007
To cite this Article: Amiot, Emmanuel (2007) 'David Lewin and maximally even sets', Journal of Mathematics and Music, 1:3, 157 - 172
To link to this article: DOI: 10.1080/17459730701654990
URL: http://dx.doi.org/10.1080/17459730701654990

Journal of Mathematics and Music
Vol. 1, No. 3, November 2007, 157–172

David Lewin and maximally even sets

EMMANUEL AMIOT*

1 rue du Centre, F 66570 St Nazaire, France

(*Received 4 July 2006; in final form 27 August 2007*)

David Lewin originated an impressive number of new ideas in musical formalized analysis. This paper formally proves and expands one of the numerous innovative ideas published by Ian Quinn in his dissertation, to the import that Lewin might have invented the much later notion of *Maximally Even Sets* with but a small extension of his very first published idea, where he made use of Discrete Fourier Transform (DFT) to investigate the intervallic differences between two pc-sets. Many aspects of Maximally Even Sets (ME sets) and, more generally, of generated scales, appear obvious from this original starting point, which deserves, in our opinion, to become standard. In order to vindicate this opinion, we develop a complete classification of ME sets starting from this new definition. As a pleasant by-product we mention a neat proof of the hexachord theorem, which might have been the motivation for Lewin's use of DFT in pc-sets in the first place. The nice inclusion property between a ME set and its complement (up to translation) is also developed, as occurs in actual music.

Keywords: Maximally even sets; Discrete Fourier transform; David Lewin

1. Notation

- The cyclic group of order c is $\mathbb{Z}_c$. It models a chromatic universe with c pitch classes, and it is, as usual, pictured as a regular polygon on the unit circle. In most actual examples, c will be equal to 12.
- x/y indicates the integer x divides y. For the sake of readability we generally use the same notation for integers and their residue classes, the context usually making clear whether a computation occurs in $\mathbb{Z}$ or in $\mathbb{Z}_c$.
- The greatest common divisor of x, y is denoted by gcd (x,y).
- We will use indiscriminately 'Fourier transform', 'discrete Fourier transform', or 'DFT'.
- The bracket notation $\lfloor \ldots \rfloor$ is for the floor function.
- The symbol $X \oplus Y$ indicates 'all possible sums of an element of X and an element of Y', each result being obtained in a unique way.

*Email: manu.amiot@free.fr

Journal of Mathematics and Music
ISSN 1745-9737 print/ISSN 1745-9745 online
http://www.tandf.co.uk/journals
DOI: 10.1080/17459730701654990

2. Fourier transform of pc-sets

Part of our claim that Fourier transforms provide the best way to define Maximally Even Sets relies on the great musical significance of the DFT of pc-sets in general. This was salient in [1] for the special pc-sets that Quinn collected as 'prototypes', among which are the ME sets; this has since been confirmed by many other cases. We thus feel it important to spend some time on the general DFT of pc-sets before turning to the main topic, that is its application to ME sets proper.

2.1. History

In a short paper, Lewin [2] investigated intervallic relationships between two 'note collections' and proved that, except in several listed exceptional cases, the interval function between the 'note collections' enables us to reconstruct one from the other. He cursorily motivates the five exceptional cases by a final note, wherein he puts forward that

1. the interval function is a convolution product (of characteristic functions), and
2. the Fourier transform of such a product is the ordinary product of Fourier transforms.

This shows that (when the Fourier transform of the characteristic function of a pc-set A is non-vanishing) knowledge of A and of the interval function from A to a pc-set B yields complete knowledge of the characteristic function of B.

Defining the interval function between A, $B \subset \mathbb{Z}_c$ as

$$IFunc(A, B)(t) = \mathrm{Card}\{(a, b) \in A \times B,\ \ b - a = t\},$$

and the characteristic function of X as

$$1_X(t) = \begin{cases} 1, & \text{if } t \in X, \\ 0, & \text{if} t \notin X, \end{cases}$$

IFunc appears immediately as the convolution product of the characteristic functions of $-A$ and B:

$$1_{-A} * 1_B : t \mapsto \sum_{k \in \mathbb{Z}_c} 1_{-A}(k) 1_B(t-k) = \sum_{k \in \mathbb{Z}_c} 1_A(k) 1_B(t+k) = IFunc(A, B)(t),$$

as $1_A(\mathrm{k})1_B(t+k)$ is nil except when $k \in A$ and $t+k \in B$. Hence, from the general formula for the Fourier transform of a convolution product,

$$\mathcal{F}(IFunc(A, B)) = \mathcal{F}(1_{-A}) \times \mathcal{F}(1_B),$$

where $\mathcal{F}(f)$ is the discrete Fourier transform of map f.

We will not quote the formula given by Lewin himself, as it is hardly understandable: his notation is undefined and the computations extremely cursory. Of course, this is not for lack of rigour: as the following quotation suggests, Lewin did not really hope to be understood when making use of mathematics.

> The mathematical reasoning by which I arrived at this result is not communicable to a reader who does not have considerable mathematical training. For those who have such a training, I append a sketch of the proof: consider the group algebra [...] [2]

Reading Lewin's paper gives one a strong feeling that he wrote as little as possible on the mathematical tools that underlay his results. Indeed, what little he mentioned did arouse some readers to righteous ire in the next issue of JMT.

Nowadays, such a 'considerable mathematical training' will be considered basic by many readers of this journal; for instance, Vuza made essential use of the above equation in the 1980s in the course of his seminal work on rhythmic canons (see [3], Lemma 1.9 sqq), wherein he stressed the importance of Lewin's use of the DFT of characteristic functions.

As we will endeavour to prove, this approach enables us to define ME sets (in equal temperament) in a way perhaps more suggestive, and even intuitive, than historical/usual definitions.

2.2. A quick summary of Fourier transforms of subsets of $\mathbb{Z}_c$

2.2.1. *First moves*

Definition 2.1 Following Lewin, we will define the Fourier transform of a pc-set $A \in \mathbb{Z}_c$ as the Fourier transform of its characteristic function 1_A:

$$\mathcal{F}_A = \mathcal{F}(1_A): t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}.$$

The values $\mathcal{F}_A(t)$, $t \in \mathbb{Z}_c$, are the Fourier coefficients.

1_A is a map from $\mathbb{Z}_c$ to $\mathbb{C}$, the DFT of which is well defined for t mod c as $\mathcal{F}_A(t+c) = \mathcal{F}_A(t)$.

The DFT of a single pc a is a single exponential function $t \mapsto e^{-2i\pi at/c}$, and the DFT of the whole chromatic scale is $\mathcal{F}_{\mathbb{Z}_c}(t) = \Sigma_{k=0}^{c-1} e^{-2i\pi kt/c} = 0$ for all $t \in \mathbb{Z}_c$ except $t = 0$.

But $\mathcal{F}_A + \mathcal{F}_{\mathbb{Z}_c \setminus A} = \mathcal{F}_{\mathbb{Z}_c}$, hence:

Lemma 2.2 *The Fourier transforms of a pc-set A and of its complement $\mathbb{Z}_c \setminus A$ have opposite values, except when $t = 0$:*

$$\forall t \in \mathbb{Z}_c, t \neq 0, \quad \mathcal{F}_{\mathbb{Z}_c \setminus A}(t) = -\mathcal{F}_A(t).$$

Furthermore, we obtain $\mathcal{F}_{\mathbb{Z}_c \setminus A}(0) = \mathcal{F}_A(0)$ if and only if Card $A = c/2$, as:

Lemma 2.3 *The Fourier transform of A in 0 is equal to the cardinality of A*: $\mathcal{F}_A(0) = \text{Card } A$.

For other coefficients, taking into account Lemma 2.2 and the triangular inequality, one obtains:

Lemma 2.4 $\forall t \in \mathbb{Z}_c, t \neq 0 \Rightarrow |\mathcal{F}_A(t)| \leq min(d, c-d)$.

The DFT $\mathcal{F}_A$ characterizes the pc-set A by the following identity (inverse Fourier transform):

$$1_A(t) = \frac{1}{c} \sum_{k \in \mathbb{Z}_c} e^{+2ikt\pi/c} \mathcal{F}_A(k),$$

which is easily derived from the definition of $\mathcal{F}_A$. Thus, the DFT yields the same information as the pc-set, but *in a form that stresses musically relevant concepts*. More precisely, there is preservation of the absolute value of the DFT under all usual* musical transformations. For instance,

* Less usual transformations, such as $t \mapsto 7t$ mod 12, *permute* the Fourier coefficients

Theorem 2.5 *The* length *of the Fourier transform, i.e. the map* $|\mathcal{F}_A|: t \mapsto |\mathcal{F}_A(t)|$, *is invariant by (musical) transposition or inversion of the pc-set* A. *More precisely, for any* $p, t \in \mathbb{Z}_c$:

- $\mathcal{F}_{A+p}(t) = e^{-2ip\pi t/c}\, \mathcal{F}_A(t)$ *(invariance under transposition)*,
- $\mathcal{F}_{-A}(t) = \overline{\mathcal{F}_A(t)}$ *(invariance under inversion)*,

and also under complementation (except in 0 when Card $A \neq c/2$*).*

Let us state that A, B are Lewin-related when maps $|\mathcal{F}_A|$ and $|\mathcal{F}_B|$ are identical. This is the case whenever A, B are exchanged by the T/I group of musical transformations, but the reverse is not true (see below). All the same, the map $|\mathcal{F}_A|$ appears to be a very good snapshot of the relevant musical information of a given pc-set: by dropping the information of the *phase* of the Fourier coefficients and retaining only the *absolute value*, we seem to keep the best part, in a way reminiscent of the Helmholtzian approach of sound, which showed that the phase of a sine wave can (mostly) be neglected, as the frequency is the part that generates the perception of pitch. This strongly vindicates, and to some measure extends, Quinn's [1] notion of 'chord quality', which appears in the last section of his dissertation with a value that is precisely $|\mathcal{F}_A(d)|$ ($d =$ Card A), and is measured in 'lewins'.

As a nice application of these invariance properties, we may characterize periodic subsets:

Proposition 2.6 $A \subset \mathbb{Z}_c$ *is periodic, meaning* $A + \tau = A$ *for some* τ, *if and only if* $\mathcal{F}_A(t) = 0$ *except when* t *belongs to some subgroup of* $\mathbb{Z}_c$.

The proof is left to the reader (see also Supplementary 2, available via the Multimedia link on the online article webpage).

Remark 1

- Some may well claim that this proposition is obvious: a subset $A \in \mathbb{Z}_c$ is the set of residues of a periodic set $\hat{A} \subset \mathbb{Z}$, with period c. This periodicity means precisely that 1_A (or $1_{\hat{A}}$, with the same formula) can be expressed as a combination of c exponential functions, $t \mapsto e^{2i\pi kt/c}$: this is the inverse Fourier transform formula and the very reason Fourier transform works. The existence of a smaller period $m|c$ means that m exponential functions only are sufficient, e.g. $t \mapsto e^{2i\pi kt/m}$.
- In $\mathbb{Z}_{12}$, the octatonic scale (0 1 3 4 6 7 9 10) is an interesting example of such a periodic subset. Its group of periods is 3 $\mathbb{Z}_{12}$. Periodic subsets of $\mathbb{Z}_{12}$ are well known as Messiaen's *Modes à Transposition Limitées.*

2.2.2. *DFT and intervallic content.* The following theorem is based on the idea of interpreting the multiplicities of pc intervals within a pc-set A as complex numbers (such as we did with the values 0 and 1 of the characteristic functions 1_A). The interval content is treated as a function from $\mathbb{Z}_c$ to the complex numbers and is defined on the c (oriented) possible intervals.*

* Usually, textbooks define interval content for T/I classes of intervals.

THEOREM 2.7 *(Lewin's Lemma) Define the interval content of a subset* $A \in \mathbb{Z}_c$ *as*

$$IC_A(k) = IFunc(A, A)(k) = Card\{(i,j) \in A^2, i - j = k\}.$$

Then the DFT of the intervallic content is equal to the square of the length of the DFT of the set:

$$\mathcal{F}(IC_A) = |F_A|^2.$$

Proof: Let A be a pc-set; as Lewin observed (for the more general interval function between two subsets), the 'intervallic function' from pc-set A to itself is* the convolution product

$$IC_A = 1_{-A} * 1_A.$$

But as we recalled earlier, the Fourier transform of a convolution product is the ordinary product of Fourier transforms, i.e. (using the last part of Theorem 2.5)

$$\mathcal{F}(IC_A) = \mathcal{F}_A \times \mathcal{F}_{-A} = \mathcal{F}_A \times \overline{\mathcal{F}_A} = |F_A|^2.$$

□

Note that the Fourier transform of any IC is a real positive-valued function, an uncommon occurrence among the DFT of integer-valued functions.† Now we see that the Lewin relation is the equivalence closure of the Z-relation.

PROPOSITION 2.8 $A, B \subset \mathbb{Z}_c$ *are Lewin-related* ($|\mathcal{F}_A| = |\mathcal{F}_B|$) *if and only if they share the same interval content.*

The equivalence stands because $|\mathcal{F}_A|$ holds all the information concerning IC_A by inverse Fourier transform‡—this case follows directly from Theorem 2.5.

From this we also obtain a very short proof of the hexachord theorem, one of the most striking mathematical results in music theory.

At the time he published his first paper, Lewin had come to work with Milton Babbitt, who was trying to prove the hexachord theorem (see figure S1 in Supplementary 2, available online).

THEOREM 2.9 *If two hexachords (i.e. six note subsets of* $\mathbb{Z}_{12}$*) are complementary pc-sets in* $\mathbb{Z}_{12}$*, then they share the same intervallic content (same numbers of same intervals).*

A simple derivation of this theorem in $\mathbb{Z}_c$ for any even c ensues from the elementary properties of the DFT already listed.

* This relation has been quoted in a musical contex, by several authors: for Vuza [3] it might be the most important single contribution by David Lewin: 'It is therefore my conviction that in the near future music theory will integrate convolution and fourier transform as effective investigation tools, music theorists being able to use them in the same way as presently they make use of groups, homomorphisms, group actions, and so forth'; it also appears, for instance, in the recent paper by Jedrzejewski [4].

† The DFT of a real-valued function is non-real in general, it only verifies $\mathcal{F}(f)(-t) = \overline{\mathcal{F}(f)(t)}$.

‡ Note that we endeavour here to define a true equivalence relation, contrarily to the fortean tradition which excludes the 'easy case', when A, B are T/I related. This traditional position is weird; another argument against it is that some classes of 'Z-related'. Chords are indeed exchanged through the action of a group larger than T/I, such as the two famous all-intervals (0 1 4 6) and (0 1 3 6) in $\mathbb{Z}_{12}$, which are affine-related (see [5], pp. 102 sqq)—and this is a general situation, as any affine transform of an all-interval set will be Z-related. Jon Wild pointed out to me that the reverse is false.

Proof: If $A \in \mathbb{Z}_c$ has $c/2$ elements, then, as mentioned above, $\mathcal{F}_{\mathbb{Z}_c \setminus A} = -\mathcal{F}_A$. Therefore,

$$\mathcal{F}(IC_A) = |F_A|^2 = |F_{\mathbb{Z}_c \setminus A}|^2 = \mathcal{F}(IC_{\mathbb{Z}_c \setminus A}).$$

Hence (by inverse DFT), $IC_A = IC_{\mathbb{Z}_c \setminus A}$. □

As far as I know, this short proof was first published in [6] after I mentioned it during the John Clough Memorial Symposium (Chicago, July 2005). But considering the coincidence in time of Lewin's first paper and his meeting with Babbitt, it is almost certain that he was aware of it. Perhaps the harsh reactions to the mathematics in his first paper explain why he did not publish it. It is left to the reader, as a good and entertaining exercise, to prove in the same way the Generalized Hexachord Theorem, as expounded in [4,5,7], and by many others.

3. Maximally even sets and their Fourier transforms

The attribute 'maximally even' applies to pitch class sets, which—in comparison to all pitch class sets of the same cardinality—are distributed as evenly as possible within $\mathbb{Z}_c$. This is obviously the case for totally regular sets, which exist only for cardinalities d dividing the number c of pitch classes. The opposite special case—where d and c are mutually coprime—was well studied in [8]. The point of departure for the extensive study of the general case in [9] is an explicit construction of generalized diatonic sets in [8]. The formula for this construction was later termed the J-function. It departs from the arithmetic series of rational numbers $0, c/d, \ldots, (d-1)\, c/d$ and 'digitizes' them within $\mathbb{Z}_c$ in terms of the residue classes of the floor values of these ratios mod c: $0, \lfloor c/d \rfloor, \ldots, \lfloor (d-1)(c/d) \rfloor \mod c$. The J-function includes a translation parameter α:

$$J^{\alpha}_{c,d}: k \mapsto \left\lfloor \frac{kc + \alpha}{d} \right\rfloor, \quad k = 0 \ldots d-1.$$

In this section we accomplish the theory of maximally even sets with an alternative definition via Fourier coefficients and derive the main known results directly from this definition. Our 'Lewinesque' definition matches the semantics of the term 'maximally even' better than the explicit J-function, which lacks the aspect of comparison. See Supplementary 1, available online, for a compilation of facts and arguments around maximally even sets, or the recent publication [10].

3.1. An illuminating remark by Ian Quinn

Discussing a general typology of chords (or pc-sets), Ian Quinn noticed [1] (3.2.1) that what he calls 'generic prototypes' are the ME sets, and that they share an extremal property in terms of Fourier 'weight'.* This is what we will now adopt as a definition; Quinn's impressive survey and classification of the landscape of all chords was not focused exclusively on ME sets, and as his redaction voluntarily avoided, to quote him, the 'stultifying' quality inherent to dry mathematical generalizations, he left room for a formal proof that this definition is equivalent to the traditional ones (we will prove the

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* 'We note that generic prototypicality may be interpreted as maximal imbalance on the associated Fourier balance—at least to the extent that a generic prototype tips its associated Fourier balance more than any other chord of the same cardinality possibly can'.

following definition is equivalent to the classical description, up to and including the formula with J-functions; see [9,10] for equivalence between all previous definitions).

Moreover, and this is in itself sufficient justification for what follows, many properties of ME sets will now appear obvious from this starting point. Finally, the only quantity involved is $|\mathcal{F}_A|$, the invariant of the Lewin relation, which is, as we have seen, in many ways the most natural musical invariant for pc-sets.

3.2. A Lewinesque definition of ME sets and derived properties

Definition 3.1 The pc-set $A \subset \mathbb{Z}_c$, with cardinality d, is a ME set, if the number $|\mathcal{F}_A(d)|$ is maximal among the values $|\mathcal{F}_X(d)|$ for all pc-sets X with cardinality d:

$$\forall X \subset \mathbb{Z}_c, \quad \mathrm{Card} X = d \Rightarrow |\mathcal{F}_A(d)| \geq |\mathcal{F}_X(d)|.$$

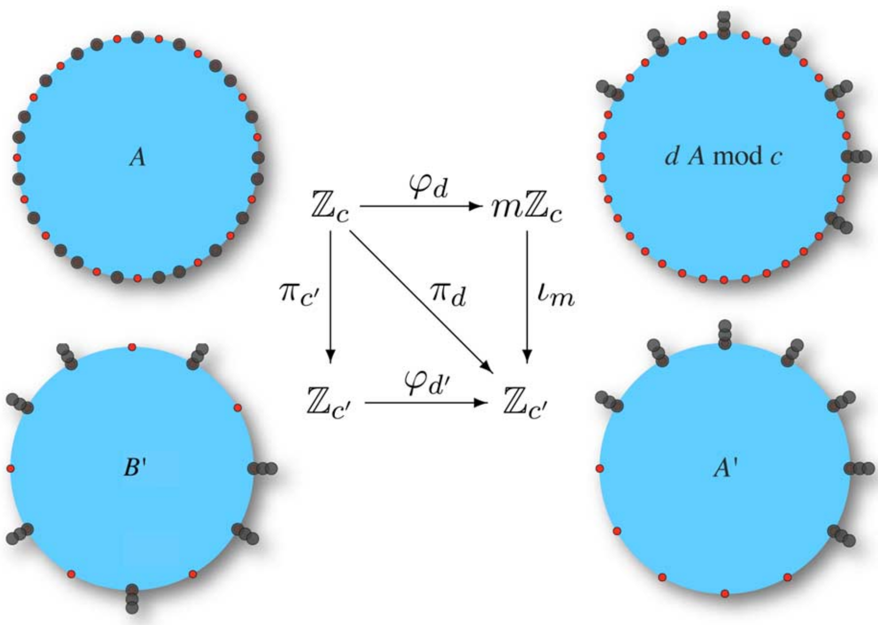
As the number of pc-sets is finite, a solution must exist. Remember that $|\mathcal{F}_A(d)| = (\mathcal{F}(IC_A)(d))^{1/2}$ (see section 2). Therefore, maximal evenness is also manifest in the DFT of the interval vector as a maximality condition for $\mathcal{F}(IC_A)(d)$.

From the invariance of the 'Fourier profile' $|\mathcal{F}_A|$ under musical operations (see Theorem 2.5 and Lemma 2.2 concerning complementation) we readily obtain the following proposition.

Proposition 3.2 *The transposition, inversion and complementation of a ME set still yield a ME set.*

3.3. Notation and maps

Throughout the remainder of this section let $m = gcd(d,c)$ denote the greatest common divisor of d and c and let $d' = d/m$ and $c' = c/m$ denote the associated quotients.



$$\begin{tikzcd} \mathbb{Z}_c \arrow[r, "\varphi_d"] \arrow[d, "\pi_{c'}"'] \arrow[dr, "\pi_d"] & m\mathbb{Z}_c \arrow[d, "\iota_m"] \\ \mathbb{Z}_{c'} \arrow[r, "\varphi_{d'}"] & \mathbb{Z}_{c'} \end{tikzcd}$$

Figure 1. Notations and morphisms.

Let $\varphi_d:\mathbb{Z}_c \to m\,\mathbb{Z}_c$ and $\varphi_{d'}:\mathbb{Z}_{c'} \to \mathbb{Z}_{c'}$ denote the linear multiplication maps $\varphi_d(l) = d\cdot l$ and $\varphi_{d'}(k) = d'\cdot k$, respectively. Further, let $\pi_{c'}:\mathbb{Z}_c \to \mathbb{Z}_{c'}$ denote the reduction of the finer residue classes mod c to the coarser residue classes mod c′, i.e. $\pi_{c'}(l) = l \bmod c'$. Finally, let $i_m : m\mathbb{Z}_c \to \mathbb{Z}_{c'}$ denote the isomorphism identifying the submodule $m\mathbb{Z}_c$ of $\mathbb{Z}_c$ with $\mathbb{Z}_{c'}$:$i_m(mk) := k \bmod c'$.

Note that the multiplication by d is a concatenation of the multiplications by m and by d'. Thus, if we concatenate the maps φ_d and i_m into a map $\pi_d := i_m \circ \varphi_d$, we see that the map i_m 'undoes' the previous multiplication by m. Therefore, $i_m \circ \varphi_d = \varphi_{d'} \circ \pi_{c'}$, which means that the diagram shown in figure 1 commutes.

3.4. Pitch class sets and related multisets

Our goal is to translate the maximality condition for the absolute value $|\mathcal{F}_A(d)|$ of the dth Fourier coefficient for pc-sets A into an equivalent maximality condition for the absolute value $|\mathcal{F}_{A'}(1)|$ for associated pc-multisets A'. To that end we investigate the image of a pc-set $A \subset \mathbb{Z}_c$ under the map π_d in a refined way. The refinement of the image $\pi_d(A)$ is a *multiset* that controls the multiplicity of each single image $l = \pi_d(k) \in \mathbb{Z}_{c'}$ for $k \in A$, i.e. the cardinality of the pre-image $\pi_d^{-1}(l) \cap A$. A suitable definition of the concept of a multiset is given in terms of a generalized concept of a characteristic function.

Recall that the ordinary characteristic function $1_A:\mathbb{Z}_c \to \{0,1\} \subset \mathbb{C}$ serves as an alternative representation of the set A. In this way, the set of subsets of $\mathbb{Z}_c$ appear as the subspace of complex-valued functions on $\mathbb{Z}_c$, with the condition that the values are only 0,1. The Fourier transform is an automorphism of this last algebra.

As an extrapolation of this idea, we consider the function $v_A^d : \mathbb{Z}_{c'} \to \{0,\ldots,m\} \subset \mathbb{C}$ with

$$v_A^d(l) := \mathrm{Card}(\pi_d^{-1}(l) \cap A) = \mathrm{Card}(\{k \in A | d\cdot k = l\}).$$

The *multiset* associated with A consists of the elements of $\pi_d(A)$, each being repeated with multiplicity $v_A^d(l)$. For the non-elements of $\pi_d(A)$, i.e. for all $l \in \mathbb{Z}_{c'} \backslash \pi_d(A)$, the multiplicity vanishes: $v_A^d(l) = 0$. In order to manipulate this multiset like an ordinary set, we attach the multiplicity of each element as a superscript: $A' := \{{}^{v_A^d(l)}l | l \in \pi_d(A)\}$. For instance, the multiset associated with $c = 12$, $d = 3$ and the regular set (augmented triad) $A = \{0,4,8\} \subset \mathbb{Z}_{12}$ is the multi-singleton set $A' = \{{}^3 0\}$ (with $0 \in \mathbb{Z}_4$). The multiset associated with $c = 12$, $d = 8$ and the octatonic set $A = \{0,1,3,4,6,7,9,10\} \subset \mathbb{Z}_{12}$ is $A' = \{{}^4 0, {}^4 2\}$ (with $0,2 \in \mathbb{Z}_3$).

The straightforward following lemma relates the dth Fourier coefficient of the set A to the first Fourier coefficient of the function v_A^d. When the meaning of A' is clear, we may adopt the notation from pc-sets and write $\mathcal{F}_{A'} := \mathcal{F}(v_A^d)$ and call this *the Fourier transform of the multiset A′*.

LEMMA 3.3 *With the above notation we have* $\mathcal{F}_A(d) = \mathcal{F}_{A'}(1)$.

Proof: We need to re-interpret a Fourier coefficient defined over $\mathbb{Z}_c$ as a Fourier coefficient over $\mathbb{Z}_{c'}$:

$$\mathcal{F}_A(d) = \sum_{k\in A} e^{-2\pi ikd/c} = \sum_{k\in A} e^{-2\pi ikd'/c'} = \sum_{l\in\mathbb{Z}_{c'}} \sum_{k\in A\cap\pi_d^{-1}(l)} e^{-2\pi il/c'} = \sum_{l\in\mathbb{Z}_{c'}} v_A^d(l) e^{-2\pi il/c'}$$
$$= F_{A'}(1).$$ □

In order to faithfully translate the maximality conditions from sets in $\mathbb{Z}_c$ to multisets in $\mathbb{Z}_{c'}$, we need to determine the correct collection of multisets involved. The following definition and lemma clarify this issue.

DEFINITION 3.4 A $m|d$-multiset in $\mathbb{Z}_{c'}$ is a function $\xi : \mathbb{Z}_{c'} \to \{0, \ldots, m\}$ satisfying $\Sigma_{k \in \mathbb{Z}_{c'}} \xi(k) = d$.

LEMMA 3.5 *$m|d$-multisets are exactly the multisets associated with a subset A with cardinality d.*

Proof: We represent $\mathbb{Z}_c$ as a disjoint union of the pre-images $\pi_d^{-1}(l)$ of single residue classes $l \in \mathbb{Z}_{c'}$ under the surjective map π_d, i.e. $\mathbb{Z}_c = \pi_d^{-1}(0) \sqcup \pi_d^{-1}(1) \sqcup \cdots \sqcup \pi_d^{-1}(c'-1)$, and we list the m elements of each of these pre-images in some arbitrary way: $\pi_d^{-1}(l) := \{k_{l,1}, \ldots, k_{l,m}\}$ for each $l \in \mathbb{Z}_{c'}$. Now for $A = \{k_{0,1}, \ldots, k_{0,\xi(0)}\} \sqcup \{k_{1,1}, \ldots, k_{1,\xi(1)}\} \sqcup \cdots \sqcup \{k_{c'-1,1}, \ldots, k_{c'-1,\xi(c')}\}$, we easily see that $v_A^d = \xi$.

Conversely, the kernel of φ_d is the subgroup $c'\mathbb{Z}_c$, with m elements, so the multiplicity of any element of $\pi_d(A)$ is at most d. And, of course, the sum of multiplicities is Card $A = d$. □

COROLLARY 3.6 *The absolute value $|\mathcal{F}_A(d)|$ of the dth Fourier coefficient of a pc-set $A \subset \mathbb{Z}_c$ is maximal among the values $|\mathcal{F}_X(d)|$ for all d-element subsets $X \subset \mathbb{Z}_c$ iff the absolute value $|\mathcal{F}_{A'}(1)| = |\mathcal{F}(v_A^d)(1)|$ of the first Fourier coefficient of the associated multiset A' is maximal among the values $|\mathcal{F}(\xi)|$ for all $m|d$-multisets ξ in $\mathbb{Z}_{c'}$.*

3.5. Huddling Lemma

This subsection is dedicated to the analysis of the maximality condition for the absolute values of the first Fourier coefficients for multisets associated with pc-sets A.

LEMMA 3.7 *(Huddling Lemma) The absolute value of the first Fourier coefficient* $|\mathcal{F}(\zeta)(1)|$ of an *$m|d$-multiset A' with characteristic function ζ is maximal among the values* $|\mathcal{F}(\xi)(1)|$ *for all $m|d$-multisets ξ in $\mathbb{Z}_{c'}$ iff ξ is a contiguous cluster of d' pitch classes of multiplicity m*, i.e. *iff there is a $l_0 \in \mathbb{Z}_{c'}$ such that ζ is of the form*

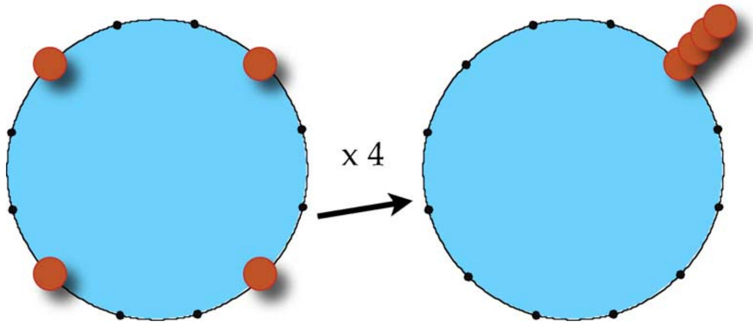


Figure 2. All exponentials superimposed.

$$\zeta(l) = \begin{cases} m, & \text{for } l - l_0 \in \{0, \ldots, d' - 1\}, \\ 0, & \text{for } l - l_0 \in \{d', \ldots, c' - 1\}. \end{cases}$$

For illustration, we point out the two simple subcases.

- When c,d are coprime, π_d is bijective and $A' = dA$ is an ordinary subset of $\mathbb{Z}_c$. The definition of ME sets, corollary 3.6 and the Huddling Lemma above mean that A' is a chromatic cluster, i.e. some translate of $\{1,2, \ldots, d\}$. Hence, $A = d^{-1}A'$ is an arithmetic sequence with ratio d^{-1}, as has been well known since [8]. The seminal example is the major scale, generated by a cycle of fifths.
- When d is a divisor of c, then A' is a multi-singleton set $\{^d a'\}$, as then the value $|\mathcal{F}(\zeta)(1)| = d$ is clearly maximal—here the Huddling Lemma is obvious. This means that A is a saturated pre-image, i.e. $A = \pi_d^{-1}(a') = a + c'\mathbb{Z}_c = a + \ker \pi_d$, with $\pi_d(a) = a'$, i.e. A is a regular polygon: see figure 2.*

Now for the technical proof of the Huddling Lemma. It relies basically on the very old geometrical fact that the sum of two vectors making an acute angle is greater than both.

Proof: We consider an $m|d$-multiset A' in $\mathbb{Z}_{c'}$, such that ξ does not have the contiguous form given in the lemma, and prove that $|\mathcal{F}(\xi)(1)| = |\mathcal{F}_{A'}(1)|$ is not maximal; the heuristic idea is that 'filling in the holes' increases the length of the sum.

Let us enumerate the elements of A' as r real integers in some increasing order: $k_1 < k_2 < \cdots k_r < k_1 + c$ (the span $k_r - k_1$ could be chosen minimal, but it is sufficient that it be $< c$). Assume that A' is not a translate of $\{^m0, ^m1, ^m2, \ldots ^m d-1\}$, then there must be some element $k \in [k_1, k_r]$ with multiplicity $0 \le \xi(k) < m$ (and $r > d'$).

- Say there is such a k with multiplicity $< m$, aka a 'hole', with $k_1 < k < k_r$; I claim that $|\mathcal{F}_{A'}(1)|$ strictly increases when (say) k_1 is replaced by k, i.e. when $\xi(k)$ is incremented while $\xi(k_1)$ is decremented: in so doing, the sum $S = \mathcal{F}_{A'}(1) =$

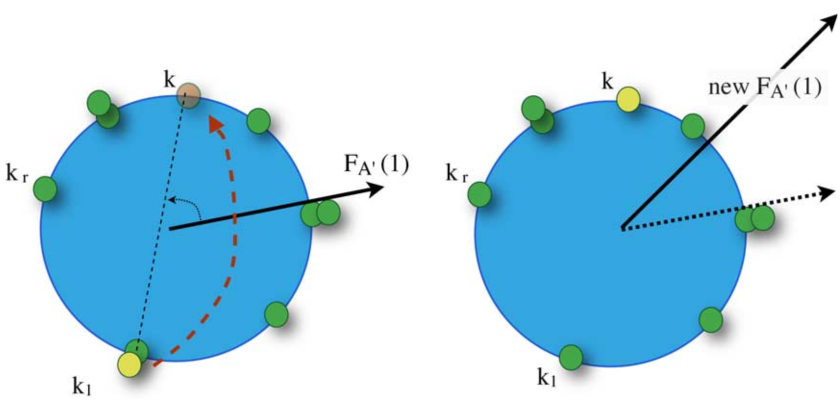


Figure 3. Maximizing the sum on a multiset.

* This shows that the Lewinesque definition aims at looking for the best approximation to a regular polygon—obviously, it will be only an approximation when d does not divide c, for instance there is no regular heptagon inside the 12 note universe. Indeed, the solution (the major scale $A = (0\ 2\ 4\ 5\ 7\ 9\ 11)$ or any translate thereof) achieves $|\mathcal{F}_A(7)| = 2 + \sqrt{3} \approx 3.73$, still far from the unattainable value 7 (ot rather 5, for the complement), but still the largest value possible.

$\Sigma_{l\in\mathbb{Z}_c}\ \xi(l)e^{-2i\pi l/c'}$ is replaced by $S' = S + e^{-2i\pi k/c'} - e^{-2i\pi k_1/c'}$. If $S=0$, then clearly $|S'|>|S|$. If not, let $S=re^{-i\theta}$. We can choose a determination of θ mod 2π (or rather choose the k_i's) such that $2\pi k_1/c' < \theta < 2\pi k_r/c'$, and I will assume that θ is closer to $2\pi k_1$ (if not, the proof is the same but with k_r), i.e. $0<\theta - 2\pi k_1/c' < \pi$. As $V = e^{-2i\pi k/c'} - e^{-2i\pi k_1/c'} = 2\sin\{[\pi(k-k_1)]/c'\}\,e^{-i\pi(k+k_1)/c'+i\pi/2}$ and $0<\theta - \pi(k+k_1)/c' + \pi/2 < \pi/2$ by our assumption that θ is 'close' to $2\pi k_1/c'$, the vectors S, V with directions, respectively, $-\theta$ and $-\pi(k+k_1)/c'+\pi/2$ make an acute angle. Hence, their sum S' is longer than both, qed (see figure 3). This can be done until no 'holes' remain between k_1 and k_r, i.e. $\xi(k)=m$ for all $k_1<k<k_r$.

- Eventually, we reach the last case: the vector of multiplicities must then be

$$\xi(k_1)=\mu, \qquad \xi(k_2)=m=\xi(k_3)=\cdots\xi(k_d),\ \xi(k_{d+1})=m-\mu.$$

Say, for instance, $\mu \geq m-\mu$. Then the direction θ of

$$\mathcal{F}_{A'}(1) = re^{-i\theta} = m\sum_{k=1}^{d+1} e^{-2i\pi k/c} + (m-\mu)(e^{-2i\pi k_1/c} - e^{-2i\pi k_{d+1}/c})$$

lies between $2\pi k_1/c$ and the mean value $\pi(k_1+k_{d+1})/c$ (convexity). Hence, as above, moving one point from position k_{d+1} to position k_1, i.e. incrementing $\xi(k_1)$ while decrementing $\xi(k_d)$, i.e. adding $\mathrm{e}^{-2i\pi k_1/c'} - \mathrm{e}^{-2i\pi k_{d+1}/c'}$ to S, increases its length, as the two vectors makes an acute angle.

Iteration of this process increases S strictly until it is no longer possible, which occurs when A' consists of d' consecutive points with multiplicity m. □

Remember that, for $m=1$, the maximal solution is simply a chromatic cluster: A' is an ordinary set with d consecutive points.

3.6. *Maximally even sets revisited*

It remains to be justified that our Lewinesque definition of maximal evenness is indeed equivalent to the traditional definitions. In the following subsection we recover the definition via J-functions. In the present subsection we explore the pre-images $\pi_d^{-1}(\zeta)$ of contiguous clusters as described in the Huddling Lemma. This leads to the well-know taxonomy of maximally even sets.

- *The regular polygon type.* When $m=d$ and hence $d'=1$, as mentioned above the associated multiset v_A^d is a multi-singleton $\{^m l_0\}$ of multiplicity m which corresponds to the complete pre-image $\pi_d^{-1}(l_0) = k_0 + \{0, c', \ldots, (m-1)c'\}$ for some $k_0 \in \pi_d^{-1}(l_0)$ and hence is a regular polygon in $\mathbb{Z}_c$.
- *The Clough/Myerson type.* When $m=1$ and hence $c=c'$ the map $\pi_d = \varphi_d = \varphi_{d'}$ is an automorphism of $\mathbb{Z}_c$ and the associated multiset v_A^d is the characteristic function of an ordinary cluster of cardinality d coprime with c. We find again the result of [8], e.g. that maximally even sets of cardinality d which are coprime with the chromatic cardinality c are generated by the inverse d^{-1} mod c.*
- *The general Clough/Douthett type.* From our construction,

* The contiguous order of cluster $A'=l_o+\{0,\ldots,d-1\}$ represent the generation order of ME set $A=l_o d^{-1}+\{0, d^{-1},\ldots,(d-1)d^{-1}\}$.

$$A = \pi_d^{-1}(\{{}^m l_0, \ldots {}^m l_{d'-1}\}) = \pi_d^{-1}({}^m l_0) \sqcup \cdots \pi_d^{-1}({}^m l_{d'-1}) = (a_0 + m\ \mathbb{Z}_c) \sqcup \cdots (a_{d'-1} + m\ \mathbb{Z}_c)$$
$$= \{a_0, \ldots a_{d'-1}\} \oplus m\ \mathbb{Z}_c,$$

meaning, in accordance with the known facts from [9], that general maximally even sets are Cartesian products of the two previous types, i.e. bundles of regular polygons which are anchored in a Clough/Myerson type maximally even set. For example, with the octatonic scale, we have $A' = \{{}^4 0, {}^4 2\}$, with pre-images 0, 3, 6, 9 for 4 and 1, 4, 7, 11 for 2: $A = \{0,1\} \oplus \{0,3,6,9\} = B \oplus 3\mathbb{Z}_{12}$.

There is a nice Fourier interpretation of this last and most complicated case: as seen above, A is periodic with period c'.

Let us introduce for clarity

$$B = \{0, 1 \ldots c' - 1\} \cap A = \{0, 1 \ldots c' - 1\} \cap \pi_d^{-1}(A') = \pi_{c'}(A) \subset \mathbb{Z}_{c'}.$$

We have shown that:

THEOREM 3.8 *A is a ME set in $\mathbb{Z}_c$ if and only if $A = B \oplus m\ \mathbb{Z}_c$* and *$B$ is a ME set in $\mathbb{Z}_{c'}$.*

This is pleasantly related to the following simple equation between Fourier transforms.

Remark If $A = B \oplus m\ \mathbb{Z}_c$, then $\mathcal{F}_A(d) = m\mathcal{F}_B(d')$ (B being considered as a subset of $\mathbb{Z}_{c'}$).

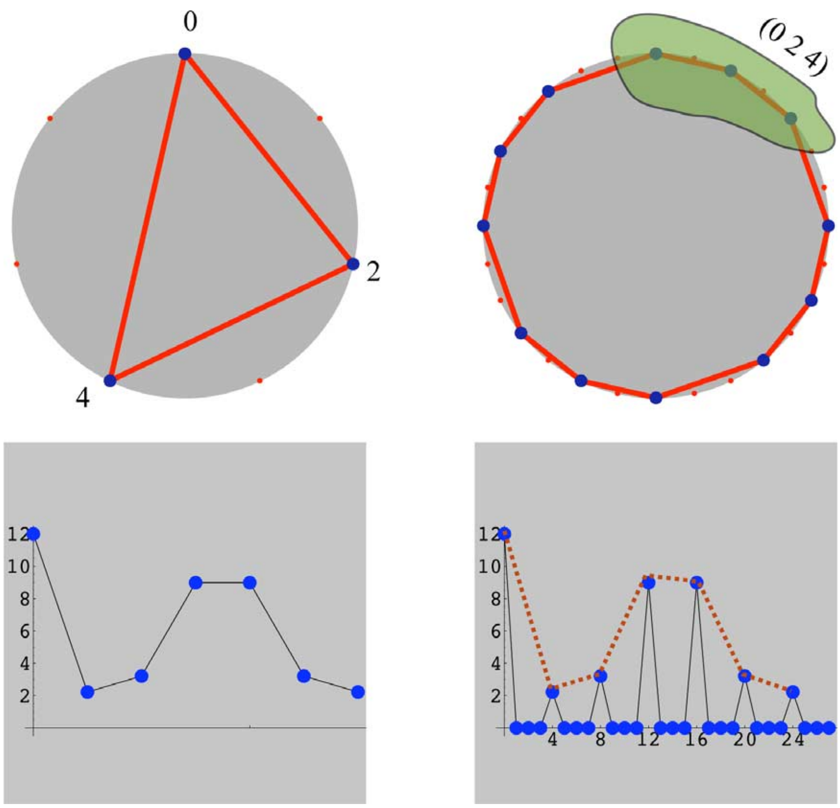


Figure 4. Maximizing for B is maximizing for A.

This number is of maximal length if and only if B is a ME set in $\mathbb{Z}_{c'}$, which is precisely the above theorem.

Indeed, the Fourier coefficients of B are (up to the m factor) the meaningful values of $\mathcal{F}_A(d)$, as when A is c'-periodic, all coefficients $\mathcal{F}_A(k)$ vanish for k not a multiple of c' (Proposition 2.6). This is clearly visible in figure 4, with Fourier transforms of the ME set (0 2 4) in $\mathbb{Z}_7$ and its counterpart (0 2 4) $\oplus$(0 7 14 21) in $\mathbb{Z}_{28}$. This argument seems to us more illuminating than purely algebraic computations, as it enhances the fact that the 'characteristic domain' B concentrates its energy in the sense of the Huddling Lemma, in order for A to do the same.

We obtain from this the complete enumeration of ME sets, which is developed at the end of Supplementary 1, available online.

3.7. *Expression by way of J-functions*

For the sake of completeness we add this technical but quick derivation of all ME sets.

THEOREM 3.9 *Let $A \subset \mathbb{Z}_c$ be the pc-set whose elements are given by the J-function, i.e.*

$$A = \{J^{\alpha}_{c,d}(k) | k = 0 \dots d-1\} = \left\{ \left\lfloor \frac{kc+\alpha}{d} \right\rfloor,\ k = 0 \dots d-1 \right\}.$$

Then $\pi_d(A)$ is a contiguous cluster of d' pitch classes of multiplicity m, i.e. A is maximally even.

Proof: We compute values of the floor function in $\mathbb{Z}$, but interpret the results in $\mathbb{Z}_c$ and $\mathbb{Z}_{c'}$. Further, we assume $\alpha = 0$ w.l.o.g.

From the equations

$$\left\lfloor \frac{(k+d')c}{d} \right\rfloor = \left\lfloor \frac{kc}{d} + \frac{d'c}{d} \right\rfloor = \left\lfloor \frac{kc}{d} \right\rfloor + c',$$

we conclude first that A is a disjoint union of m translates of the set $B = \{\lfloor kc/d \rfloor, k = 0 \dots d'-1\}$, with multiples of c' as displacements, i.e. $A = B \oplus c'\mathbb{Z}_c$. Thus, each element in the multiset $\pi_d(A)$ has multiplicity m. It remains to be shown that $\pi_d(B)$ is a contiguous cluster.

We will use the fact that the fractional parts of the rational numbers $kc/d = kc'/d'$ take d' different values when k runs from 0 to $d'-1$. This is true because c' and d' are co-prime. To see this, choose $0 \le k,\ k' < d'$:

$$\frac{k'c'}{d'} - \frac{kc'}{d'} = n \in \mathbb{Z} \Rightarrow (k'-k)c' = d'n \Rightarrow d' | (k'-k) \Rightarrow k' = k, \quad \text{since } |k'-k| < d'.$$

From the d' different fractional parts $0 \le kc'/d' - \lfloor kc'/d' \rfloor < 1$ we obtain d' different integers $0 \le kc' - d'\lfloor kc'/d' \rfloor \le d'-1$, which are in fact all the integers $0, \dots, d'$. Reduction of the elements $kc' - d'\lfloor kc'/d' \rfloor$ modulo c' yields the set $-\pi_d(B) = -d'B \bmod c'$.

Thus, $\pi_d(B)$ is a cluster, namely $\pi_d(B) = \{c'-d'+1, c'-d'+2, \dots, c'\}$. $\square$

4. Generated sets and groups generated by a set

The Fourier approach offers further directions of investigation. Here we restrict ourselves to maximality conditions for the absolute values for the Fourier coefficients. As we have seen in section 3, it is the index $d \in \mathbb{Z}_c$, i.e. the residue class of the chords cardinality, to which the maximality condition for maximal evenness is attached.

What about the other coefficients? It is illuminating to investigate the maximal Fourier coefficients among invertible indices $t \in \mathbb{Z}_c^*$ as well as among all non-zero indices $t \in \mathbb{Z}_c \backslash \{0\}$. In the definition below we exclude the index $t = 0$, because the maximum $|\mathcal{F}_A(0)| = d$ is shared by all sets A with d elements.

Definition 4.1 For any pitch class set $A \in \mathbb{Z}_c$ let $\|\mathcal{F}_A\| = \max_{t \in \mathbb{Z}_c, t \neq 0} |\mathcal{F}_A(t)|$ and $\|\mathcal{F}_A\|^* = \max_{t \in \mathbb{Z}_c^*} |\mathcal{F}_A(t)|$ be, respectively, the maximal absolute value among all Fourier coefficients at non-zero indices, and the maximal value of Fourier coefficients at *invertible* indices.

First notice that if $f : x \mapsto a\,x + b$, $a \in \mathbb{Z}_{c'}$, is a bijective affine map, then for any subset A

$$\|\mathcal{F}_{f(A)}\|^* = \|\mathcal{F}_A\|^*, \quad \text{as } \forall t \in \mathbb{Z}_c \quad |\mathcal{F}_{f(A)}(t)| = |\mathcal{F}_A(a\ t)|$$

(the Fourier coefficients are permuted by affine maps). The same for $\|\mathcal{F}_A\|$: these quantities are invariant on affine orbits of subsets.

There are three plausible values for the maximum $\|\mathcal{F}_A\|$ or $\|\mathcal{F}_A^*\|$. The first is the value characterizing ME sets.

Proposition 4.2 *Fix a cardinality d coprime with c. Let* $\mu(c, d) = |\mathcal{F}_B(d)|$ *for some* (c, d) *ME set B. For all d-element subsets of* $A \subset \mathbb{Z}_c$, *we find that* $\|\mathcal{F}_A\|^* \leq \mu(c, d)$. *The equality occurs iff* $A = r \cdot B + t$ *for suitable* $r \in \mathbb{Z}_c^*$ *and* $t \in \mathbb{Z}_c$, *or, equivalently,* $A = a_0 + \{0, f, \ldots, (d-1)f\}$ *is generated by a residue* $f \in \mathbb{Z}_c^*$ *(coprime with c).*

The second plausible value is $\sin(\pi\ d/c)/\sin(\pi/c)$, which is equal to $|\mathcal{F}_C(1)|$ for C a cluster, e.g. $C = \{1, 2 \ldots d\}$. The affine images of C are the generated scales with cardinality d, and we have a similar proposition.

Proposition 4.3 *Let* $\rho(c, d) = |\mathcal{F}_{\{1,2\ldots d\}}(1)|$. *For all d-element subsets of* $A \subset \mathbb{Z}_c$, *we find that* $\|\mathcal{F}_A\|^* \leq \rho(c, d)$. *The equality* $\|\mathcal{F}_A\|^* = \rho(c, d)$ *occurs if and only if* $A = a_0 + \{0, f, \ldots, (d-1)f\}$, *i.e. A is generated by a residue* $f \in \mathbb{Z}_c^*$ *coprime with c.*

The last interesting value is d itself, as we have seen that $\mathcal{F}_A(t)| \leq d\,\forall t$. First of all, remember that from Lemma 2.2, $\|\mathcal{F}_{\mathbb{Z}_c \backslash A}\| = \|\mathcal{F}_A\|$ is at most the lowest of d, $c - d$, so it is sufficient to work out the case $d \leq c/2$: dealing with a 'large' ME set ($d > c/2$) is equivalent to dealing with a 'small' one ($d \leq c/2$), its complement. Henceforth we will assume the latter case.

Proposition 4.4 $\|\mathcal{F}_A\| = d$ *iff A is contained in a regular polygon, i.e.*

$$\exists r \in \mathbb{N}, a_0 \in \mathbb{Z}_c, \quad 1 < r < c, \quad A \subset a_0 + r\mathbb{Z}_c.$$

Notice that, although this includes the generated scales that we missed in the last proposition, other cases are possible: $C = \{0, 2, 6\} \in \mathbb{Z}_{12}$ also checks $\mathcal{F}_C(6) = 3$.

The proofs of these propositions, and a discussion of the remaining chords with maximal $\|\mathcal{F}_A\|$ which are not of the previous types, are to be found in Supplementary 3, available online.

5. Chopin's theorem

As the inverse of a ME set (in the musical sense) is also maximally even, either $f' = d'^{-1}$ or its opposite $-f'$ will generate a $\langle c', d' \rangle$ ME set.* This has a consequence on complementary ME set classes: as $\gcd(c, c-d) = \gcd(c, d) = m$, when one replaces d by $c-d$, one obtains the same c', and replaces d' by $(c-d)/m = c' - d' \equiv -d' \bmod c'$; hence

LEMMA 5.1 *The same generator f' can be used for the construction of both $\langle c, d \rangle$ and $\langle c, \ c-d \rangle$ ME sets.*

For instance, the fifth $f' = f = 7$ generates both the pentatonic and the major scales, when $c = 12$. For, say, $c = 20$ and $d = 8$, one obtains $m = 4$, $d' = 2$, $c' = 5$, $f' = 3$ and the generated ME sets with eight and 12 elements are $\{0, 3\} \oplus \{0, 5, 10, 15\}$ and $\{0, 3, 6, 9\} \oplus \{0, 5, 10, 15\} = \{0, 3, 1, 4\} \oplus \{0, 5, 10, 15\}$. More generally,

THEOREM 5.2 *Let $1 < d \leq c/2$, then any given $\langle c, \ c-d \rangle$ ME set contains several (exactly $c' - 2d' + 1$) $\langle c, d \rangle$ ME sets. In other words, any 'small' ME set is contained in several translates of its complement.*

Proof: A $\langle c, d \rangle$ ME set is constructed by truncating to just d' consecutive values the sequence $\{f', 2f', \ldots (c' - d')f'\}$ mod c', which generates (adding up $c'\mathbb{Z}_c$) the given $\langle c, \ c-d \rangle$ ME set A. This can be done in precisely $c' - 2d' + 1$ ways.

From this, as seen in Theorem 3.8, it suffices to add $c' \ \mathbb{Z}_c$ to obtain *both* whole ME sets, since c' is the same for d and $c-d$, preserving the inclusion relation all the time. □

We would like to baptize this result CHOPIN'S THEOREM in reference to the ETUDE OP 10 N°5 (see figure S2 in Supplementary 2, available online) where the right hand plays the pentatonic (black keys only) while the left hand wanders through several keys, G flat and D flat major, for instance. This result has been observed (especially in this pentatonic $\subset$ major scale case) and commented on,† although perhaps it has not been stated and proved as a quality of all ME sets (or, alternatively, generated scales).

So David Lewin, who almost invented ME sets as we have seen, might also have originated set-complex *Kh*-theory in one fell swoop.

6. Coda

We have examined the definition of the DFT of a pc-set, according to David Lewin. Several interesting features of the pc-set are encapsulated in the absolute value of this function.

Following Ian Quinn, we were led to advance an original definition of Maximally Even Sets, which appears to be geometrical, concise, elegant, and illuminating.‡ We hope that this definition will become a productive one.

* The interesting question of *all* generators of a scale (not only for ME sets) will be elucidated in [11].

† For instance, in [1] (2.3): 'all secondary prototypes are Kh-related to one another', which seems to be a statement equivalent to the theorem above.

‡ Although less general than [10], which allows *all* possible strictly convex measures on the unit circle to be chosen indifferently.

Acknowledgements

First of all I thank Ian Quinn, who not only spelled out the property which forms the gist of this paper, but also drew our attention, through his comprehensive study of the chords landscape, to the impressive advantages of the DFT of chords, and not only ME sets and other 'prototypes'. David Clampitt kindly explained the subtleties of WF scales vs. ME sets and most of the history of these fascinating notions. Equally important to the field of 'mathemusical' knowledge is the continued contribution of Jack Douthett (with the late John Clough and other partners). Several reviewers were instrumental in bringing this paper up to the quality required by the Journal, an indomitable task for a lone writer. I would like to thank especially Dmitri Tymocsko, Robert Peck and, particularly, Thomas Noll, in that respect.

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Journal of Mathematics and Music
Vol. 1, No. 3, November 2007

David Lewin and maximally even sets

EMMANUEL AMIOT

1 rue du Centre, F 66570 St Nazaire, France

Supplementary 1: concerning maximally even sets

0.1. *A short history of ME sets*

Maximally Even Sets, or ME sets for short, were defined in [1], generalized in 2 and later extended to Well Formed Scales, which also exist in non-equal temperaments [3]. The name refers to the intuitive feature of being 'as evenly distributed in the chromatic circle as possible'. As we shall see, it is not so easy to make this idea rigorous: many different, although equivalent, definitions exist, and our main objective in this paper is to firmly ground the notion of ME sets on a DFT-based definition. We include a short paragraph for readers who might still be unfamiliar with the notion, followed by a discussion of several existing definitions. A very thorough paper on state-of-the-art applications of ME sets is [4].

Originally, Clough, Myerson and, soon after, Douthett observed this yet informal notion of 'maximal evenness' in a collection of famous scales: whole tone scale, major scale, pentatonic, octatonic, etc. For musicological reasons, and perhaps also because of mathematical difficulties we shall refer to below, their definition was rather indirect.

In the minor scale there are three different values of intervals between consecutive notes. Not so for the major scale, or the melodic (ascending) minor, but the latter features three different fifths.

From these examples, and others, ME sets were defined with regard to the different (some say 'diatonic') possible values of intervals inside the scale: for instance, the major scale and the pentatonic alike have only two different interval sizes between consecutive notes—tones and semi-tones for the one, tones and minor thirds for the other. Also note that the two semi-tones in the major scale, for instance, are as far from one another as possible. This has some relevance to the organization of black and white keys on a keyboard, and hence to traditional musical notation in staves.

The common original definition (here reworded) states that

Definition 0.7 Let A be a subset of $\mathbb{Z}_c$. For convenience, let use call a 'second' any interval between two adjacent elements of A, a 'third' an interval between every odd note, and so on. Then A is maximally even if, and only if, there are at most two different kinds of 'seconds', 'thirds', 'fourths', etc.

This definition suffers from the common blemish of many formalized musicological definitions that take for granted many notions with intuitive, musical support (such as

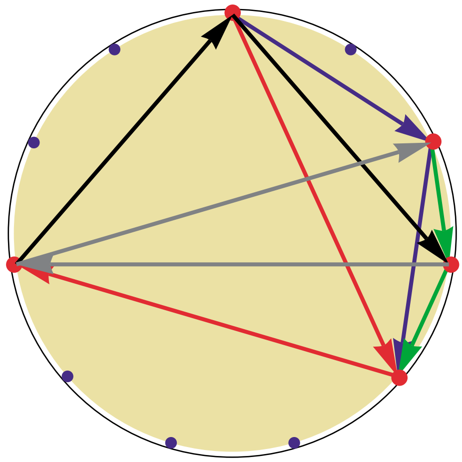

Figure S1. All intervals come in two sizes.

diatonic intervals, adjacency of notes, etc.) which are not so obvious to define mathematically.*

To state it with numbers: if an *ordered* scale† is $A = \{a_1, a_2, \ldots a_d\}$ with indexes taken modulo d and values taken modulo c, for each value of k there should be at most two different values of $a_{i+k} - a_i$ when i varies (figure S1).

This was named the 'Myhill property' in [1]‡) and it is not at all straightforward.

Worse still, in our opinion, this definition necessitates an ordering, or reordering, of the notes: (C E D G A) is not a ME set, although (C D E G A) is! This verges on the unsatisfactory, if one is interested in pc-sets and not (ordered) scales.

Many geometrical criteria have been given, and proved equivalent [4]; we especially like the 'black and white' definition in [2], very intuitive, although hardly practical (see figure S2): plot two regular polygons, one white with d vertexes and one black with c–d vertexes. Then rearrange all the vertexes, preserving order, with identical distance between consecutive points. Both black and white subsets are ME sets.

The most effective way to actually compute ME sets is as follows: taking c as the cardinality of the ambient chromatic space, d the number of notes of the looked-for set, and α some arbitrary number, the J-functions

$$J^{\alpha}_{c,d} : k \mapsto \left\lfloor \frac{kc + \alpha}{d} \right\rfloor, \quad k = 0 \ldots d-1,$$

already introduced in [1], give all ME sets with cardinality d by their sets of values

$$J^{\alpha}_{c,d}(0), J^{\alpha}_{c,d}(1), \ldots J^{\alpha}_{c,d}(d-1)$$

(taken modulo c): for instance, with $c = 12$, $d = 5$, $\alpha = 12$, one obtains the pentatonic (0 2 4 7 9), but relevance to the intuitive idea of maximum evenness, or even to sizes of intervals, is less than obvious.

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* To be fair, pre-Hilbert mathematics (and also some post-Hilbert) often relied too heavily on intuitions of the physical world, as the quarrel concerning non-Euclidean geometries made clear.

† We skip a formal definition of 'ordered' in $\mathbb{Z}_c$, which will be useless in our approach.

‡ Note that, in general, it is not sufficient that the Myhill property holds for adjacent notes, e.g. having only two kinds of 'seconds' does not ensure we have a ME set, as shown by the example of the melodic minor scale.

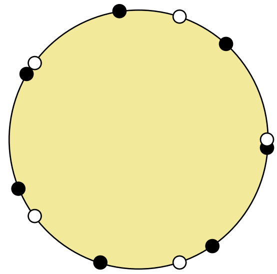

Figure S2. Rearranging the points of two intertwined regular polygons.

The most natural definition might be to try and maximize the mutual distances between all the notes, e.g. $\Sigma_{a,a' \in A}\ \delta(a, a')$, but the result depends on the chosen distance function δ, and is not satisfactory for the (arguably) most natural one, the interval metric:

$$\delta(u, v) = \min_{k \in Z} |u - v + kc|,$$

as several unexpected* extraneous solutions crop up, as in figure S3. A 'good' definition would be expected to give one characteristic shape for a given pair (c,d), not so many. This exemplifies why there is no universal, or obvious, definition for the naïve concept of 'Evenness'.

It is because none of these definitions (or others) appears completely satisfactory, in our opinion, that we ventured to propose another one.

0.2. *Symmetries of ME sets*

This is the sequel of subsection 3.6.

COROLLARY 0.8 *The number of different ME sets of cardinality d in $\mathbb{Z}_c$ is $c' = c/\gcd(c,d)$ (the number of different possible B's). All are translates of one another (the group of translations acts transitively on ME sets).*†

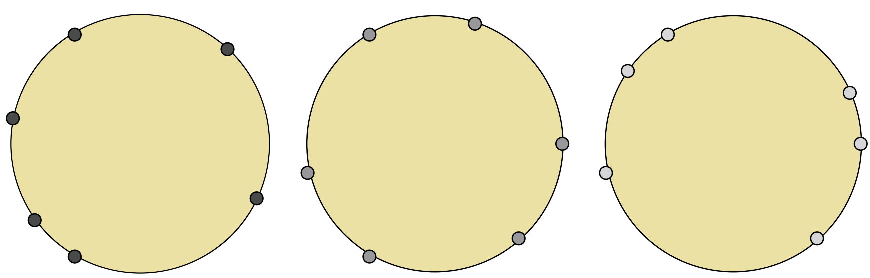

Figure S3. Some sets maximizing the sums of differences for the interval metric $-c = 15$, $d = 6$.

* But all *strictly* convex distance functions on the unit circle will give maximums on the same pc-sets, which are the ME sets, as shown in [5]. Nonetheless, such a distance (like the chordal distance, length of the line segment between two points of the circle) has little musical meaning.

† Only when $m = 1$ do we have simple transitivity, i.e. an interval group in the sense of [6].

For each couple (c,d) there is but one translation class of ME sets with d points in $\mathbb{Z}_c$. Henceforth, we will denote such a ME set *class* as $\langle c, d \rangle$. An actual ME set will be 'a $\langle c, d \rangle$ ME set'. For example, there are exactly three different $\langle 12, 8 \rangle$ ME sets, i.e. the octatonic scales.

Remark 1 Each individual $\langle c, d \rangle$ ME set is invariant under the m translations of step c' and multiples. We have seen (Theorem 2.5) that the inversion operation preserves the class of ME sets: this means that the inverse of a ME set is one of its translates. Indeed, a ME set is its own image under exactly‡ $2 \times m$ operations, m translations and m inversions in the dihedral group T/I of transformations of type $x \mapsto x + \tau$ and $x \mapsto \ell - x$ in $\mathbb{Z}_c$. For instance, inversions $x \mapsto -x, 3 - x, 6 - x, 9 - x$ preserve the above octatonic.

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‡ The stabilizer of any pc-set in T/I, isomorphic to the dihedral group D_c, is either a cyclic or a dihedral group. For $\langle c, d \rangle$ it is always D_m.

Journal of Mathematics and Music
Vol. 1, No. 3, November 2007

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EMMANUEL AMIOT

1 rue du Centre, F 66570 St Nazaire, France

Supplementary 2: pictures and proofs

In figure S1 with the two complementary hexachords, the fifths have been signalled with arrows. Each hexachord has the same number of fifths, three in this example (figure S2).

Proof: (proof of Proposition 2.6) From Theorem 2.5 we have

$$A \text{ is } \tau-\text{periodic} \iff \forall t \in \mathbb{Z}_c \quad \mathcal{F}_A(t) = e^{-2i\pi\tau t/c}\mathcal{F}_A(t) \iff \forall t \in \mathbb{Z}_c \quad (1 - e^{-2i\pi\tau t/c})\mathcal{F}_A(t) = 0.$$

Unless $e^{-2i\pi\tau t/c} = 1$, this compels $\mathcal{F}_A(t)$ to be 0. Now the condition $e^{-2i\pi\tau t/c} = 1$ is equivalent to $c \mid \tau t$, i.e. t multiple of $m = c/\gcd(c, \tau)$—this makes sense for any representative of the residue classes τ and t. This is compatible with reduction modulo c, and means $t \in m\,\mathbb{Z}_c \subset \mathbb{Z}_c$. Conversely, if $\mathcal{F}_A$ is nil except on a subgroup, say $m\,\mathbb{Z}_c$ with $0 < m \mid c$ in $\mathbb{Z}$ (we recall all subgroups of $\mathbb{Z}_c$ are cyclic), then, by inverse Fourier transform,

$$\forall k \in \mathbb{Z}_c \qquad 1_A(k) = \frac{1}{c}\sum_{t \in \mathbb{Z}_c} \mathcal{F}_A(t)\, \mathrm{e}^{2i\pi kt/c} = \frac{1}{c}\sum_{t' \in m\,\mathbb{Z}_c} \mathcal{F}_A(t')\, \mathrm{e}^{2i\pi k,t'/c}$$

$$= \frac{1}{c}\sum_{t''=1\ldots c/m} \mathcal{F}_A(m\,t'')\, \mathrm{e}^{2i\pi kt''m/c},$$

and this is obviously periodic with (the residue class of) c/m as a period, as each term in the sum is c/m periodic. □

Proof: (proof of Proposition 3.2) For transposition and inversion it is Theorem 2.5. For complementation we see that

$$|\mathcal{F}_{\mathbb{Z}_c}(c-d)| = |\mathcal{F}_{\mathbb{Z}_c}(-d)| = |-\overline{\mathcal{F}_A(d)}| = |\mathcal{F}_A(d)|$$

holds for any subset A. So one value is maximal whenever the other is, e.g. A is a ME set iff $\mathbb{Z}_c \diagdown A$ is maximally even. □

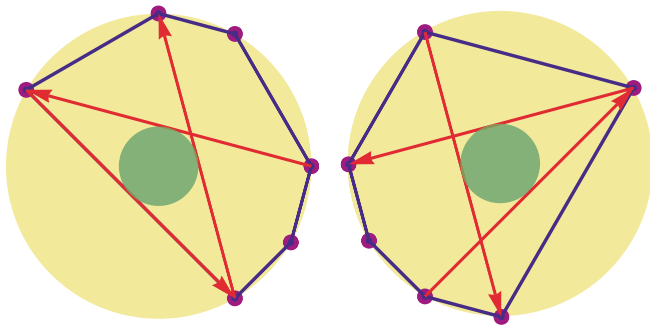

Figure S1. These two hexachords share intervallic content.

Proof: (proof of remark 1 (subsection 3.6)) Linking the Fourier coefficients of A and its reduction B mod c$'$:

$$\mathcal{F}_A(d) = \sum_{k \in A} \mathrm{e}^{-2i\pi dk/c} = \sum_{k'' \in B} \sum_{\ell=0}^{m-1} \mathrm{e}^{-2i\pi d(k''+\ell c')/c} = \sum_{k'' \in B} \mathrm{e}^{-2i\pi dk''/c} \sum_{\ell=0}^{m-1} \mathrm{e}^{-2i\pi \ell}$$
$$= m \sum_{k'' \in B} \mathrm{e}^{-2i\pi d'k''/c'} = m\mathcal{F}_B(d').$$

□

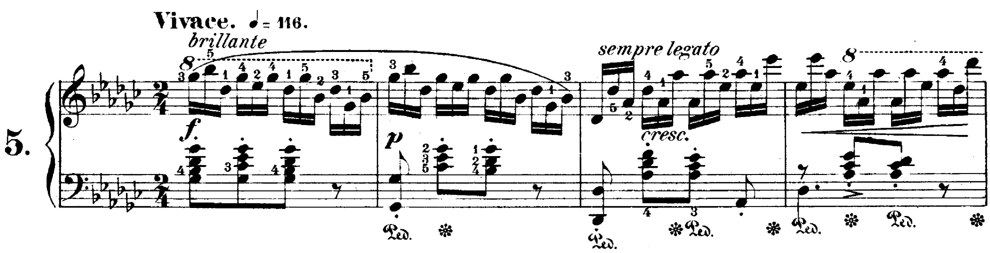

Figure S2. Etude N° 5 opus 10, Frédéric Chopin.

Journal of Mathematics and Music
Vol. 1, No. 3, November 2007

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EMMANUEL AMIOT

1 rue du Centre, F 66570 St Nazaire, France

Supplementary 3: concerning other maximums of Fourier coefficients

When d is coprime with c, generated $\langle c, d \rangle$ ME sets (the Clough–Myerson type) obtain their maximum Fourier coefficient value in d:

$$\|\mathcal{F}_A\| = \|\mathcal{F}_A\|^* = |\mathcal{F}_A(d)| = \mu(c, d) = \frac{\sin(\pi d/c)}{\sin(\pi/c)}.$$

Any generated scale with a generator coprime with c will share the same value of $\|\mathcal{F}_A\|^*$, as

- any ME set A is in affine bijection with any such generated scale, both being affine images of the cluster $\{0, 1, 2 \ldots d-1\}$, and
- if $f : x \mapsto a\,x + b$, $a \in \mathbb{Z}_{c'}$, is a bijective affine map, then $\|\mathcal{F}_{f(A)}\|^* = \|\mathcal{F}_A\|^*$, as $\forall\, t \in \mathbb{Z}_c\ |\mathcal{F}_{f(A)}(t)| = |\mathcal{F}_A(at)|$ (the Fourier coefficients are permuted by affine maps).

We can reformulate Proposition 4.2 in more detail.

PROPOSITION 0.7 *Fix a cardinality d coprime with c. For all d-element subsets of* $A \subset \mathbb{Z}_c$ *we find that* $\|\mathcal{F}_A\|^* \leq \mu(c, d)$. *With regard to equality the following three conditions are equivalent:*

(i) $\|\mathcal{F}_A\|^* = \mu(c, d)$;
(ii) $A = r \cdot M(c, d) + s$ *for suitable* $r \in \mathbb{Z}_c^*$ *and* $s \in \mathbb{Z}_c$ *and* $M(c, d)$ *as in definition 3.1*;
(iii) $A = a_0 + \{0, f, \ldots, (d-1)f\}$ *is generated by a residue* $f \in \mathbb{Z}_c^*$ *coprime with c.*

Proof: Choose $t \in \mathbb{Z}_c^*$ such that $\|\mathcal{F}_A\|^* = |\mathcal{F}_A(t)|$. Then we have $|\mathcal{F}_A(t)| = |\mathcal{F}_{d^{-1}t \cdot A}(d)| \leq \mu(c, d)$. To prove (i) $\Leftrightarrow$ (ii) we argue that the equality $\|\mathcal{F}_A\|^* = \mu(c, d)$ holds iff $d^{-1}t \cdot A = M(c, d) + s'$ or, equivalently, iff $A = t^{-1}d \cdot M(c, d) + t^{-1}d\, s'$. To prove (ii) $\Leftrightarrow$ (iii) we remember that $M(c, d) = k_0 + \{0, d^{-1}, \ldots, (d-1)d^{-1}\}$, hence $A = r \cdot M(c, d) + s = (rk_0 + s) + \{0, d^{-1}r, \ldots, (d-1)d^{-1}r\}$. □

When $c,\ d$ are no longer coprime this is no longer true. The value of $\mu(c, d) = |\mathcal{F}_A(d)|$ for a $\langle c, d \rangle$ ME set is now $m \sin(d'\pi/c')/\sin(\pi/c')$ (this comes from Theorem 3.8), which is larger than $\rho(c, d) = \sin(d\pi/c)/\sin(\pi/c)$ because (by concavity) $\sin(\pi/c') = \sin(m\ \pi/c) \leq m\ \sin(\pi/c)$.

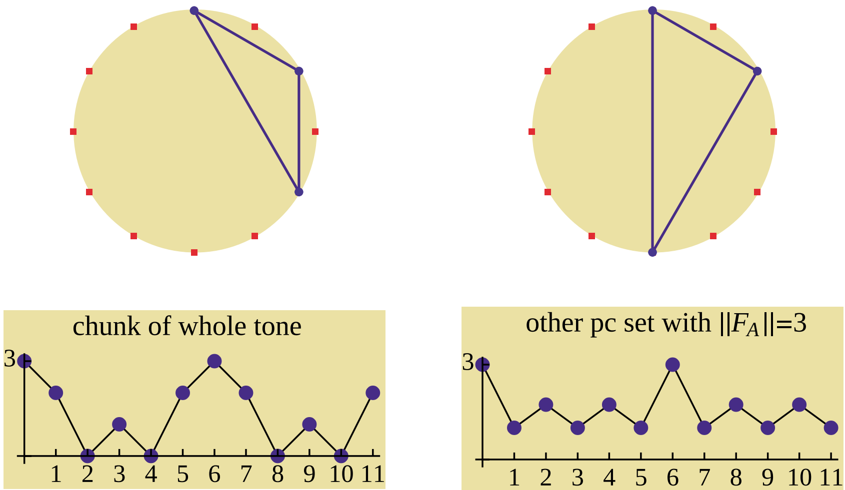

Figure S1.

But in that more general case, and with this value, we can characterize scales generated by some *invertible* generator (among which are the chromatic clusters): this is proposition 4.3, the proof of which follows.

Proof: Choose $t_0 \in \mathbb{Z}_c^*$ such that $\|\mathcal{F}_A\|^* = |\mathcal{F}_A(t_0)|$. Then we have $|\mathcal{F}_A(t_0)| = |\mathcal{F}_{t_0 A}(1)| \leq \mu(c, d)$ by the Huddling Lemma in the simple case $m = 1$. The maximal case is that of a cluster, i.e. $t_0 A = \tau + \{0, 1 \ldots d-1\}$ is a cluster. Multiplying by $f = t_0^{-1}$ we obtain $A = a_0 + f\{0, 1 \ldots d-1\}$. □

We do not find a characterization of *all* generated scales, i.e. also for generators not coprime with c. This is because, for instance, the chunk of whole tone scale $A = \{0, 2, 4\} \subset \mathbb{Z}_{12}$, generated though certainly not maximally even, realizes $\mathcal{F}_A(6) = 3$, clearly an unbeatable value (note that $|\mathcal{F}_A(3)| = 1 < 3$) (figure S1).

In order to understand better the maximality condition for $\|\mathcal{F}_A\|$, it is useful to inspect the subgroup of $\mathbb{Z}_c$ that is generated by the intervals of a pitch class set A.

Definition 0.8 For any pitch class set $A \subset \mathbb{Z}_c$ let $G[A] \subset \mathbb{Z}_c$ denote the interval group* of A. It is generated by the differences in A : $G[A] := \langle A - A \rangle = \{r \cdot (k_1 - k_2) | k_1, k_2 \in A, r \in \mathbb{Z}_c\}$. One can see that $G[A] = \langle \{a_0 - k | k \in A\} \rangle$ independently of the choice of $a_0 \in A$ (cf. [1], p. 125).

It will be impossible to reach $\|\mathcal{F}_A\| = d$ for a 'large' ME set, i.e. when $d > c/2$, as, in general, $\|\mathcal{F}_A\| \leq \min(d, c-d)$. So we work with the case $d \leq c/2$.

Theorem 0.9 $\|\mathcal{F}_A\| = d \iff G[A] \neq \mathbb{Z}_c$. *Any subgroup of $\mathbb{Z}_c$ being cyclic, say $G[A] = r\,\mathbb{Z}_c$ (taking r minimal); this means $A \subset a_0 + r\mathbb{Z}_c$. This can happen if and only if d is lower than some strict divisor $c' = c/r$* of c *(for instance, whenever c is even).*

Proof: Assume $\|\mathcal{F}_A\| = d$. Then $|\mathcal{F}_A(t_0)| = |\Sigma_{k \in A}\, e^{-2i\pi\, k\, t_0/c}| = d$ for some $t_0 \neq 0$; but from Cauchy–Schwarz inequality's case of equality, this means that all exponentials, each with length 1, are equal. In other words, multiset $t_0\, A = \{{}^d a\}$ is a singleton with multiplicity d (and t_0 cannot be invertible). Hence A is a subset of the pre-images of a,

* In a more general contex, Mazzola [1] (pp. 125–127) calls this the module of a local composition.

i.e. $A = a_0 + \ker\varphi_{t_0}$ i.e. $G[A] = \ker\varphi_{t_0}$. As we have seen when studying maps φ_d, this kernel is a regular polygon with $c' = c/\gcd(c, t_0)$ elements. So $\mathrm{Card}(A) \leq c'$, a strict divisor of c.

Conversely, assume $d \leq c' = c/m$, a strict divisor of c. Then there are subsets A of $c' \, \mathbb{Z}_c$ with cardinality d, any of which will check $|\mathcal{F}_A(m)| = d$. It is notable that, in that case, the maximum is reached for members of a subgroup:

$$|\mathcal{F}_A(t)| = d \iff t \in m\,\mathbb{Z}_c.$$

□

Note that although this includes generated scales, other cases are possible: $C = \{0, 2, 6\} \in \mathbb{Z}_{12}$ also checks $\mathcal{F}_C(6) = 3$ (see figure S1). This includes the 'secondary' and many ternary 'prototypes'* in [2] (2.4), as it appears Quinn had noticed. This class of maximal pc-sets includes generated scales, but is somewhat wider.

The general question now arises: *for a given pair* (c, d), *what are the subsets* $A \subset \mathbb{Z}_c$ *with cardinality d that yield the maximal value* $\mathcal{M}$ *of all* $\|\mathcal{F}_A\|$? There are three cases.

(i) The maximum value is d: this is well understood from the last theorem. It happens whenever d is lower than some strict divisor of c.

(ii) The maximum value is the one for ME sets, i.e. $m \sin(\pi d'/c') / \sin(\pi/c')$. This is very often the case. For instance, for $c = 11$; $d = 4$ the cluster (0 1 2 3) is not maximal for $\|\mathcal{F}_A\|$: the winner is the $\langle 11, 4\rangle$ ME set (0 3 6 9).

(iii) Sometimes the maximum is not one of the previous types. For instance, for $c = 75$, $d = 27$ when d is larger than all divisors of c, one obtains $\mu(c, d) = 21.6581$ for ME sets or their affine image; the value for clusters or generated scales is lower, $\rho(c, d) = 21.6075$ (this is general), but for

$$A = \{0, 4\} \cup \{0, 3, 6, 9, 12, \ldots, 66, 69, 72\},$$

i.e. $3\mathbb{Z}_c \cup \{1, 4\}$, one obtains $\|\mathcal{F}_A\| = \sqrt{579} = 24.062188$. The principle involved is, just as in the Huddling Lemma but in greater generality, to have for some k the multiset $A' = k\,A \subset \mathbb{Z}_c$ as 'clustered' as possible (we depart here from the definition of A' in figure 1 of the main article). Ideally, from the Huddling Lemma philosophy, one should aim at a few multipoints as close as possible in $k\mathbb{Z}c$, with maximum multiplicities. Let us clarify this without working out the complete theory.

We are working with a subset A with cardinality d, greater than any strict divisor of c, which is an odd number. The maximum value of $|\mathcal{F}_A|$ is some $|\mathcal{F}_A(k)| = |\mathcal{F}_{A'}(1)|$ where A' is the multiset kA; k is not coprime with c or else we obtain a previous case, and, as permutations of Fourier coefficients are irrelevant, we can assume (up to an affine transform of A) that k is a divisor of c.

The kernel of π_k: $t \mapsto kt \bmod c$ is $c'\mathbb{Z}_c$ with $c' = c/k$, and it has k elements; this is the maximum possible multiplicity for a point of A'. The distance between consecutive points of A' (in $\mathbb{Z}_c$) will be k, hence the number of different points in A' is $d' = \lceil d/k \rceil$. All these points will have multiplicity k, except one on the border with multiplicity $d - kd'$. The maximum will be found among the different subsets obtained in this way, checking on all values of $k \mid c$.

In the example, A' has one point (0) with multiplicity $k = 25$ and another one (25) with multiplicity 2. B' has two points, with multiplicity 15 and 12, respectively, and C'

* (0 2 6) mod 10 falls under the last theorem, but not (0 1 2 3 4 5 7 8 9).

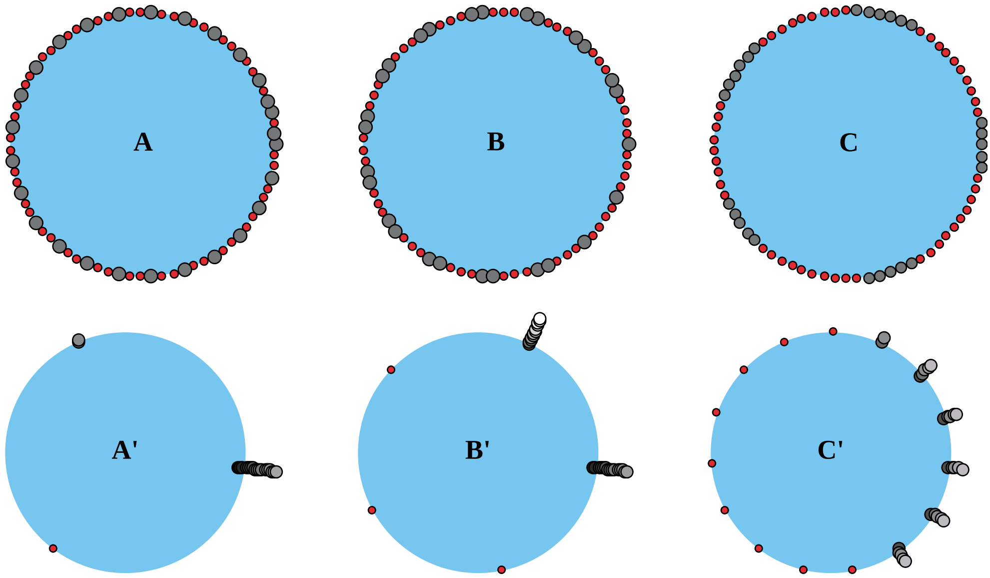

Figure S2.

has five points with multiplicity 5 and one with multiplicity 2 (other possible values of k have been left aside).

As it happens, the respective values are

$$\|\mathcal{F}_A\| = \sqrt{579} = 24.0624, \quad \|\mathcal{F}_B\| = \sqrt{909 - 180\sqrt{5}} = 22.5057, \qquad \|\mathcal{F}_C\| = 21.529.$$

In that case, the winner is A, corresponding to the greatest divisor ($k = 25$). But, in other cases, it may pay to restrict the angular span of the associated multiset–this means that $(d'-1)(2\pi/c')$ is made as low as possible, by choice of the divisor k of c. For instance, with $c = 75$, $d = 29$, the set analogous to B (i.e. one point with maximum multiplicity $k = 15$ and another one with multiplicity 14) is the winner, as $\|\mathcal{F}_A\| = 23.2594 < \|\mathcal{F}_B\| = 23.4689$ and $\|\mathcal{F}_C\| = 22.3884$.

So there is an algorithm, but no hard and fast rule for constructing the subsets with the largest $\|\mathcal{F}_A\|$. The solutions approximate (affine transforms of) ME sets, add or drop a few points. Perhaps these chords, or scales, which generalize ME sets in a way (look at the multisets in figure S2), and are defined modulo the action of the affine group, should be catalogued as, for instance, the tiling subsets of $\mathbb{Z}_n$ have been. This is one of many interesting directions for future research on the subject of musically relevant features of the DFT of discrete structures.

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