EXPONENTIAL FREQUENCY MODULATION BANDWIDTH CRITERION FOR VIRTUAL ANALOG APPLICATIONS

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ABSTRACT

Cross modulation or Exponential FM is a sound synthesis technique associated with modular analog subtractive synthesizers. It differs from the more well-known linear FM synthesis technique in that the modulation is an exponential function of the control voltage. Its spectrum shape is more complex, thus giving it a larger bandwidth with respect to the modulation depth. Thus, the prevention of aliasing distortion requires different conditions than Carson's rule as used with linear FM. A suitable equation will be presented in this paper.

1. INTRODUCTION

Research into virtual implementations of the structures of analog subtractive synthesizers has been a popular topic for the last few years. There have been a number of algorithms proposed to create bandlimited versions of the classic analog oscillators [1]. Alongside this, there have been a number of papers that derive models for the Voltage controlled filters associated with particular analog synthesizers. The designs for these have been based on an explicit circuit analysis, [2] for example, or those that create a version of the original using standard digital filter elements [3]. However, other elements of subtractive synthesis systems have not received such in-depth treatment such as the Attack-Decay-Sustain-Release (ADSR) envelope generators, Filter FM effects, and other oscillator modulation configurations. Although on modern analog synthesizers linear Frequency modulation (FM) between oscillators is sometimes a feature this was not always that case. In fact, Linear Frequency Modulation is really associated with digital synthesis and was viewed as the synthesis technology that overtook subtractive synthesis in the 1980s [4]. Linear Frequency modulation was unavailable in early modular subtractive synthesis systems for two primary reasons: Firstly, it was difficult to implement because of the Volts/Octave control voltage concept on which the elements of these synthesizers were interconnected. This system creates a non-linear relationship between any change in signal voltage and the pitch [5]. Thus, to force it to behave in a linear manner was difficult. Secondly, the tuning instabilities associated with early analog synthesizers because of component drift and ambient temperature fluctuations would have resulted in inconsistent generation of stable linear FM signals [6]. This is particular important in the case when the linear FM signal is desired to be harmonic; to achieve this, the frequencies of the carrier and modulator signals must strictly be in an integer relationship. Any deviation from this can seriously affect the perception of the sound.

The nonlinear Volts/Octave control voltage relationship of these analog synthesizers meant that modulation of one oscillator by another actually produced an Exponential FM waveform. One notable feature of this technique was that when the modulation depth was varied dynamically, a pitch shifting of the sound was perceived. This issue meant that for musicians Exponential FM was most often used to produce special effect sounds that were clangorous rather than for melodic lines. An excellent treatment of the theory of Exponential FM was written by [7]. This paper provided a description of the signal and its spectrum for sine-wave modulators. It also offered a configuration of analog modules that introduced a pitch correction factor that could be used to produce a harmonic version of Exponential FM. However, the work in [7] was written for implementation on an analog synthesizer system and thus assumed an infinite output bandwidth. This is not the case for digital implementations and algorithm designers always have to be aware of limitations on signal bandwidth imposed by the sampling frequency so as to minimise aliasing distortion. Therefore, this paper will examine the implementation of Exponential FM from a digital perspective. It will examine the spectra produced by the Exponential FM system in an effort to produce a guideline for its digital implementation. Section 2 will introduce the theory behind Exponential FM and will provide the spectral bandwidth analysis. Section 3 will contain the conclusions.

2. THEORY OF EXPONENTIAL FM

First of all, the Volts/Octave control signal representation in analog synthesizers means that the pitch of a note doubles as the control voltage doubles [5]. Thus, the relationship can be expressed

$$f \propto 2^{V} \tag{1}$$

where f is the note pitch and V is the control voltage.

For example, if 2 volts produces a pitch of 110Hz, then 3 volts will produce a note an octave higher of 220Hz.

Next, although the well-known equation for linear FM synthesis is actually phase modulation, on a modular synthesizer system Exponential FM is implemented as a true frequency modulation. The modulator signal voltage is interpreted as frequency variation that is used to define the control voltage associated with frequency of the carrier. The modulator is thus an instantaneous frequency signal that for exponential FM is defined as in [7] to be

$$\dot{\theta}(t) = f_c 2^{V(t)} \quad (2)$$

where f_c and f_m are the Carrier and Modulation frequencies in hertz respectively, and $V(t)$ represents the modulating signal. Note that the dot on the term on the left hand side indicates that it is a differential (of the phase).

The Modulating signal can be written as the combination of a DC term, V_0 and a time-varying quantity, assumed to be a cosine here, of amplitude V_m, which can also be termed as the Modulation Depth [7],

$$V(t) = V_0 + V_m \cos(2\pi f_m t) \quad (3)$$

Substituting (3) into (2) and using logarithms to write the power term

$$\dot{\theta}(t) = f_c e^{(V_0 + V_m \cos(2\pi f_m t))\ln(2)} \quad (4)$$

which can be rewritten

$$\dot{\theta}(t) = f_{ce} e^{(V_m \cos(2\pi f_m t))\ln(2)} \quad (5)$$

with

$$f_{ce} = f_c e^{V_0 \ln(2)} \quad (6)$$

To convert the instantaneous frequency signal into a phase the exponential term in (5) must be integrated

$$\theta(t) = \int \dot{\theta}(t) \quad (7)$$

However, it is not possible to integrate the exponential term in (5) directly and instead it must be expanded as set of Modified Bessel functions [7]

$$e^{V_m \cos(2\pi f_m t)\ln(2)} = I_0(V_m \ln(2)) + 2\sum_{k=1}^{\infty} I_k(V_m \ln(2)) \cos(2\pi k f_m t) \quad (8)$$

Substituting (8) back into (5)

$$\dot{\theta}(t) = I_0(V_m \ln(2)) f_{ce} + \left(2\sum_{k=1}^{\infty} I_k(V_m \ln(2)) \cos(2\pi k f_m t)\right) f_c \quad (9)$$

Integrating to obtain the phase as shown by (7) and assuming a sinusoidal carrier will produce the time domain signal [7]

$$y(t) = \sin(\theta(t)) = \sin\left(2\pi I_0(V_m \ln(2)) f_{ce}\, t + \left(\sum_{k=1}^{\infty} \frac{2 f_c}{k f_m} I_k(V_m \ln(2)) \sin(2\pi k f_m t)\right)\right) \quad (10)$$

The carrier frequency of (11) can be written as

$$f_E = I_0(V_m \ln(2)) f_{ce} \quad (11)$$

and the frequency deviation of each term is

$$D_k = \frac{2 f_c}{k f_m} I_k(V_m \ln(2)) \quad (12)$$

The expression in (10) is a multi-component complex FM signal.

2.1. Carrier Frequency Analysis

From (6) and (10) it can be seen that the final carrier frequency of the Exponential FM signal is a function of the exponent of the DC term V_0 and the value of the zeroth modified Bessel function that has the Modulation Depth in its argument. This is different to the linear FM case where the carrier frequency is independent of the modulation. This relationship should be taken into account for the digital version of Exponential FM. For example, assuming for convenience that the DC term V_0 is zero it is possible to show graphically how the carrier frequency increases with increasing modulation amplitude. This is illustrated in Figure 1 where the input carrier frequency f_c is plotted along the x-axis, the Modulation Depth V_m on the y-axis, and the actual carrier frequency given by (11) on the z-axis. The maximum possible value of the input carrier was assumed to be 8372Hz, corresponding to midi-note #120.

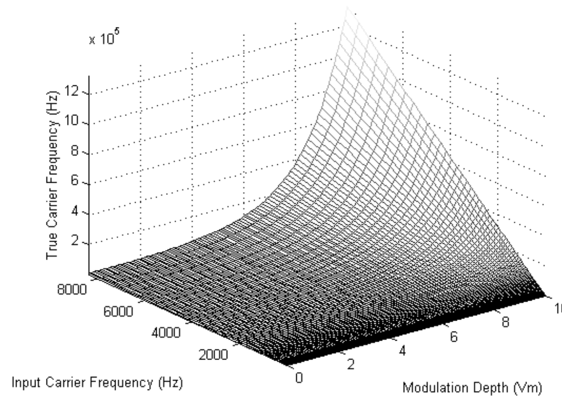

Figure 1: *The relationship between the input carrier frequency, the modulation depth and the actual carrier frequency.*

From Figure 1 it can be seen the actual carrier frequency increases quite rapidly as the Modulation Depth V_m reaches values of 8 or more. This illustrates the difference between linear FM and Exponential FM well. It also hints at the problems that can occur with the digital implementation of Exponential FM and warns that care must be taken when setting a sampling frequency for any implementation so that it is commensurate with the width of the Modulation Depth control. Lastly, looking at (11) it can be seen that including a DC term (V_0) in the modulating signal adds further complications in that it can raise the carrier frequency significantly. For example, for V_0=5 the carrier frequency will be scaled by a factor of 7.17.

2.2. Computing the Spectrum of Exponential FM

To obtain the spectrum of the Exponential FM signal in (10) there are a number of possible approaches. These can be numerical, analytical, or a hybrid of the two. Note though what is more useful here is the spectrum envelope rather than the actual spectrum itself. When attempting to compute the bandwidth it is much easier to work with the envelope because any gaps that exist between the partials in the signal that can disrupt an automated spectral analysis process to find a significant low energy spectral region.

The most obvious numerical technique to use is the Fast Fourier Transform (FFT). This is readily available in most software environments. However, this does not produce the envelope itself and further processing is required for this. Methods to achieve this include autoregressive analysis or Cepstral techniques.

Another approach is to employ the semi-analytic method of [8] that expresses the frequency modulation itself as a piecewise linear function. The total spectrum of the modulated output is the summation of the spectra for each piecewise-linear modulated part of the entire waveform. In essence this models the modulated waveform as a succession of linear Chirp signals, and thus the spectrum is the combination of the spectra for these chirps. In [8] it is proposed that the spectrum of the chirp signal is obtained using a numerical evaluation of the Fresnel equations. A faster estimate can be obtained using a Stationary Phase Approximation (SPA) and it also does not have any associated numerical integration issues [9]. However, efforts to apply this technique were not successful. It resulted in an approximate spectrum that had a blocky appearance which was not a particularly good match to the FFT based spectrum. It neither captured the true height of the various spectral components or the complete width of the spectrum.

Aside from the quasi-static approach for spectral approximation that is allowable under very particular conditions [10], the most general analytical approach is to expand the modulation term of (10) using Bessel functions [10]. Rewriting (10) by substituting (11) and (12)

$$y(t) = \sin\left(2\pi f_E\, t + \left(\sum_{k=1}^{\infty} D_k \sin(2\pi k f_m t)\right)\right) \quad (13)$$

It can be expanded as [10]

$$\sin\left(\omega n + \sum_{i=1}^{k} D_k \sin(\omega_k n + \varphi_k)\right) = \sum_{k_k} \cdots \sum_{k_1}\left(\prod_{i=1}^{k} J_{k_i}(D_k)\right)\sin\left(\omega n + \left(\sum_{i=1}^{k} k_i(\omega_i n + \varphi_i)\right)\right) \quad (14)$$

An example of using (14) to produce the magnitude spectrum is given in Figure 2 for f_c=440Hz and f_m=44Hz and V_m=2. In the figure the reflection of the negative frequencies back into the positive frequency region was not carried out as these always introduce spectral zeros for this particular signal case, while what is required is a smooth spectral envelope. It can be seen in Figure 2 that the envelope has a number of resonant peaks that become wider with respect to increasing frequency. The most significant component is below about half the carrier frequency. Although the higher frequency components are smaller than the carrier the spectrum does not have a true lowpass shape, but rather curves upwards at the last resonant peak.

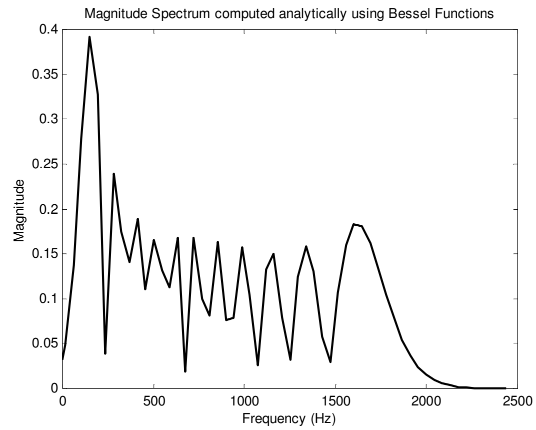

Figure 2: *Magnitude spectrum of an Exponential FM signal using a Bessel Function Analysis.*

There are two difficulties with the Bessel function based spectral analysis. Firstly, given that (14) is a complicated expression involving product terms, it is only possible to evaluate it numerically. This means that it cannot provide an intuitive compact expression. Secondly, the complexity of the evaluation of the equation grows exponentially with each additional sinusoidal term in the modulation signal. Some savings can be made by eliminating the evaluation of low amplitude Bessel function terms from the computation, and using programming optimisations to avoid the nested loop implementation.

2.3. Spectral Bandwidth Evaluation

The spectral bandwidth such that aliasing components would be of sufficiently low magnitude was defined to be point at which the spectrum was 80dB below the peak value. This was a reasonably strict criteria and much stronger than Carson's rule [10]. The intention was to express the Bandwidth as a function of the carrier frequency and the Modulation Depth.

First, using (14) the spectra of Exponential FM signals were computed for different values of carrier frequency and with fixed values for the modulation frequency and modulation depth. It was found that under these conditions the spectra were simply translated in relation to the carrier meaning shape invariant to the carrier frequency.

Next, spectra were again generated with the modulation frequency was expressed as a ratio of the carrier frequency from 0.1 up to 1 for a fixed value of Modulation Depth, and the bandwidth measured by an automated analysis in each case. This was repeated for other values of Modulation Depth. Figure 3 shows a plot of the results for values of Modulation Depth V_m=1, 2 and 3. The Bandwidth relative to the carrier frequency is shown on the *y*-axis as a multiple of the carrier frequency. From the plot it can be seen that the relationship between the Modulation frequency and relative Bandwidth is almost linear for all values of V_m. Thus, a simple linear fit can be made to characterize the relationship.

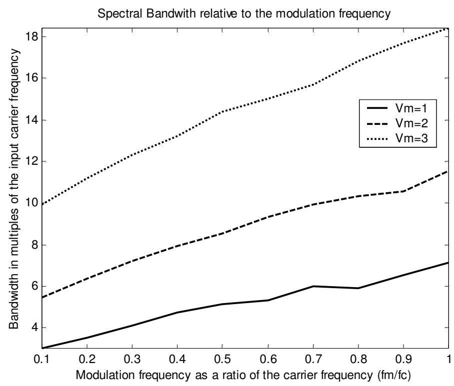

Figure 3: *Spectral Bandwidth shown as a function of the Modulation frequency expressed as a ratio of the carrier frequency for three different values of Modulation Depth.*

For the three different values of V_m in the figure the coefficients calculated are given in Table 1

V_m	$\boldsymbol{p_0}$	$\boldsymbol{p_1}$
1	2.771	4.3030
2	5.168	6.4606
3	9.3841	9.2485

Table 1: *Coefficients linear fit to data in Figure 4 for different values of Modulation Depth.*

The equation for the relative Bandwidth (to be multiplied by f_c for the value in Hz) was

$$BW(f_c, f_m, V_m) = p_0 + p_1(f_m/f_c) \tag{15}$$

To create a more general expression that also includes V_m on the right hand side of (15) the values in Table I can be examined. It can be seen that as V_m increases the value for $\boldsymbol{p_1}$ increases approximately by a factor V_m. Similarly, the value for $\boldsymbol{p_0}$ increases by about V_m -1 to the power of 2. Incorporating this along with possible scaling of f_c by the DC term V_0, (15) gives the final expression for the -80dB bandwidth in Hertz

$$BW_{-80dBHz}(f_c, f_m) = f_c e^{V_0 \ln(2)} 2^{V_m - 1} p_{01} + (p_{11} + V_m) f_m \tag{16}$$

where $p_{01} = 2.771$ and $p_{11} = 4.3030$.

Two examples are given in Figure 4 to illustrate the validity of this expression. These are shown in Figure 4. In the upper panel the values to generate the Exponential FM signal were f_c=100Hz and f_m=250Hz, V_0=1 and V_m=4. Its actual spectrum was computed using a Hanning windowed FFT and is displayed using a dB scale. The equation in (16) was used to find the bandwidth in Hz. This is plotted using the dashed vertical line in the figure. It clearly marks a point close to -80dB in front of the primary spectral region. In the lower panel the values were f_c=10Hz and f_m=40Hz, V_0=0 and V_m=7. Again, the dB magnitude FFT was found and the bandwidth was plotted using a dashed vertical line. In this case it actually estimates the -80dB to be at a greater location in frequency. However, this error is acceptable as it is an overestimation rather than an underestimation.

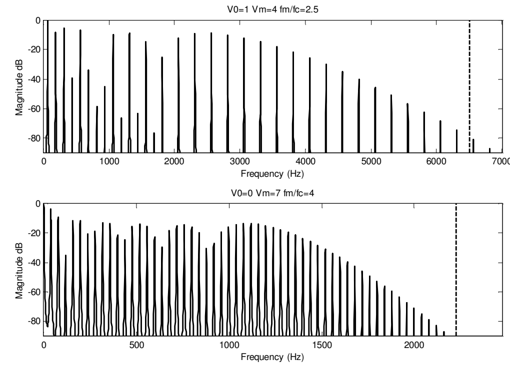

Figure 4: *Example plots of FFT spectra of Exponential FM signals with the -80dB Bandwidth in Hz highlighted using a dashed line in both panels. Upper panel parameters were* f_c=100Hz *and* f_m=250Hz, V_0=1 *and* V_m=4 *and for the lower panel were* f_c=10Hz *and* f_m=40Hz, V_0=0 *and* V_m=7.

3. CONCLUSIONS

This paper has presented an expression to evaluate the -80dB bandwidth of a cosine modulated Exponential FM signal. Examples were given to demonstrate its effectiveness. It should be a useful formula for low-aliasing digital implementations of Exponential FM. Future work aims to produce an exact analytical expression for the spectrum of the Exponential FM signal. It will also investigate bandwidth criteria for cases when the modulation is not sinusoidal, such as for sawtooth and square waves.

4. REFERENCES

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