PHYSICAL MODEL OF THE STRING-FRET INTERACTION

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ABSTRACT

In this paper, a model for the interaction of the strings with the frets in a guitar or other fretted string instruments is introduced.

In the two-polarization representation of the string oscillations it is observed that the string interacts with the fret in different ways. While the vertical oscillation is governed by perfect or imperfect clamping of the string on the fret, the horizontal oscillation is subject to friction of the string over the surface of the fret.

The proposed model allows, in particular, for the accurate evaluation of the elongation of the string in the two modes, which gives rise to audible dynamic detuning. The realization of this model into a structurally passive system for use in digital waveguide synthesis is detailed.

By changing the friction parameters, the model can be employed in fretless instruments too, where the string directly interacts with the neck surface.

1. INTRODUCTION

Accurate physically inspired synthesis of musical instrument require realistic models of all the parts of the instrument that significantly contribute to the production of the characteristic timber and its evolution, together with sufficiently general models of the interaction of the player with the instrument [1].

This work is a piece of a broader project whose aim is to closely emulate the playing of a guitar, with extension to other instruments in the family of plucked strings. In previous papers, the author, together with other researchers, introduced models for the plucking of the string, both with finger and plectrum, for the collisions of the string with the neck and other objects and for the synthesis of harmonic or flageolet tones [2, 3, 4, 5, 6, 7]. The models were introduced for immediate application in digital waveguide synthesis of the guitar, but they are also usable in other type of synthesis techniques such as finite difference time domain (FDTD).

In this paper, the issue of modeling the fret-string interaction is considered, which influences the sound produced by the synthesis algorithm. Disregarding longitudinal modes, a guitar string is represented by coupled wave equations, each pertaining to a polarization mode, i.e. to one of the orthogonal axes in the planes transversal to the string. In a fretted instrument, when a player's finger pushes the string against the frets in order to produce the desired tone, perfect or near perfect clamping only occurs in the direction normal to the fret surface.

In the horizontal direction, i.e. in the direction parallel to fret and tangent to this – in many electric guitars the fret is curved – the string is free to move but subject to friction on the fret surface. As a result, not only the two polarization modes show different decay times but also their fundamental frequencies differ and vary with time, due to the unequal elongation of the string. In fretless instruments the dynamic is similar but the string is pushed directly against the neck. As a result, clamping is less perfect in the vertical direction and the string is still free to move but subject to friction on the neck surface in the horizontal direction. Furthermore, the direct contact with the finger of the player introduces a considerable amount of damping and collisions with the neck are more likely due to smaller string to neck distance.

A new model for the fret-string interaction is presented in this paper, which is based on recent advances in modeling friction with dynamic systems [8, 9]. From the equations governing the string motion in the contact area with fret or neck we derive a structurally passive junction to be included as a fret-string interaction module in a pair of double rail digital waveguides, one for each polarization mode.

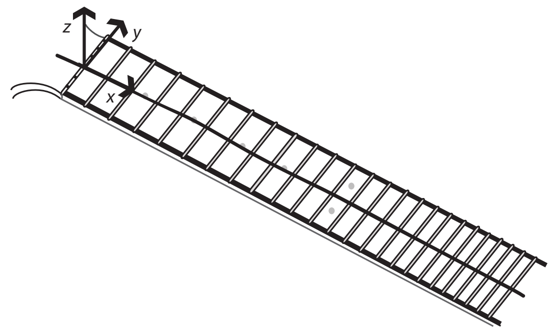

Figure 1: *Coordinate system aligned with the string rest position.*

2. FRETBOARD FINGERING

In this paper we choose the coordinate system shown in Figure 1 where the x axis is directed from nut to bridge along the string rest position. The y axis representing the horizontal direction is parallel to the frets, while the z axis is orthogonal to the other two axes and represents the vertical direction. In order to fix our ideas, we assume that the instrument is designed for a right-hand player, where strings are plucked with the right hand and the player pushes fingers of the left hand on the strings against the fretboard.

In conventional fretted instrument, frets are spaced on the fingerboard in order to achieve equally tempered tuning. This is achieved by placing the active (topmost) edge of the fret at co-

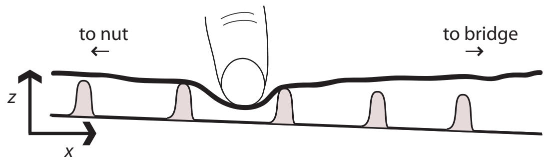

Figure 2: *Side view of the vertical shape of the string when pressed against frets by a player's finger.*

ordinates

$$x_n = L_s \left(1 - 2^{-n/12}\right), \tag{1}$$

where L_s is the length of the string and n the fret number counted from nut to bridge [10].

Since the bridge is taller than the nut, the neck of the instrument, which supports the fretboard, appears as slightly tilted with respect to the string rest position. As a result, when the player pushes the string against the fretboard, the string rests on two frets (or one fret and the nut for the lowest fingered tone $n = 1$), as shown in Figure 2. The leading fret is the one closest to the bridge (rightmost) and is responsible for tuning by reducing the length of the active portion of the string (from fret to bridge). The trailing fret (leftmost) further blocks residual vibrations from reaching the inactive portion of the string. Occasional collisions with other frets may also occur if the string is vigorously plucked in the vertical direction.

Fingering on the fretboard produces a deflection of the string that slightly modifies the string length. Characteristics influencing the intonation of fretted guitar tones are described in [11]. The player does not need to push the finger all the way against the fingerboard: in order to produce proper tones it suffices that the string rests quite firmly on the leading fret. This is generally achieved by placing the finger as close as possible to the leading fret. The frequency of the tone slightly depends on how much the string is pushed towards the fingerboard.

With respect to the vertical polarization mode, the string appears as clamped to the fret. In the horizontal direction, however, the string is quite free to slip over the fret, as shown in Figure 3. The motion is subject to friction force in the direction opposite to velocity and to the restoring tensile forces of the string along the y direction. The horizontal oscillations of the string are further coupled to the finger behind the leading fret, which essentially acts as an elasto-plastic spring damper. Residue oscillations further travel toward the trailing fret, subject to further friction, and toward the nut and back. However, the amplitude of oscillation in this trailing path is negligible due to the damping and clamping introduced by the finger pressing the string towards the neck.

3. FRICTION MODEL

In order to simulate the stick-slip motion of the string over the fret in the horizontal polarization mode, a suitable model for the friction is necessary. A sufficiently general scheme is derived from [8, 12], where the effect of friction is modeled as a dynamic system, know as the Lu-Gre model, which generalizes the Coulomb model of friction. In this model, the surfaces are thought of as being randomly coated by elastic bristles, which deflect as two contacting surfaces are set in relative motion.

An extension [12] of the bristle based model has been previously used in sound synthesis to capture the dynamics of the violin bow [13] or to model general friction interactions among rigid bodies [14] with modal synthesis. Here we apply a similar model to the string-fret interaction and provide a realization for use in digital waveguide simulation of the string.

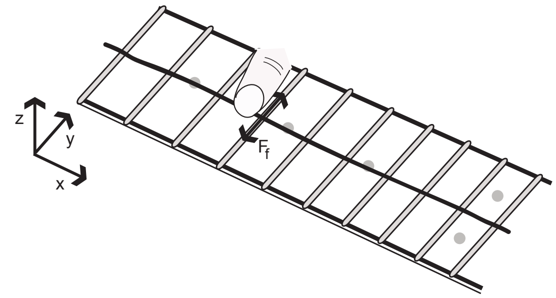

Figure 3: *Top view of the guitar fretboard showing string-fret friction force in the horizontal polarization mode of the string.*

Although remarkable generalizations of the friction model have been introduced [9], which allow us to capture subtle phenomena such as friction hysteresis, our aim is to obtain a simple system capturing the main characteristics of the string-fret interaction at reasonable computational costs. Unlike in friction driven sound, the friction noise in the fret-string interaction is not a main audible feature but friction does contribute to the dynamics of the string, which is audible through modulation of the elongation and slowdown of the string. Its inclusion contributes to a more naturally sounding model.

3.1. Bristle-Based Friction Models

Following [12], the average deflection ξ of elasto-plastic bristles can be modeled by the following first order differential equation:

$$\frac{d\xi}{dt} = v_{rel}\left(1 - \alpha(v_{rel}, \xi)\frac{\xi}{\xi_{ss}(v_{rel})}\right) \tag{2}$$

where v_{rel} denotes the relative velocity of the contacting surfaces (the string and the fret in our case). The function $\alpha(v, \xi)$ allows us to capture the elasto-plastic behavior of the bristles for large displacement. In a simplified model (Lu-Gre) one can let $\alpha(v_{rel}, \xi) = 1$.

The function $\xi_{ss}(v)$ provides the limit value for the deflection in steady state where the relative velocity v, instanciated by v_{rel} in (2), and the average bristle deflection are constant.

The friction force f_f can be written in terms of bristle displacement and relative velocity as follows:

$$f_f(\xi, \dot{\xi}, v_{rel}) = \sigma_0\xi + \sigma_1\frac{d\xi}{dt} + \sigma_2 v_{rel} \tag{3}$$

where σ_0 represents the stiffness of the bristles' spring, σ_1 is a damping coefficient and σ_2 is the viscous friction coefficient and $\dot{\xi} = \frac{d\xi}{dt}$.

In the Lu-Gre parametrization one provides $\xi_{ss}(v)$ as follows:

$$\xi_{ss}(v) = \frac{\mathrm{sign}(v)}{\sigma_0}\left(f_c + (f_s - f_c)\,e^{-(v/v_s)^2}\right) \tag{4}$$

where f_c is the magnitude of Coulomb friction force, f_s is the magnitude of the static friction (*stiction*) force and v_s is the Stribeck

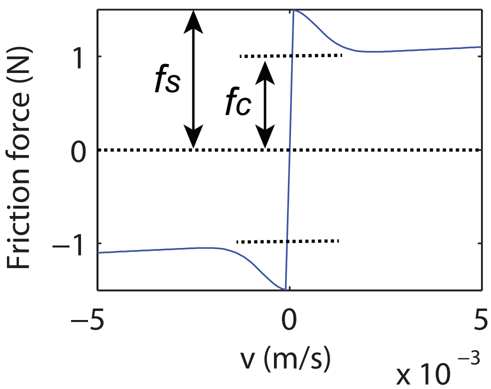

Figure 4: *Typical steady state friction force versus velocity.*

velocity, which controls the characteristics of the Stribeck effect where friction continuously decreases as relative velocity increases in the low velocity regime. A typical plot of the steady state friction force versus velocity is shown in Figure 4.

The bristle model is sufficiently general to capture most of the phenomena associated to friction. In particular, due to the dependency of the force on the relative velocity of the contact surfaces, the model is able to produce stick-slip motion, continuously switching from static to kinetic friction according to velocity regimes.

4. DIGITAL WAVEGUIDE SIMULATION OF FRET-STRING INTERACTION

In this section we consider the friction model reviewed in Section 3.1 to simulate the behavior of the string pressed against the fret in the vertical z direction but free to move in the horizontal y direction. We will first derive the continuous time system describing the string-fret interaction and then provide a discrete version of the model based on bilinear transform. Furthermore, we provide a scheme to compute the solution of the nonlinear difference equation describing the string-fret node, which is based on the so-called K-method [15].

4.1. Continuous Time String-Fret Node

Let us denote by $u_y(x,t)$ and $u_z(x,t)$, respectively, the value of the string displacement at time t and position x along the string for the y and z polarization modes.

Disregarding nonlinear [16] and dispersive effects [17, 18], the wave equation holds for segments of the string not in contact with other objects such as the plectrum or the player's finger and the fret. Assume that the only object in contact with the string is the fret, touching the string on a segment of width Δ and centered at coordinate x_f, then for a string of length L_s we have

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}; \qquad x \in \left]0, x_f - \tfrac{\Delta}{2}\right[\cup \left]x_f + \tfrac{\Delta}{2}, L_s\right[, \quad (5)$$

where $u(x,t)$ denotes any of the two polarization displacement $u_y(x,t)$ or $u_z(x,t)$, while $c = \sqrt{K_0/\mu}$ is the propagation velocity, K_0 is the tension of the string, and μ is the linear mass density, and they are all assumed to be constant. Here we have disregarded all propagation losses along the string, as these can be consolidated at one of the extremities and embedded in the bridge model [19].

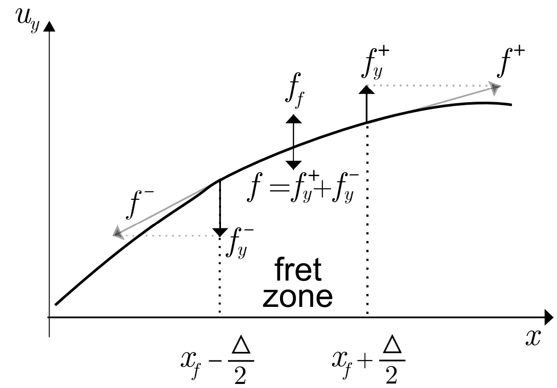

Figure 5: *String segment subject to tensile forces f^- and f^+ and friction f_f against the fret. For the y polarization, the tensile forces are projected along the y direction, obtaining f_y^- and f_y^+. The resultant $f = f_y^- + f_y^+$ of the projected tensile forces is considered as acting at the point x_f.*

The solution of (5) can be written in D'Alembert form as a superposition of a left-going u^- and a right-going u^+ wave:

$$u(x,t) = u^-(x,t)+u^+(x,t) = u_l(t+x/c)+u_r(t-x/c), \quad (6)$$

where $u_l(x/c) = u_r(x/c) = u(x,0)/2$ for a static initial displacement condition.

For the vertical polarization mode u_z the portion of the string in contact with the fret can be largely assumed to be clamped in normal playing conditions. In this case the left-going wave u_z^- is perfectly reflected at the fret back towards the bridge, i.e., $u_z^+(x_f,t) \approx -u_z^-(x_f,t)$. As already remarked, the same is not true for the horizontal polarization mode u_y. On the string-fret contact segment, which we will also refer to as the fret zone shown in Figure 5, the equilibrium equation of the string with the bristle based dynamic system modeling friction (2) is enforced:

$$\mu\Delta\frac{\partial^2 u_y}{\partial t^2} = f(t) - f_f(\xi,\dot{\xi},v_y) \qquad x \in \left]x_f - \tfrac{\Delta}{2}, x_f + \tfrac{\Delta}{2}\right[, \quad (7)$$

where the force $f(t)$ is the resultant of the transversal component of the tensile force of the string acting at the extreme points of the contact segment and $f_f(\xi,\dot{\xi},v_y)$ is the friction force (3). The velocity v_y is the relative velocity of the string over the fret, which coincides with string displacement velocity in the y-polarization mode. It is therefore convenient to rewrite (7) all in terms of $v_y(x,t) = \frac{\partial u_y}{\partial t}$:

$$\mu\Delta\frac{\partial v_y}{\partial t} = f(t) - f_f(\xi,\dot{\xi},v_y) \qquad x \in \left]x_f - \tfrac{\Delta}{2}, x_f + \tfrac{\Delta}{2}\right[. \quad (8)$$

At small string displacements, for the tensile force we have:

$$f(t) = K_0\left(\left.\frac{\partial u_y}{\partial x}\right|_{x=x_f+\frac{\Delta}{2}} - \left.\frac{\partial u_y}{\partial x}\right|_{x=x_f-\frac{\Delta}{2}}\right). \quad (9)$$

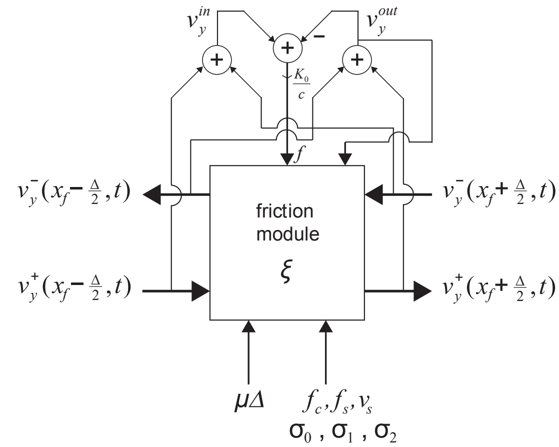

Figure 6: *Block diagram representing the friction node for the simulation of string-fret interaction.*

Since from (6) we have

$$\frac{\partial u}{\partial x} = \frac{1}{c}\left(v^{-}(x,t) - v^{+}(x,t)\right) \tag{10}$$

where

$$\begin{array}{l} v^{-}(x,t) = \frac{\partial u^{-}}{\partial t} \\ v^{+}(x,t) = \frac{\partial u^{+}}{\partial t} \end{array} \tag{11}$$

then (9) can be rewritten as follows:

$$f(t) = \frac{K_0}{c}\left(v_y^{in}(t) - v_y^{out}(t)\right), \tag{12}$$

where we have defined v_y^{in} as the velocity wave entering the fret zone and v_y^{out} as the velocity wave leaving the fret zone, i.e,

$$\begin{array}{l} v_y^{in}(t) = v_y^{-}(x_f + \frac{\Delta}{2}, t) + v_y^{+}(x_f - \frac{\Delta}{2}, t) \\ v_y^{out}(t) = v_y^{+}(x_f + \frac{\Delta}{2}, t) + v_y^{-}(x_f - \frac{\Delta}{2}, t). \end{array} \tag{13}$$

Assimilating $v_y(x_f, t)$ to $v_y^{out}(t)$, i.e., shrinking the system (8) to a point, while retaining the finite mass $\mu\Delta$, we obtain the string-fret node equation:

$$\mu\Delta\frac{dv_y^{out}}{dt} = \frac{K_0}{c}\left(v_y^{in}(t) - v_y^{out}(t)\right) - \sigma_0\xi - \sigma_1\frac{d\xi}{dt} - \sigma_2 v_y^{out}(t), \tag{14}$$

where we have substituted (3) and (12) in (7) after establishing that $v_{rel} = v_y^{out}$. A block diagram of the string-fret interaction node is shown in Figure 6.

The bristle displacement function ξ in (14) must satisfy equation (2). Defining a state vector

$$\mathbf{x} = \begin{bmatrix} v_y^{out} \\ \xi \end{bmatrix}, \tag{15}$$

equations (14) and (2) can be put in the form of a nonlinear state space system:

$$\left\{ \begin{array}{l} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}v_y^{in} + \mathbf{e}\phi \\ \phi = \rho(\mathbf{x}) \end{array} \right., \tag{16}$$

where

$$\begin{array}{l} \mathbf{A} = \frac{-1}{\mu\Delta}\begin{bmatrix} \sigma_2 + \frac{K_0}{c} & \sigma_0 \\ 0 & 0 \end{bmatrix} \\ \mathbf{b} = \frac{K_0}{c\mu\Delta}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \mathbf{e} = \begin{bmatrix} -\frac{\sigma_1}{\mu\Delta} \\ 1 \end{bmatrix} \end{array} \tag{17}$$

and

$$\rho\left(\begin{bmatrix} v_y^{out} \\ \xi \end{bmatrix}\right) = v_y^{out}\left(1 - \alpha(v_y^{out}, \xi)\frac{\xi}{\xi_{ss}(v_y^{out})}\right) \tag{18}$$

is a scalar function of the state vector.

In the form (16) the system describing the string-fret node is ready for suitable discretization required in digital simulations of strings.

4.2. Discrete Time Computation of the String-Fret Node

In this section we carry out the discretization of the system (16) using the bilinear transformation. This method has the advantage of preserving passivity of the system, which prevents the introduction of instability due to numerical approximation of the derivatives. We will also show how to handle the delay-free loops in the computation.

The system in (16) is characterized by first order derivatives. In Laplace transform a differentiator is equivalent to multiplication by the Laplace variable s. By bilinear transformation, s is replaced by $2(1-z^{-1})/T(1+z^{-1})$, where T is the sampling time interval. Accordingly, a first order differential equation of the type

$$\dot{\eta}(t) = f(\eta(t), t) \tag{19}$$

is led by bilinear transformation to the recurrence

$$\eta(n) = \eta(n-1) + \frac{T}{2}\left[f(\eta(n), n) + f(\eta(n-1), n-1)\right], \tag{20}$$

where we dropped the factor T in the arguments of the functions.

Using this rule, it is easy to discretize the system (16). The discrete version of equation for the first state component expresses the current value of the output velocity $v_y^{out}(n)$ in terms of past values of v_y^{out}, present and past values of the input velocity v_y^{in}, present and past values of ϕ and present and past values of bristle deflection ξ. The discrete version of the equation for the second component of the state becomes the recurrence:

$$\xi(n) = \xi(n-1) + \frac{T}{2}(\phi(n) + \phi(n-1)). \tag{21}$$

This recurrence can be substituted in the first state component recurrence in order to remove the dependency from the present value of ξ, obtaining

$$\begin{array}{l} v_y^{out}(n) = c_1 v_y^{out}(n-1) + c_2\xi(n-1) \\ +c_3\left[v_y^{in}(n) + v_y^{in}(n-1)\right] + c_4\left[\phi(n) + \phi(n-1)\right] \end{array} \tag{22}$$

where

$$
\begin{aligned}
c_1 &= \frac{2\,\mu\,\Delta c - Tc\sigma_2\ - TK_0}{2\,\mu\,\Delta c + Tc\sigma_2\ + TK_0} \\
c_2 &= -\frac{2Tc\sigma_0}{2\,\mu\,\Delta c + Tc\sigma_2\ + TK_0} \\
c_3 &= \frac{TK_0}{2\,\mu\,\Delta c + Tc\sigma_2\ + TK_0} \\
c_4 &= -\frac{Tc\left(\sigma_1\ + \frac{T}{2}\sigma_0\right)}{2\,\mu\,\Delta c + Tc\sigma_2\ + TK_0}.
\end{aligned}
\tag{23}
$$

We are then left with a recurrence for v_y^{out} that depends on known values, except for that of $\phi(n)$. Yet, the equation for ϕ requires the value of $v_y^{out}(n)$ in order to be computed, which is a delay-free loop of the system. This delay-free loop must be properly handled in order to be able to find the solution, as described next.

Substituting the recurrence for $v_y^{out}(n)$ and that for $\xi(n)$ in the vector argument of the function ρ in (16), one obtains an equation of the type

$$\phi(n) = g(\phi(n), n), \tag{24}$$

where g is a known function, which is built from ρ by isolating the dependency on ϕ and reducing all other dependencies to an explicit dependency on time index n. This equation can be solved by finding, at any sample index n, a local zero of the function

$$\zeta - g(\zeta, n), \tag{25}$$

which can be achieved by means of Newton-Raphson root finding method. Look-up tables for the roots can be precalculated in order to ease real-time computation [15]. The root ζ of (25) is assigned to $\phi(n)$ and all other quantities are known in order to compute $v_y^{out}(n)$ and $\xi(n)$, which describes how to handle the delay-free loop in the computation.

4.3. Fret Junction in Digital Waveguides

The discrete time realization of the fret-string interaction block illustrated in the previous section is directly usable as a block in digital waveguides for the synthesis of strings based on velocity waves. The block is only included in the waveguide simulating the horizontal y-polarization mode. The input velocity v_y^{in} is obtained by summing the input velocities v_{in}^{+} and v_{in}^{-} from the two rails of the waveguide. The output velocity v_y^{out} obtained from the fret-string system is equally fed to the two rails of the waveguide. In order to force the output velocity at the fret contact point, a scattering junction of the type

$$\begin{bmatrix} v_{out}^{-} \\ v_{out}^{+} \end{bmatrix} = \mathbf{S_c} \begin{bmatrix} v_{in}^{-} \\ v_{in}^{+} \end{bmatrix} + \frac{v_y^{out}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{26}$$

where

$$\mathbf{S_c} = \frac{1}{2} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \tag{27}$$

is included, similar to what described in [6] in order to force after-collision displacement.

As this paper is part of a larger project for the accurate simulation of the guitar, and as in our system displacement waves are preferred for their ease of use in the detection of string-neck or string-fret collisions, differentiator and integrator blocks have to be introduced in order to obtain the input velocity. These blocks can be realized, respectively, by directly taking the first order difference of the incoming signal and by a discrete time leaky integrator.

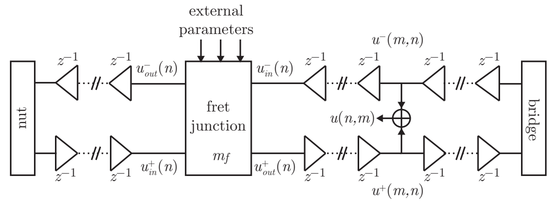

Figure 7: *Inclusion of a fret junction in a digital waveguides for string displacement waves (horizontal polarization).*

A valid alternative is to use a differentiator and an integrator derived by applying the bilinear transformation to the analog differentiator and integrator, similar to what described in Section 4.2. The bilinear differentiator is given by the following recurrence:

$$v(n) = -v(n-1) + \frac{2}{T}\left(u(n) - u(n-1)\right) \tag{28}$$

while the bilinear integrator is

$$u(n) = u(n-1) + \frac{T}{2}\left(u(n) + u(n-1)\right) \tag{29}$$

as in (20). Bilinear integrator and differentiator also have the advantage of being inverse of each other.

The insertion of the fret junction in a displacement wave based digital waveguide is shown in Figure 7.

The parameters of the underlying friction model are the magnitudes f_c of the Coulomb and f_s of the stiction force, the Stribeck velocity v_s, together with bristles' stiffness σ_0, damping σ_1 and viscous friction coefficient σ_2. Also, the elasto-plastic map function $\alpha(v_{rel}, \xi)$ needs to be specified. In first approximation we disregarded elasto-plastic phenomena and enforced a simplified Lu-Gre model setting $\alpha(v_{rel}, \xi) = 1$.

The string-fret friction parameters can and should be measured accurately from the string-fret friction characteristics. String-fret friction measurement will be the object of further studies. A special laboratory set up is required in which a free piece of guitar string is pulled, at several constant velocities, over a single fret. The friction force is measured by means of a miniature accelerometer.

In our preliminary experiments we used reference values for these parameters as follows. For the σ parameters we let $\sigma_0 = 10^5$ N/m, $\sigma_1 = 300\ Ns/m$ and $\sigma_2 = 10^{-3}\ Ns/m$. For the stiction force we used a 50% increment of the Coulomb force level, i.e., $f_s = 1.5 \times f_c$, where forces are measured in Newtons N.

The Coulomb force can be estimated as the the friction coefficient for metal, about 0.5, times the normal force at the fret. However, we found that the value of the friction coefficient for metals is too high for the simulation of the string. This is due to the fact that both string and fret are rounded and smooth surfaces, more resembling ball bearing than flat surfaces in contact. Indeed the string is also allowed to roll over the fret to some extent, generating torsional effects on the string. Large friction coefficients tend to stop the string and / or introduce noise that is not typical of this type of interaction.

The normal force at the fret can be estimated from the vertical component of the tensile force due to the bending of the string at the fret, given by the force $F_{1,z}$ as shown in Figure 8. This force is essentially given by the slope of the string at the fret times the

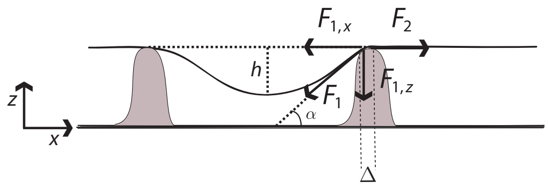

Figure 8: *Components of the tensile force due to the bending of the string by the player's finger near the fret.*

string tension K_0. It consists of two components: a static component due to the bending of the string by the finger and a dynamic one due to string motion in the vertical z polarization. The dynamic component introduces non-linear coupling of the z and y polarizations. However, in usual playing modes, all dynamic variations can be disregarded as they only provide a very minor variation of the normal force due to the largely bent string at the fret.

Finally, for the Stribeck velocity we used a reference value of $v_s = 10^{-3}\ m/s$.

The fret junction must be completed by a model of the player's finger placed next to the fret. An accurate model can be derived from the damped spring-mass system presented in [7] for the finger plucking where, in the case of the finger over the fret, the coordinates of the fingers are static. However, the effect of the finger behind the fret has no dramatical influence on the sound. Thus, a simpler reflector with damping can be suitably employed in normal playing conditions. A further completion of the model requires the inclusion of a second fret junction corresponding to the trailing fret on which the string is resting. However, this is quite unnecessary provided that one suitably blocks the oscillations on the right portion of the string to propagate to the left portion of the string with respect to the fret.

The dynamic model of friction contributes to provide an accurate simulation of the deflection of the string over the fret in the horizontal polarization. The string simulation is completed by a tension modulation module that allows us to obtain the pitch-bending characteristic of plucked guitar tones, which is otherwise absent in the wave equation as one assumes there that the tension is constant. Alternate implementations of tension modulation methods can be found in [20, 21].

5. CONCLUSIONS

In this paper we have considered the sliding of the string over the fret as a phenomenon to be modeled for the accurate synthesis of fretted string instruments. We provided a discrete model derived from bristle based models of friction. The continuous time model is described by a nonlinear state-space system, which is discretized by means of the bilinear transformation. The computation requires the solution of a nonlinear equation, which can be achieved by Newton-Raphson root finding method.

The acoustical results are very realistic and require very moderate amount of friction, as controlled by the friction coefficient in the Coulomb force, which is a parameter of the model. High friction coefficients tend otherwise to stop the string too early in the vertical polarization and introduce unnatural noise.

Sound examples can be found at `http://staffwww.itn.liu.se/~giaev/soundexamples.html`.

6. ACKNOWLEDGMENTS

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